# Functional Derivative Techniques in the Theory of Superconductivity*

STANLEY ENGELSBERG
*Department of Physics, University of Illinois, Urbana, Illinois*

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The equations for the Green's functions of the theory of superconductivity are developed in an iterative scheme in which the two-particle function is rewritten as the functional derivative of the single-particle function with respect to external-source terms. When a source coupled to electron pairs in addition to an external potential coupled to the charge density is included in the Hamiltonian, the possibility for a gap in the single-particle excitations is found. The first order in the iteration scheme for the solution to the Green's function equations is independent of the way in which the two-particle function is generated, as the final result must be. Equations for the vertex functions are obtained and used to find the linear response of the current to an applied electromagnetic field. An exact solution to the vertex equation of interest is used to show that no current will be induced by a static longitudinal vector potential. In addition, all the functions calculated are found to have translationally invariant solutions. Thus, the original invariances of the Hamiltonian are restored when the source terms are set equal to zero. Corrections to the self-energy are estimated by using another of the solutions to the vertex equations.

## I. INTRODUCTION

OUR present understanding of superconductivity is based on an electron-pairing process which takes place for attractive interactions. There have been several techniques successfully applied to the theory once the basic mechanism was understood by Bardeen, Cooper, and Schrieffer.[1] We present here another approach to the theory in which the process of electron pairing is accounted for by generating the Green's function equations with an electron-pair source term.[2] It is noted that the two-particle function which appears in the interaction term of the equation for the one-particle function may be generated in either of two ways. Conventional perturbation theory follows when the two-particle function is generated from the functional derivative of a one-particle function with respect to an external potential coupled to the charge density. However, the two-particle function may also be generated as a functional derivative of the one-particle function with respect to a source term which is coupled to electron pairs. When both representations of the two-particle function are retained, a set of four coupled equations for various single-particle functions are obtained. These four equations are rewritten as a single equation for a matrix of single-particle functions.[3] Since there are terms added to the Hamiltonian which destroy gauge invariance, this matrix contains functions which would be zero if particle number were conserved. However, we find that the physical results obtained when the source terms are set equal to zero satisfy the invariances, both gauge and translational, of the original Hamiltonian.

At the first stage in an iterative procedure for solving the Green's function equations, the results are found to be independent of the way in which the two-particle function was generated. The integral equations satisfied by the vertex functions in this approximation is next considered. The solution to one of the vertex equations is used to show, in a manner similar to that of Nambu,[3] that a static longitudinal electromagnetic vector potential will not induce a current. Corrections to the self-energy are obtained in terms of four vertex functions. When this correction is estimated, it is found that its inclusion is unjustified without reliable calculations of the functions which appear in the lowest order equation for the self-energy.

## II. DEVELOPMENT OF THE GREEN'S FUNCTION EQUATIONS

The Hamiltonian chosen is that which includes two-particle interactions depending only on the distance between the particles, ($\hbar=1$).

$$H=-\sum_{\sigma}\int d^3r\,\psi_\sigma^\dagger(\mathbf{r})\left(\frac{\nabla^2}{2m}+\mu\right)\psi_\sigma(\mathbf{r})+\tfrac{1}{2}\sum_{\sigma,\sigma'}\int d^3rd^3r'\,V(\mathbf{r}-\mathbf{r}')\psi_\sigma^\dagger(\mathbf{r})\psi_{\sigma'}^\dagger(\mathbf{r}')\psi_{\sigma'}(\mathbf{r}')\psi_\sigma(\mathbf{r}), \tag{1}$$

where $\mu$ is the chemical potential of the system and $\sigma$ refers to the spin of the particle. We adjoin to this Hamiltonian the source terms by which the solution will be generated.

$$\Delta H=\int d^3rd^3r'\,A(\mathbf{r},\mathbf{r}')\psi_\uparrow(\mathbf{r})\psi_\downarrow(r')+\int d^3rd^3r'\,A^*(\mathbf{r},\mathbf{r}')\psi_\downarrow^\dagger(\mathbf{r}')\psi_\uparrow^\dagger(\mathbf{r})+\sum_\sigma\int d^3r\,U_\sigma(\mathbf{r})\psi_\sigma^\dagger(\mathbf{r})\psi_\sigma(\mathbf{r}). \tag{2}$$

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[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957); **108**, 1175 (1957); referred to as BCS.

[2] N. N. Bogolyubov, D. N. Zubarev, and Y. A. Tserkovnikov [Soviet Phys.—JETP **12**, 88 (1961)] introduced similar terms into the BCS reduced Hamiltonian to exclude the trivial gap solution.

[3] Y. Nambu [Phys. Rev. **117**, 648 (1960)] has introduced an equivalent matrix.

