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Asymptotic Properties of a System with Nonzero Total Mass*

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When certain geometrical figures, which in a flat space would be closed, such as parallelograms of geodesics, are constructed far from the massive source of a gravitational field, they will fail to close in the Riemann-Einstein manifold considered here. If, on removing such a figure to infinity we scale it up proportionally to its distance from the source, the size of the gap will have an asymptotic value of the order of the Schwarzschild radius of the massive source. Accordingly one cannot construct an asymptotically geodesic coordinate system in the whole region far from the source that will mesh properly everywhere at spatial infinity. Coordinate systems in actual use are not as nearly geodesic as would be compatible with the local curvature of the manifold.

1. INTRODUCTION

THE Einstein equations for the gravitational field and its potentials possess some solutions in which the metric tensor, for suitably chosen coordinates, approaches the Minkowski values if we proceed toward infinity in any space-like direction or along the light cone. The best known of these solutions is Schwarzschild's metric, which represents the field produced by a spherically symmetric mass distribution outside the region containing the mass. Other static solutions with axial symmetry and similar boundary conditions at spatial infinity had been discovered by Bach and Weyl and by Levi-Civita by 1920. There is also some evidence that there exist nonstatic solutions, with radiative properties, that are asymptotically flat in the same sense. In all these solutions, the deviation of the metric field from the Minkowski values decreases in any spatial or null direction at least with the first negative power of the coordinate distance from the coordinate origin. The components of the curvature tensor decrease as fast as r^{-3} in the static solutions but as r^{-1} in the null directions for cases with radiation.

The totality of all these known or conjectured solutions is usually described as "asymptotically flat." Appropriate coordinate systems are denoted by the same terminology; and coordinate transformations that lead from one asymptotically flat coordinate system to another are occasionally called "asymptotic Lorentz transformations." Recently these transformations were examined by group-theoretical methods; an attempt was made to see whether it was possible to construct quantities that transform with respect to these transformations as if they were actual Lorentz transformations. Into the transformation law of such quantities would then enter only the six parameters of the homogeneous and the four parameters of the inhomogeneous Lorentz transformations but not the arbitrary curvilinear relations in the interior. The result of this examination was that there are indeed quantities that will transform under the homogeneous Lorentz group —the classical examples are the free vectors representing total energy and total linear momentum, and similar integrals—but none that will transform under rigid translations as well while being invariant with respect to the curvilinear transformations confined to finite regions of space-time.[1]

If this result is to be accepted, it implies that in these asymptotically flat solutions it is possible to define mutually-parallel quadruples of orthonormal vectors ("Vierbeine") everywhere at spatial infinity, that the affine connection is asymptotically integrable at great distances from the world tube representing the source of the gravitational field. But by the same token, it is not possible to define an asymptotically-integrable rectilinear coordinate system. Because the results of a formal argument such as the one developed in the earlier paper may well depend on the adoption of mathematically convenient but physically unfortunate assumptions, it seemed worthwhile to explore in a more intuitive manner whether this result is indeed true.

The distinction between an integrable quadruple of directions and an integrable rectilinear coordinate system may be explained thus: To establish the integrability of parallel displacement of directions, or vectors, we must known that a vector carried parallel to itself along a closed curve will upon return to the point of departure be unchanged. The change of a vector on such a closed circuit will be proportional to the average Riemannian curvature multiplied by the area of the cap bounded by the circuit. In the case of static solutions, at any rate, this product will be of the order $O(r^{-1})$ for curves whose linear dimensions are $O(r)$. For the construction of rectilinear coordinate systems, it is necessary that a parallelogram consisting of segments of geodesic curves that are pairwise parallel and of equal length be asymptotically closed. Any gap at one corner will be caused by a discrepancy in the parallelism of the tangent vectors of the corresponding sides, multiplied by the length of each side, and hence roughly proportional to the product of the average

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[1] P. G. Bergmann, Phys. Rev. **124**, 274 (1961).

curvature, the area of the cap bounded by the parallelogram, and the length of a side. Hence, again for the Schwarzschild case, the gap will not decrease for a figure all of whose dimensions are proportional to r, but it will be of asymptotically constant amount, and of the order of the Schwarzschild radius, $2Gm/c^2$, of the physical system that gives rise to the gravitational field, multiplied by the ratio between the area of the cap and the square of the distance from the origin. In Sec. 2, we shall show by means of three simple examples that this rough estimate is correct.