The usual perturbation series may be developed by use of the term with the external potential $U(r)$. When $U(r)$ is arbitrary, the translation invariance of the starting Hamiltonian is broken, thereby violating momentum conservation. The terms which include the source of electron pairs break the gauge invariance of the starting Hamiltonian, thus violating both charge and current conservation. We must require that after the solutions to the equations of motion and the physical quantities of interest are obtained, with $A$, $A^*$, and $U$ set equal to zero, the original conservation requirements demanded by the Hamiltonian are satisfied.

The equations of motion for the field operators are

$$\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu-U_\uparrow(x)\right)\psi_\uparrow(x)=\sum_\sigma\int d^4x''\,V(x-x'')\psi_\sigma^\dagger(x'')\psi_\sigma(x'')\psi_\uparrow(x)-\int d^4x''\,A^*(x,x'')\psi_\downarrow^\dagger(x''), \tag{3}$$

and

$$\left(-i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu-U_\downarrow(x)\right)\psi_\downarrow^\dagger(x)=\sum_\sigma\int d^4x''\,V(x-x'')\psi_\downarrow^\dagger(x)\psi_\sigma^\dagger(x'')\psi_\sigma(x'')+\int d^4x''\,A(x'',x)\psi_\uparrow(x''). \tag{4}$$

We have allowed the external sources to be functions of time by writing $V(x-x'')$ and $A(x,x'')$, with the understanding that they contain delta functions which set the two time indices equal. Of course $V(x-x')$ may be chosen to depend on the time difference in a nontrivial way, as is true when one starts with the electron–phonon interaction and eliminates the phonon field in favor of the electron–electron interaction. In this case $V(x-x')$ would correspond to the phonon propagator plus the direct Coulomb interaction. We introduce the notation

$$\langle X\rangle\equiv\langle\Phi_0,+\infty|X|\Phi_0,-\infty\rangle/\langle\Phi_0,+\infty|\Phi_0,-\infty\rangle,$$

where $|\Phi_0,-\infty\rangle$ to the Heisenberg state vector of the ground state of the system specified by a complete set of commuting observables whose eigenvalues are given at the time $-\infty$. The time-ordered Green's functions of interest are then defined by

$$\begin{aligned}G_{11}(x,x')&=-i\langle T(\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle,\\ G_{12}(x',x)&=-i\langle T(\psi_\downarrow(x')\psi_\uparrow(x))\rangle,\\ G_{21}(x,x')&=-i\langle T(\psi_\downarrow^\dagger(x)\psi_\uparrow^\dagger(x'))\rangle,\\ G_{22}(x',x)&=-i\langle T(\psi_\downarrow(x')\psi_\downarrow^\dagger(x))\rangle.\end{aligned} \tag{5}$$

Equation (3) may be used to arrive at the equation

$$\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu-U_\uparrow(x)\right)G_{11}(x,x')=\delta(x-x')-\int d^4x''\,A^*(x,x'')G_{21}(x'',x')$$
$$-i\sum_\sigma\int d^4x''\,V(x-x'')\langle T(\psi_\sigma^\dagger(x'')\psi_\sigma(x'')\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle. \tag{6}$$

Equation (6) contains the two-particle functions

$$\langle T(\psi_\uparrow^\dagger(x'')\psi_\uparrow(x'')\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle+\langle T(\psi_\downarrow^\dagger(x'')\psi_\downarrow(x'')\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle.$$

The action principle[4] allows us to generate the first term with the external potential $U$.

$$\langle T(\psi_\uparrow^\dagger(x'')\psi_\uparrow(x'')\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle=i(\delta/\delta U_\uparrow(x''^+))\langle T(\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle+\langle T(\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle\langle\psi_\uparrow^\dagger(x'')\psi_\uparrow(x'')\rangle. \tag{7}$$

However, the second term may be generated in either of two ways:

$$\begin{aligned}\langle T(\psi_\downarrow^\dagger(x'')\psi_\downarrow(x'')\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle&=i(\delta/\delta U_\downarrow(x''^+))\langle T(\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle+\langle T(\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle\langle\psi_\downarrow^\dagger(x'')\psi_\downarrow(x'')\rangle\\ &=-i(\delta/\delta A(x^-,x''^-))\langle T(\psi_\downarrow^\dagger(x'')\psi_\uparrow^\dagger(x'))\rangle+\langle T(\psi_\downarrow^\dagger(x'')\psi_\uparrow^\dagger(x'))\rangle\langle\psi_\downarrow(x'')\psi_\uparrow(x)\rangle.\end{aligned} \tag{8}$$