2. SAMPLE CALCULATIONS

In what follows we shall not construct "parallelograms" but figures that can be analyzed simply by the utilization of symmetry properties of the Schwarzschild metric. We shall denote the Schwarzschild radius itself by R,

$$R=2Gm/c^2, \tag{1}$$

where G is Newton's constant of gravitation, m the mass, and c the speed of light. Symbols such as r and $\bar{r}$ will be reserved for lengths, or coordinate distances, that will be made to increase without limit as we displace our figures toward spatial infinity. Numerical factors will be omitted in all formulas.

(a) A Purely Space-Like Figure

In the three-dimensional space that is normal to the time-axis of the Schwarzschild metric (this hypersurface is defined invariantly), we consider a plane circle[2] that lies in a plane through the coordinate origin and is concentric about the origin. This circle is a curve of constant curvature and without torsion. If desired, it may be approximated by a plane regular polygon whose sides are geodesics. If the coordinate radius of this circle in the customary Schwarzschild line element is denoted by r, then the radius of curvature (the reciprocal magnitude of the curvature vector, which is space-like) equals

$$1/k=r/(1-R/r)^{\frac{1}{2}}. \tag{2}$$

On this curve we choose an arbitrary point of departure and then proceed a distance equal to $2\pi/k$. If we were to perform this operation in a flat space, we should return precisely to the point of departure. In this case we do not but overshoot by an amount that is proportional to R and asymptotically independent of r (the amount of asymptotic overshoot actually equals πR).

(b) Circular Trajectories

Consider the motion of a test particle along a circular trajectory at a distance r from the coordinate origin.[3] In the Newtonian approximation, which is sufficient for our purposes, the speed on such a circular trajectory is

$$v=c(R/2r)^{\frac{1}{2}}. \tag{3}$$

Suppose we set off two particles in opposite directions and let them go for a length of time t. By that time, the angular distance between them has become

$$\theta=2vt/r=2ct(R/2r^3)^{\frac{1}{2}}. \tag{4}$$

If this angle is small, we may calculate the difference between their distance as a result of the circular motion and the distance that they should have reached under similar circumstances in field-free space by means of simple power expansions of the trigonometric functions involved. This difference turns out to be of the order

$$\begin{aligned}\Delta l &\approx R\theta^3\approx A\times ct\times R/r^3,\\ A&\approx lct,\end{aligned} \tag{5}$$

where R/r^3 is the order of the Riemann-Gauss curvature, A the "area" (in cm^2) bounded by the two trajectories and the distance between the end points, and ct the "length" (in cm) of a trajectory.

(c) Radial Trajectories

Let us consider at the same time two points lying along a straight line through the origin, and start two particles along escape trajectories along the radius. The connecting radial segment and the two escape trajectories are all geodesic curve segments. After a time t the two particles will be separated by a different distance, the radius again being a geodesic curve segment. If we call the initial coordinate distances from the origin r_1 and r_2, respectively, the escape velocities, in the Newtonian approximation, will be

$$v_1=c(2R/r_1)^{\frac{1}{2}},\quad v_2=c(2R/r_2)^{\frac{1}{2}}. \tag{6}$$

As a result of these different initial velocities, the two test particles will change their mutual distance in the course of time. However, there will be an additional change in distance because of the different accelerations they suffer. If both the time t during which their motion is being followed, and their initial distance satisfy the inequalities

$$t\ll r/v\approx(r^3/2c^2R)^{\frac{1}{2}},\quad |r_1-r_2|\ll r_1+r_2, \tag{7}$$

we can again expand into power series. We find that the difference between the final distance of the two test particles and the distance that they should have if the experiment had been performed in field-free space with the same initial separation and the same initial velocities is of the order

$$\begin{aligned}\Delta l&\approx\frac{c^2R}{r^3}|r_1-r_2|t^2\approx\frac{R}{r^3}Act,\\ A&\approx ct|r_1-r_2|.\end{aligned} \tag{8}$$

Except for numerical factors of the order of unity, the

[2] For the definition of circular paths see Appendix A.

[3] This trajectory is not a "circular path" but a helix in four-dimensional space.

results (5) and (8) are the same.

3. CONCLUSION

Given the kind of space-time manifold in which an isolated accumulation of matter produces a Schwarzschild-like metric field, an attempt to construct a rectilinear coordinate system in the region far from the concentration of matter will lead to discrepancies in the assignment of coordinate values to world points that are a distance r from the mass concentration and a distance $\bar{r}$ from the proposed origin of the coordinate system of the order

$$\Delta x \approx (\bar{r}^2/r^2)R, \quad \Delta t \approx (1/c)\Delta x, \tag{9}$$

if we connect the world point in question with the proposed origin by more than one path consisting of geodesic curve segments, even though none of these paths approaches the mass concentration closer than a distance of the order of r. If we extend our construction in all space-like directions so as to surround the mass concentration by a coordinate network, r and $\bar{r}$ will be of the same order of magnitude, and the assignment of coordinate values to world points throughout the region will be uncertain by an amount of the order of R, the gravitational radius of the mass concentration. For a geometric visualization, see Appendix B.

This uncertainty is by any ordinary standard exceedingly small. For a nucleon the Schwarzschild radius is of the order of 10^{-52} cm, for the whole earth 1 cm, and for the sun 10^5 cm (1 km). These distances are many orders of magnitude less than present-day experimental tolerances in the fixation of frames of reference. Nevertheless, any uncertainty in the determination of physical quantities inherent in the theoretical foundations tends to teach us something about the theory itself, regardless whether it has immediate observational consequences or not.

We hope to extend these investigations also to metric fields that involve some gravitational radiation. By a qualitative argument one would expect that the uncertainties in the construction of a rectilinear coordinate system should increase (rather than assume constant limiting values) with increasing distance from the source, and that in the presence of gravitational radiation the affine connection itself is not asymptotically integrable.

APPENDIX A

In this Appendix we generalize the notion of a circle. In an n-dimensional Riemannian space with at least two space-like dimensions we call a three-times-continuously-differentiable space-like curve a "space-like circular path"[4] if its tangential vector $u^a = dx^a/ds$ fulfills the equation

$$\delta^2 u^a/\delta s^2 = -\lambda^2 u^a, \quad \lambda \neq 0; \quad u^a u_a = -1.$$

[4] E. Cartan, Chem. Revs. **184**, 138 (1927).

$\delta/\delta s$ means invariant differentiation along the curve. If we assume that the curve is suitably imbedded in a congruence of curves with the tangential vector field u^a, we may write this equation as

$$(u^a{}_{;b}u^b)_{;c}u^c = -\lambda^2 u^a, \quad \lambda \neq 0.$$

A semicolon again denotes covariant differentiation. Introducing the vector $k^a = u^a{}_{;b}u^b$ of the geodesic curvature, we can write the condition also

$$\delta k^a/\delta s = k^a{}_{;c}u^c = -\lambda^2 u^a, \quad \lambda \neq 0.$$

Since u^a is a unit vector, k_a is orthogonal to u^a; thus $k_a u^a = 0$. Therefore we get for the derivative of the curvature $\kappa^2 = -k_a k^a$ along the curve

$$\delta(\kappa^2)/\delta s = 2\kappa\delta\kappa/\delta s = -2k_a \delta k^a/\delta s = 2\lambda^2 k_a u^a = 0.$$