If the functional equations could be solved exactly, either representation of the two-particle function could be used. However, since we are going to solve the equations by means of an iteration scheme, both representations must be used. The mixture of generating functions to be used is

$$\langle T)\psi_\downarrow^\dagger(x'')\psi_\downarrow(x'')\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle$$
$$=\vartheta\left[i\frac{\delta}{\delta U_\downarrow(x''^+)}\langle T(\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle+\langle T(\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle\langle(\psi_\downarrow^\dagger(x'')\psi_\downarrow(x''))\rangle\right]$$
$$+(1-\vartheta)\left[-i\frac{\delta}{\delta A(x^-,x''^-)}\langle T(\psi_\downarrow^\dagger(x'')\psi_\uparrow^\dagger(x'))\rangle+\langle T(\psi_\downarrow^\dagger(x'')\psi_\uparrow^\dagger(x'))\rangle\langle\psi_\downarrow(x'')\psi_\uparrow(x)\rangle\right], \tag{9}$$

which obviously is still exact. At this point we may consider $\vartheta$ to be a variational parameter which is to be determined by the criterion that the ground-state energy be minimized. At nonzero temperatures the equivalent condi-

[4] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452, 455 (1951).

tion would be minimizing the free energy. Substituting the above expressions for the two-particle function in the Green's function equation, we obtain

$$\left[i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu-U_\uparrow(x)+i\int d^4x''\,V(x-x'')\left(G_{11}(x'',x''^+)+\vartheta G_{22}(x'',x''^+)-\vartheta\frac{\delta}{\delta U_\downarrow(x''^+)}-\frac{\delta}{\delta U_\uparrow(x''^+)}\right)\right]G_{11}(x,x')$$
$$=\delta(x-x')+\int d^4x''\left[A^*(x,x'')+i(1-\vartheta)V(x-x'')\left(G_{12}(x''^+,x)-\frac{\delta}{\delta A(x^-,x''^-)}\right)\right]G_{21}(x'',x'). \quad (10)$$

The superscripts $(+)$ and $(-)$ are included to determine the time ordering when the coordinates are identical. The equation that $G_{21}$ obeys may be obtained by using Eq. (4).

$$\left(-i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu-U_\downarrow(x)\right)G_{21}(x,x')=\int d^4x''\,A(x'',x)G_{22}(x'',x')$$
$$-i\sum_\sigma\int d^4x''\,V(x-x'')\langle T(\psi_\downarrow^\dagger(x)\psi_\sigma^\dagger(x'')\psi_\sigma(x'')\psi_\uparrow^\dagger(x'))\rangle. \quad (11)$$

Again we develop the two-particle function by means of both types of functional derivatives; however, we assume the same parameter $\vartheta$ should be used to determine the way in which the function is generated. The equation for $G_{21}$ then becomes

$$\left[-i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu-U_\downarrow(x)+i\int d^4x''V(x-x'')\left(\vartheta G_{11}(x'',x''^+)+G_{21}(x'',x''^+)-\vartheta\frac{\delta}{\delta U_\uparrow(x''^-)}-\frac{\delta}{\delta U_\downarrow(x''^-)}\right)\right]G_{21}(x,x')$$
$$=\int d^4x''\left[A(x'',x)+1(1-\vartheta)V(x-x'')\left(G_{21}(x^+,x'')+\frac{\delta}{\delta A^*(x''^+,x^+)}\right)\right]G_{11}(x'',x'). \quad (12)$$

By defining the matrix[3]

$$G(x,x')=\begin{pmatrix}G_{11}(x,x') & G_{12}(x',x)\\ G_{21}(x,x') & G_{22}(x',x)\end{pmatrix}, \quad (13)$$

we may write the four equations of motion for the Green's functions as the single matrix equation

$$\int d^4x''\begin{pmatrix}\left[i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu-U_\uparrow(x)+i\int d^4x_1\,V(x-x_1)\left(G_{11}(x_1,x_1^+)+\vartheta G_{22}(x_1,x_1^+)-\frac{\delta}{\delta U_\uparrow(x_1^+)}-\vartheta\frac{\delta}{\delta U_\downarrow(x_1^+)}\right)\right]\delta(x-x') & A^*(x,x'')+i(1-\vartheta)V(x-x'')\left[-G_{12}(x''^+,x)+\frac{\delta}{\delta A(x^-,x''^-)}\right] \\ -A(x'',x)-i(1-\vartheta)V(x-x'')\left[G_{21}(x^+,x'')+\frac{\delta}{\delta A^*(x''^+,x^+)}\right] & \left[-i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu-U_\downarrow(x)+i\int d^4x_1\,V(x-x_1)\left(\vartheta G_{11}(x_1,x_1^+)+G_{22}(x_1,x_1^+)-\vartheta\frac{\delta}{\delta U_\uparrow(x_1^-)}-\frac{\delta}{\delta U_\downarrow(x_1^-)}\right)\right]\delta(x-x'')\end{pmatrix}$$
$$\times G(x'',x')=\begin{pmatrix}\delta(x-x') & 0\\ 0 & \delta(x-x')\end{pmatrix}. \quad (14)$$

In order to develop an equation for $G^{-1}$, we consider the action of the functional derivatives on the matrix $G$.

$$-i\int d^4x_1\,V(x-x_1)\begin{pmatrix}\frac{\delta}{\delta U_\uparrow(x_1^+)}+\vartheta\frac{\delta}{\delta U_\downarrow(x_1^+)} & 0\\ 0 & \vartheta\frac{\delta}{\delta U_\uparrow(x_1^-)}+\frac{\delta}{\partial U_\downarrow(x_1^-)}\end{pmatrix}G(x,x')$$
$$=i\int d^4x_1d^4x_2d^4x_3\,V(x-x_1)\left(\frac{\delta}{\delta U(x_1)}\right)G(x,x_2)G^{-1}(x_2,x_3)G(x_3,x'). \quad (15)$$

We have indicated the matrix on the left-hand side of the equation by $(\delta/\delta U)$, but the derivatives are to act only on $G^{-1}(x_2,x_3)$. The other set of functional derivative terms is treated similarly.

$$i\int d^4x''\, V(x-x'')\begin{bmatrix} 0 & (1-\vartheta)\dfrac{\delta}{\delta A(x^-,x''^-)} \\ -(1-\vartheta)\dfrac{\delta}{\delta A^*(x''^+,x^+)} & 0 \end{bmatrix} G(x'',x')$$

$$=-i\int d^4x''d^4x_1d^4x_2\, V(x-x'')\left(\frac{\delta}{\delta A}\right)G(x'',x_1)G^{-1}(x_1,x_2)G(x_2,x'), \quad (16)$$

where again the elements of the matrix $(\delta/\delta A)$ act only on $G^{-1}(x_1,x_2)$. We may now write the equation for the inverse of $G$,

$$G^{-1}(x,x')=G_0^{-1}(x,x')+F_{\text{ext}}(x,x')-\Sigma(x,x'), \quad (17)$$

with the definitions

$$G_0^{-1}(x,x')=\begin{bmatrix} i\dfrac{\partial}{\partial t}+\dfrac{\nabla^2}{2m}+\mu & 0 \\ 0 & -i\dfrac{\partial}{\partial t}+\dfrac{\nabla^2}{2m}+\mu \end{bmatrix}\delta(x-x'),$$

$$F_{\text{ext}}(x,x')=\begin{pmatrix} -U_\uparrow(x)\delta(x-x') & A^*(x,x') \\ -A(x',x) & -U_\downarrow(x)\delta(x-x') \end{pmatrix},$$

$$\Sigma(x,x')=\begin{bmatrix} -i\int d^4x_1\, V(x-x_1)[G_{11}(x_1,x_1^+)+\vartheta G_{22}(x_1,x_1^+)]\delta(x-x') & i(1-\vartheta)V(x-x')G_{12}(x''^+,x) \\ i(1-\vartheta)V(x-x')G_{21}(x^+,x') & -i\int d^4x_1\, V(x-x_1)[\vartheta G_{11}(x_1,x_1^+)+G_{22}(x_1,x_1^+)]\delta(x-x') \end{bmatrix} \quad (18)$$

$$-i\int d^4x_1d^4x_2\, V(x-x_1)\left(\frac{\delta}{\delta U(x_1)}\right)G(x,x_2)G^{-1}(x_2,x')+i\int d^4x_1d^4x_2\, V(x-x_1)\left(\frac{\delta}{\delta A}\right)G(x_1,x_2)G^{-1}(x_2,x').$$

There are terms in the self-energy of the form of functional derivatives which account for the same correlations as the terms which do not involve derivatives. These terms may be evaluated by calculating the functional derivatives of $G^{-1}$ in lowest order, neglecting all but the dependence of $G^{-1}$ on the forcing terms in $F_{\text{ext}}$. One of the terms which arise is

$$-i\int d^4x_1d^4x_2\, V(x-x_1)\tfrac{1}{2}(1+\tau_3)G(x,x_2)\left(\frac{\delta}{\delta U_\uparrow(x_1^+)}+\vartheta\frac{\delta}{\delta U_\downarrow(x_1^+)}\right)G^{-1}(x_2,x')$$

$$\triangleq -i\int d^4x_1d^4x_2\, V(x-x_1)\tfrac{1}{2}(1+\tau_3)G(x,x_2)\begin{pmatrix} -1 & 0 \\ 0 & -\vartheta \end{pmatrix}\delta(x_2-x_1^+)\delta(x_2-x')=iV(x-x')\begin{pmatrix} G_{11}(x,x'^+) & \vartheta G_{12}(x'^+,x) \\ 0 & 0 \end{pmatrix}.$$

Another term which occurs is

$$i\int d^4x_1d^4x_2\, V(x-x_1)\tfrac{1}{2}(\tau_1+i\tau_2)G(x_1,x_2)(1-\vartheta)\frac{\delta}{\delta A(x^-,x_1^-)}G^{-1}(x_2,x')$$

$$\triangleq i\int d^4x_1d^4x_2\, V(x-x_1)\tfrac{1}{2}(\tau_1+i\tau_2)G(x_1,x_2)(1-\vartheta)\tfrac{1}{2}(i\tau_2-\tau_1)\delta(x'-x^-)\delta(x_2-x_1^-)$$

$$=i\int d^4x_1\, V(x-x_1)\begin{pmatrix} -(1-\vartheta)G_{22}(x_1^-,x_1) & 0 \\ 0 & 0 \end{pmatrix}.$$