Accordingly the curvature κ^2 is constant along the curve. On the other hand, we have

$$-\lambda^2 u^a u_a = \lambda^2 = u_a k^a{}_{;c}u^c = (u_a k^a)_{;c}u^c - u_{a;c}u^c k^a = -k_a k^a = \kappa^2.$$

Since λ is assumed to be different from zero, the curvature $\kappa = \pm\lambda$ does not vanish. The vector of the geodesic curvature is always a space-like vector different from zero. A space-like circular path can therefore never be a geodesic.

We shall now show that the "circles" defined in a Riemannian space with axial symmetry are space-like circular paths according to the definition above. Suppose that ξ^a is a space-like vector field. Writing

$$u^a = \chi^{-1}\xi^a, \quad u_a u^a = -1, \quad \chi = (-\xi_a \xi^a)^{\frac{1}{2}}, \tag{A1}$$

we introduce the vector of geodesic curvature by

$$k_a = u_{a;b}u^b. \tag{A2}$$

Next we define the tensor U_{abc} by the equation

$$U_{abc} = 3\xi_{[a}\xi_{b;c]} + 3\xi_{(a}\xi_{b;c)} - 2\xi_a \xi_{(b;c)}, \tag{A3}$$

where round and square brackets denote, respectively, the symmetric and antisymmetric parts of the expressions. Equation (A3) may also be written as

$$U_{abc} = \xi_c \xi_{a;b} - \xi_a \xi_{c;b} + \xi_b \xi_{c;a}. \tag{A4}$$

Replacing ξ_a in this equation by χu_a, where u_a is a unit vector, we obtain

$$U_{abc} = \chi^2 u_c [u_{a;b} + (\ln\chi)_{,a} u_b + \chi^2 (u_b u_{c;a} - u_a u_{c;b}). \tag{A5}$$

Defining a two-index tensor U_{ab} by

$$U_{ab} = -\chi^{-2} U_{abc} u^c, \tag{A6}$$

we get from (A5)

$$U_{ab} = u_{a;b} + (\ln\chi)_{,a} u_b, \tag{A7}$$

since the constant length of u_a implies that $u_{c;a}u^c$ vanishes.

So far the vector field ξ_a has not been specialized beyond being required to be space-like. Assuming now

that ξ_a is a hypersurface-orthogonal Killing field, we may infer at once that U_{abc} vanishes. A hypersurface-orthogonal Killing field is defined by the equations

$$\xi_{[a}\xi_{b;c]}=0, \quad \xi_{(a;b)}=0. \tag{A8}$$

Because of these equations, the first and third terms on the right-hand side of (A3) vanish. The second term is also equal to zero since

$$\xi_{(a}\xi_{b;c)}=\xi_{(a}\xi_{(b;c))}. \tag{A9}$$

For a hypersurface-orthogonal Killing field, therefore, U_{abc} and, according to (A6) also U_{ab}, vanish.

Conversely the vanishing of U_{ab} implies that ξ_a is a hypersurface-orthogonal Killing field. This follows directly from the identities

$$U_{(ab)}=\chi^{-1}\xi_{(a;b)} \tag{A10}$$

and

$$\chi U_{[ab}\xi_{c]}=\xi_{[a;b}\xi_{c]}. \tag{A11}$$

The first shows that ξ_a is a Killing vector if $U_{(ab)}$ vanishes; the second that ξ_a is hypersurface-orthogonal if $U_{[ab]}$ vanishes.