The Pauli matrices have been denoted by

$$\tau_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad \tau_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\qquad \tau_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

When the other two terms of the same type are calculated and added to the self-energy, we obtain

$$\Sigma(x,x')=\begin{bmatrix} -i\int d^4x_1\, V(x-x_1)[G_{11}(x_1,x_1^+)+G_{22}(x_1,x_1^+)]\delta(x-x')+iV(x-x')G_{11}(x,x'^+) & iV(x-x')G_{12}(x'^+,x) \\ iV(x-x')G_{21}(x^+,x') & -i\int d^4x_1\, V(x-x_1)[G_{11}(x_1,x_1^+)+G_{22}(x_1,x_1^+)]\delta(x-x')+iV(x-x')G_{22}(x',x^+)\end{bmatrix}$$
$$-i\int d^4x_1d^4x_2\, V(x-x_1)\left(\frac{\delta}{\delta U}\right)G(x,x_2)[G^{-1}(x_2,x')-F_{\text{ext}}(x_2,x')]$$
$$+i\int d^4x_1d^4x_2\, V(x-x_1)\left(\frac{\delta}{\delta A}\right)G(x_1,x_2)[G^{-1}(x_2,x')-F_{\text{ext}}(x_2,x')].\quad(19)$$

At this stage it is important to note that the self-energy terms not involving functional derivatives are independent of $\vartheta$. It is quite unexpected that at this first stage in the iteration scheme we obtain a result independent of the parameter $\vartheta$, as the final result should be. Neglecting the functional derivative terms in Eq. (19), we may write the first approximation to the Green's function equations. The last two terms of Eq. (19) will be used to obtain corrections to this result. Thus, in this order

$$\Sigma(x,x')=-\frac{i}{2}\int d^4x_1\, V(x-x_1)\,\mathrm{Tr}[(1+\tau_3)G(x_1,x_1^+)+(1-\tau_3)G(x_1^+,x_1)]\delta(x-x')$$
$$+\frac{i}{2}V(x-x')[(1+\tau_3)G(x,x'^+)+(1-\tau_3)G(x^+,x')].\quad(20)$$

In a diagrammatic approach one would remark that Eq. (20) includes both the direct and exchange effects. The Fourier transform of $\Sigma$ is

$$\Sigma(p)=\int d^4(x-x')\, e^{-ip\cdot(x-x')}\Sigma(x,x')$$
$$=-\frac{i}{2}V(0)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}[(1+\tau_3)G(k)e^{ik^0\epsilon}+(1-\tau_3)G(k)e^{-ik^0\epsilon}]+i\int\frac{d^4k}{(2\pi)^4}V(k)G(p-k),\quad \epsilon\to 0^+.\quad(21)$$

The first term on the right-hand side of Eq. (21) diverges due to the divergence of the Coulomb interaction. However, this term, due to the electrons' Coulomb interaction, is canceled by the Coulomb interactions of the positively charged background which we have not included explicitly. We have retained the convergence factor $e^{ik^0\epsilon}$ only in the first term of Eq. (21). It will be necessary to distinguish the original time orderings whenever nonconvergent results may occur. The most general matrix form $\Sigma$ may assume is

$$\Sigma(p)=\chi(p)+p^0\zeta(p)\tau_3+i\tau_1\theta(p)+i\tau_2\phi(p).\quad(22)$$

Setting the source terms equal to zero, Eq. (17) is used to obtain

$$G^{-1}(p)=p^0[1-\zeta(p)]\tau_3-[\epsilon(p)+\chi(p)]-i\tau_1\theta(p)-i\tau_2\phi(p).\quad(23)$$

If we define

$$Z(p)=1-\zeta(p),$$
$$\bar\epsilon(p)=\epsilon(p)+\chi(p),\quad(24)$$
$$E^2(p)=\bar\epsilon^2(p)+\theta^2(p)+\phi^2(p),$$

the inverse of $G^{-1}(p)$ is found to be

$$G(p)=\frac{p^0Z(p)\tau_3+\bar\epsilon(p)-i\tau_1\theta(p)-i\tau_2\phi(p)}{p^{02}Z^2(p)-E^2(p)}.\quad(25)$$

Equations (21), (22), and (25) give two integral equa-

tions for the off-diagonal terms of $\Sigma$

$$\phi(p)=-i\int\frac{d^4k}{(2\pi)^4}\frac{V(p-k)\phi(k)}{k^{0^2}Z^2(k)-E^2(k)}. \tag{26}$$