We shall use these results to show that the trajectories with a hypersurface-orthogonal Killing field as unnormalized tangent vectors are circular paths. We form the expression U_a,

$$U_a=(U_{ab}u^b)_{;c}u^c-(U_{bc}u^bu^c)_{;a}-(\ln\chi)_{,b}U^b{}_a. \tag{A12}$$

According to (A2) and (A7), the first term on the right becomes

$$(U_{ab}u^b)_{;c}=k_{a;c}u^c-(\ln\chi)_{,a;c}u^c. \tag{A13}$$

In consequence of (A7) the second term on the right of (A13) yields

$$\begin{aligned}-(U_{bc}u^bu^c)_{;a}&=((\ln\chi)_{,b}u^b)_{;a}\\&=(\ln\chi)_{,c;a}u^c+(\ln\chi)_{,b}u^b{}_{;a}.\end{aligned} \tag{A14}$$

The last term in (A12) according to (A7) turns into

$$-(\ln\chi)_{,b}U^b{}_a=-(\ln\chi)_{,b}u^b{}_{;a}-(\ln\chi)_{,b}(\ln\chi)^{,b}u_a. \tag{A15}$$

Addition of the last three equations leads to

$$U_a=k_{a;c}u^c-(\ln\chi)_{,b}(\ln\chi)^{,b}u_a. \tag{A16}$$

As shown above, U_{ab} vanishes for a hypersurface-orthogonal Killing field. From (A12) and (A16) we conclude in this case

$$k_{a;c}u^c=(\ln\chi)_{,b}(\ln\chi)^{,b}u_a. \tag{A17}$$

This is the equation of a space-like circular path if

$$(\ln\chi)_{,b}(\ln\chi)^{,b}<0. \tag{A18}$$

We obtain for the curvature κ^2

$$\kappa^2=-(\ln\chi)_{,b}(\ln\chi)^{,b}. \tag{A19}$$

This expression for the curvature can be computed easily by use of the standard form of an axially symmetrical metric. Introducing the vector

$$v_a=\chi^{-1}u_a, \tag{A20}$$

we obtain from (A7)

$$U_{ab}=\chi(v_{a;b}+2(\ln\chi)_{(,a}v_{b)}). \tag{A21}$$

For a hypersurface orthogonal Killing field we have

$$U_{[ab]}=0, \tag{A22}$$

v_a is therefore a gradient. In every simply connected region in which our assumptions apply, we can thus introduce the coordinate v such that

$$v_{,a}=v_a. \tag{A23}$$

Choosing the $n-1$ other coordinates as independent solutions of the partial linear differential equation,

$$w_{,a}v^a=0, \tag{A24}$$

for the function w, we have a coordinate system in which all cross terms of the metric vanish. For w stands for all coordinates x^μ (Greek indices run from 1 to $n-1$) while v may be called x^n, and (A24) takes the form

$$g^{\mu n}=0. \tag{A25}$$

From (A20) we have

$$g^{nn}=g^{ab}v_{,a}v_{,b}=\chi^{-2}u_au_bg^{ab}=-\chi^{-2}. \tag{A26}$$

We have therefore

$$g_{\mu n}=0, \quad g_{nn}=-\chi^2. \tag{A27}$$

Since U_{ab} vanishes it follows from (A7) by multiplication with u^a that $\chi_{,a}v^a$ vanishes. g_{nn}, therefore, does not depend on the coordinate x^n. From (A19) and (A27) we can now infer that

$$\kappa^2=-\{\ln[(-g_{nn})^{\frac{1}{2}}]\}_{,\mu}\{\ln[(-g_{nn})^{\frac{1}{2}}]\}^{,\mu}. \tag{A28}$$

From this last formula it follows in particular that the ordinary circles in a Euclidean space are also space-like circular paths. On the other hand, it is easy to see that in a Euclidean space with two or more space dimensions every space-like circular path is an ordinary circle: We take an arbitrary point on our path and draw in this point the tangential vector and the curvature vector. Both are space-like, and orthogonal to each other. We choose the plane determined by this point and the two vectors as the x^1, x^2 plane. The index A may count the other hyperplanes orthogonal to each other and to the x^1 and x^2 hyperplanes. It follows then from the chosen initial conditions, and from the existence and uniqueness theorems for a system of ordinary differential equations, that the solution of our differential equations leads to $x^A=0$. Our curve is therefore a plane curve of constant curvature, and hence a circle.

APPENDIX B

We may form an intuitive idea of asymptotic flatness for two-dimensional surfaces with rotational symmetry.

We may call surfaces asymptotically plane if the surface imbedded into a three-dimensional Euclidean space approaches asymptotically and sufficiently smoothly a plane in the imbedding space (it may happen that such an imbedding is not possible globally, as in the case of a surface of constant negative curvature). The concept of asymptotic planeness is not intrinsic, since infinite surfaces are in general not rigid. We can imbed the surfaces in different ways so that the resulting imbedded surfaces cannot be transformed into each other by means of three-dimensional orthogonal mappings.

For surfaces with rotational symmetry, however, we can easily give a definition of asymptotic planeness that is intrinsic: We shall require that the imbedded surface is a surface of revolution in the imbedding space. It is not difficult to prove that two surfaces of revolution with the same intrinsic geometry are fixed, up to translations, rotations, and reflections in the three-dimensional Euclidean space. If we consider only these special imbedments, the concept of asymptotic planeness becomes an intrinsic property.

A simply connected regular surface with rotational symmetry may be described by the line element

$$ds^2 = dr^2 + f^2(r)d\varphi^2,$$

with $r \geqslant 0$, $0 \leqslant \varphi \leqslant 2\pi$, $f(0)=0$, $f'(0)=1$. The Gaussian curvature $K=K(r)$ is given by

$$K(r) = -f''(r)/f(r).$$

Such a surface is represented as a surface of revolution in a three-dimensional Euclidean space by $z=z(\rho)$, $\rho = +(x^2+y^2)^{\frac{1}{2}}$, where the z axis is the axis of rotation. We have $dr^2 = dz^2 + d\rho^2$, $f(r)=\rho$, $r=r(\rho)$, $r(0)=0$.

With these formulas we obtain, for instance, for a paraboloid $z=\frac{1}{2}\lambda\rho^2$, $\lambda>0$, after a simple calculation $K(r)\approx 1/4r^2$. Though the curvature falls off rather rapidly such a surface is evidently not asymptotically plane. This notion is defined by the conditions

$$\lim_{\rho\to\infty} z(\rho) = z_0 = \text{const}, \quad \lim_{\rho\to\infty} dz/d\rho = \lim d^2z/d\rho^2 = 0.$$

From the example of a cone with a rounded off vertex it may even be seen that the curvature can vanish everywhere apart from an arbitrarily small circular region, though the surface is not asymptotically plane.

In order to study rotationally symmetric surfaces with a hole around the center we use a slightly different approach. We take the line element of the surface in the form

$$ds^2 = e^{\lambda(r)}dr^2 + r^2 d\varphi^2,$$

with $r \geqslant r_0 > 0$, $0 \leqslant \varphi \leqslant 2\pi$. The one-sheeted hyperboloid is an example of such a surface. We have in the imbedding space, with $r^2 = x^2+y^2$,

$$ds^2 = dz^2 + r^2 d\varphi^2 = [(dz/dr)^2 + 1]dr^2 + r^2 d\varphi^2.$$

Therefore

$$e^\lambda = 1 + (dz/dr)^2.$$

If we take the "plane" $t=\text{const}$, $\theta=\pi/2$, of the Schwarzschild field in usual coordinates, we have

$$e^\lambda = 1/(1-2m/r).$$

It follows that[5]

$$z - z_0 = 2(2m)^{\frac{1}{2}}(r-2m)^{\frac{1}{2}}, \quad z_0 = \text{const}.$$

[5] This "plane" is therefore not asymptotically plane according to our definition. This well-known result (see, for example, M. von Laue, *Die Relativitätstheorie* (Friedrich Vieweg and Sohn, Braunschweig, Germany, 1952), Vol. 2, provides another illustration for the discussion in Sec. (2a) of this paper.