The second gap equation is identical to Eq. (26), except that $\phi$ and $\theta$ are interchanged. As a homogeneous integral equation the trivial solution is always a possible solution to Eq. (26). However, for the simple choice $V(p-k)$ constant and attractive (negative) in a band about the Fermi surface in $\mathbf{k}$ space, the well-known nontrivial BCS solution exists if we neglect $\chi$ and $\zeta$, and choose either $\theta$ or $\phi$ equal to zero. It is important to recognize that when the nontrivial solution exists, it must be chosen, for if the trivial solution is chosen the Fourier transform of any of the two-particle functions of the form $\langle T(\psi_\uparrow(x)\psi_\downarrow(x)\psi_\downarrow^\dagger(x')\psi_\uparrow^\dagger(x'))\rangle$ will not have its requisite properties, e.g., positive spectral-weight function,[2] or analyticity in the cut complex plane.[5] The failure of these properties to hold is just the mathematical restatement of the instability of the Fermi surface to the formation of Cooper pairs, which play a dominant role in the function given.

The functions $\theta$ and $\phi$ have no independent physical significance, it is the combination $\theta^2+\phi^2$ which is a gauge-invariant quantity. Let us examine the connection between $\theta$, $\phi$, and the off-diagonal elements of $G$ more closely; we define

$$a(x,x')=\langle\psi_\downarrow(x)\psi_\uparrow(x')\rangle=-\langle\psi_\uparrow(x)\psi_\downarrow(x')\rangle. \tag{27}$$

The equality written holds when $U_\uparrow(x)=U_\downarrow(x)$ due to the invariance of the Hamiltonian under the unitary transformation interchanging spin up and spin down. The Fourier transform of the Green's functions $G_{12}$ and $G_{21}$ [Eq. (5)] may be written in terms of the Fourier transform of $a(x,x')$.

$$\begin{aligned}
a(x,x')&=\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-x')}a(p),\\
G_{12}(-p)&=P\int\frac{dk^0}{2\pi}\frac{a(\mathbf{p},k^0)-a(-\mathbf{p},-k^0)}{p^0-k^0}\\
&\qquad-\frac{i}{2}[a(p)+a(-p)],\\
G_{21}(p)&=P\int\frac{dk^0}{2\pi}\frac{a^*(-\mathbf{p},-k^0)-a^*(\mathbf{p},k^0)}{p^0-k^0}\\
&\qquad+\frac{i}{2}[a^*(p)+a^*(-p)].
\end{aligned}\tag{28}$$

Note that the equalities $G_{12}(p)=G_{12}(-p)$ and $G_{21}(p)=G_{21}(-p)$ follows from the spin-flip invariance. Now using Eq. (25) to find

$$\begin{aligned}
G_{12}(-p)&=\tfrac{1}{2}\operatorname{Tr}(\tau_1-i\tau_2)G(p)\\
&=(-i\theta-\phi)/[p^{0^2}Z^2-E^2(p)],\\
G_{21}(p)&=\tfrac{1}{2}\operatorname{Tr}(\tau_1+i\tau_2)G(p)\\
&=(-i\theta+\phi)/[p^{0^2}Z^2-E^2(p)],
\end{aligned}\tag{29}$$

we may make the identification

$$\begin{aligned}
\frac{\theta(p)}{p^{0^2}Z^2-E^2}&=-P\int\frac{dk^0}{2\pi}\frac{\operatorname{Im}[a(\mathbf{p},k^0)-a(-\mathbf{p},-k^0)]}{p^0-k^0}\\
&\qquad+\frac{i}{2}\operatorname{Im}[a(p)+a(-p)],\\
\frac{\phi(p)}{p^{0^2}Z^2-E^2}&=-P\int\frac{dk^0}{2\pi}\frac{\operatorname{Re}[a(\mathbf{p},k^0)-a(-\mathbf{p},-k^0)]}{p^0-k^0}\\
&\qquad+\frac{i}{2}\operatorname{Re}[a(p)+a(-p)].
\end{aligned}\tag{30}$$

The quantity

$$G_{12}(-p)G_{21}(p)=-(\phi^2+\theta^2)/(p^{0^2}Z^2-E^2)^2 \tag{31}$$

is invariant under a gauge transformation of the first kind,

$$\psi(x)\to e^{i\alpha}\psi(x),\quad \psi^\dagger(x)\to e^{-i\alpha}\psi^\dagger(x).$$

Thus the sum $\phi^2+\theta^2$ must also be gauge invariant.

### III. VERTEX FUNCTIONS

We will next consider the equations obeyed by the vertex functions. The vertex functions will be required to continue the iteration scheme developed and are also necessary for the calculation of the two-particle response functions. There are two types of vertex functions which are of interest. One is the functional derivative of $G^{-1}$ with respect to an external source term, such as $U(x)$, depending on a single coordinate vector, the other is the functional derivative with respect to an external source term, such as $A(x,x')$, depending on two coordinate vectors. The form of the integral equation satisfied will be identical in both cases.

$$\begin{aligned}
\Gamma_F(x,x';z)\equiv\frac{\delta G^{-1}(x,x')}{\delta F(z)}&=\gamma_F(x,x';z)+\frac{i}{2}\int d^4x_1d^4x_2d^4x_3\,V(x-x_1)\delta(x-x')\\
&\quad\times\operatorname{Tr}\{(1+\tau_3)G(x_1,x_2)\Gamma_F(x_2,x_3;z)G(x_3,x_1^+)+(1-\tau_3)G(x_1^+,x_2)\Gamma_F(x_2,x_3;z)G(x_3,x_1)\}\\
&\quad-\frac{i}{2}V(x-x')\int d^4x_2d^4x_3[(1+\tau_3)G(x,x_2)\Gamma_F(x_2,x_3;z)G(x_2,x'^+)+(1-\tau_3)G(x^+,x_2)\Gamma_F(x_2,x_3;z)G(x_3,x')],
\end{aligned}\tag{32}$$

[5] L. Kadanoff and P. Martin, Phys. Rev. 124, 670 (1961).

where

$$\gamma_F(x,x';z)=\delta G_0^{-1}(x,x')/\delta F(z) \tag{33}$$

is sometimes denoted the free-particle vertex. Defining the Fourier transform of $\Gamma$ by

$$\Gamma(x,x';z)=\int \frac{d^4q d^4k}{(2\pi)^8} e^{ik\cdot(x-x')+iq\cdot(x-z)}\Gamma(k,q), \tag{34}$$

Eq. (32) may be written

$$\Gamma_F(p,q)=\gamma_F(p,q)-iV(q)\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}G(k+q)\Gamma_F(k,q)G(k)+i\int \frac{d^4k}{(2\pi)^4}V(p-k)G(k+q)\Gamma_F(k,q)G(k). \tag{35}$$

We have assumed translational invariance in Eq. (34), which is justified by Eq. (35) if $\gamma_F(x,x';z)=\gamma_F(x-x',\, x-z)$. We have also suppressed the convergence factors in Eq. (35), since we will find that the integrals are all convergent, and we may close the path of $k^0$ integration in either half-plane and obtain identical results. To obtain the solutions to Eq. (35) we follow the procedure of Nambu.[6] We introduce the functions

$$\Lambda_F(p,q)=\gamma_F(p,q)+i\int \frac{d^4k}{(2\pi)^4}V(p-k) \times G(k+q)\Lambda_F(k,q)G(k), \tag{36}$$

$$\bar{X}_F(q)=i\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}G(k+q)\Gamma_F(k,q)G(k), \tag{37}$$

and obtain the solution

$$\Gamma_F(p,q)=\Lambda_F(p,q)+\Lambda_0(p,q)V(q)\bar{X}_F(q), \tag{38}$$

where $\Lambda_0$ satisfied the integral Eq. (36) with

$$\gamma_0(p,q)=-1. \tag{39}$$

Equations (35), (36), and (37) lead to the relation

$$\bar{X}_F(q)=\frac{i\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}G(k+q)\Lambda_F(k,q)G(k)}{1-iV(q)\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}G(k+q)\Lambda_0(k,q)G(k)} \equiv \frac{X_F(q)}{1-V(q)X_0(q)}. \tag{40}$$

Let us consider the simplest, but one of the most important, choices for $\gamma$, $\gamma_0$. In this case

$$\begin{aligned}\bar{X}_0(q)&=X_0(q)/[1-V(q)X_0(q)],\\ \Gamma_0(p,q)&=\Lambda_0(p,q)/[1-V(q)X_0(q)].\end{aligned} \tag{41}$$

The solution to the homogeneous integral equation will be used to obtain the solution to Eq. (36).

$$\Lambda_H(p+\tfrac{1}{2}q,\, q)=i\int \frac{d^4k}{(2\pi)^4}V(p-k)G(k+\tfrac{1}{2}q) \times\Lambda_H(k+\tfrac{1}{2}q,\, q)G(k-\tfrac{1}{2}q). \tag{42}$$

For $q=0$ there exists the exact solution,

$$\Lambda_H(p,0)=\tau_3G^{-1}(p)-G^{-1}(p)\tau_3=2\tau_2\theta(p)-2\tau_1\phi(p). \tag{43}$$

The dispersion relation in $q$, for small $q$, is obtained from Eq. (42) in a manner identical with Nambu.[7] The condition for a solution to the homogeneous equation is

$$\int d^3\mathbf{k}\frac{(\phi^2+\theta^2)}{E(\mathbf{k})^3}\left[\left(\frac{q^0}{2}\right)^2-\left(\frac{\mathbf{k}\cdot\mathbf{q}}{2m}\right)^2-\frac{\epsilon(\mathbf{k})}{2m}\left(\frac{\mathbf{q}}{2}\right)^2\right]=0. \tag{44}$$

Since $\phi^2+\theta^2$ is nonzero only about the Fermi surface, Eq. (44) leads to the dispersion relation

$$q_0^2\approx\alpha^2\mathbf{q}^2,\quad \alpha^2=p_f^2/3m^2. \tag{45}$$

There exists an exact solution for $\Lambda_0$ when $\mathbf{q}=0$, $q^0\neq 0$.

$$\begin{aligned}\Lambda_0(p+\tfrac{1}{2}q,\, q)&=[\tau_3G^{-1}(p-\tfrac{1}{2}q)-G^{-1}(p+\tfrac{1}{2}q)\tau_3]/q^0\\ &\approx -1+(2/q^0)(\tau_2\theta-\tau_1\phi).\end{aligned} \tag{46}$$

The second approximate form for $\Lambda_0$ is obtained by neglecting the renormalization function $\zeta$, and the dependence of $\chi$, $\theta$, and $\phi$ on energy. The solution (46) for $|\mathbf{q}|=0$ and the dispersion relation for the solution to the homogeneous equation leads to the solution $\Lambda_0(p+\frac{1}{2}q,\, q)$ for $|\mathbf{q}|$ small, since when the dispersion relation is satisfied, $\Lambda_0$ must have a pole.

$$\Lambda_0(p+\tfrac{1}{2}q,\, q)\approx -1+\frac{2q^0(\tau_2\theta-\tau_1\phi)}{(q^0)^2-\alpha^2\mathbf{q}^2}. \tag{47}$$

For use in later calculations we quote the approximate value of $X_0(q)$ when $\alpha^2\mathbf{q}^2\ll\phi^2+\theta^2$, and $(q^0)^2\gg\phi^2+\theta^2$ or

[6] We will follow the notation of Nambu[3] wherever it is convenient.

[7] Note the misprint in reference 3. The term $\Delta L(p,\frac{1}{2}q)F^0(p)\times\Delta L(p,-\frac{1}{2}q)$ should be added to the right-hand side of Eq. (6.7) for $U^{(1)}(p,\frac{1}{2}q)$.

$(q^0)^2 \ll \phi^2+\theta^2$,[3]

$$X_0(q) \approx \frac{n}{m} \frac{\mathbf{q}^2}{(q^0)^2-\alpha^2\mathbf{q}^2}, \tag{48}$$

where $n$ is the electron density.

### IV. GAUGE INVARIANCE

Using the formalism developed, we will show that a static longitudinal vector potential will not induce a current. Although the underlying idea is identical with that of Nambu, the details differ: in this approach we will find that the diamagnetic and paramagnetic terms which cancel are finite, and that the vertex equation obeyed is one which includes corrections, although ineffective, neglected previously. In the presence of an external electromagnetic field $\mathbf{A}(x)$, the current in the system is given by

$$\mathbf{J}(x) = \frac{e}{2mi} \sum_\sigma (\psi_\sigma{}^\dagger(x)\nabla\psi_\sigma(x) - \nabla\psi_\sigma{}^\dagger(x)\psi_\sigma(x)) - \frac{e^2\mathbf{A}(x)}{mc} \sum_\sigma \psi_\sigma{}^\dagger(x)\psi_\sigma(x). \tag{49}$$

The expectation value of $\mathbf{J}$ is

$$\langle \mathbf{J}(x)\rangle = \lim_{x'\to x} \left\{ -\frac{e}{4m}(\nabla-\nabla') \operatorname{Tr}[(1+\tau_3)G(x,x'^+)-(1-\tau_3)G(x^+,x')]\right\} + \frac{ie^2}{2mc}\mathbf{A}(x) \operatorname{Tr}[(1+\tau_3)G(x,x^+)+(1-\tau_3)G(x^+,x)], \tag{50}$$

$$\langle n\rangle = -i \operatorname{Tr}\{(1+\tau_3)G(x,x^+)+(1-\tau_3)G(x^+,x)\}.$$

Our aim is to calculate the linear response function $K_{lm}(x,y)$,

$$\langle J_l(x)\rangle = \sum_m \int d^4y\, K_{lm}(x,y)A_m(y)$$

or

$$K_{lm}(x,y) = \left.\frac{\delta\langle J_l(x)\rangle}{\delta A_m(y)}\right|_{A=0}$$

$$= \lim_{x'\to x} \left\{ \frac{e^2}{4mc}(\nabla-\nabla')_l \int d^4x_1 d^4x_2 \operatorname{Tr}[(1+\tau_3)G(x,x_1)\Gamma_m(x_1,x_2;y)G(x_2,x'^+) - (1-\tau_3)G(x^+,x_1)\Gamma_m(x_1,x_2;y)G(x_2,x')]\right\} + \frac{ie^2}{2mc}\delta_{lm}\delta(x-y) \operatorname{Tr}[(1+\tau_3)G(x,x^+)+(1-\tau_3)G(x^+,x)], \tag{51}$$

where

$$\Gamma_m(x_1,x_2;y) \equiv \frac{\delta G^{-1}(x_1,x_2)}{\delta[(e/c)A_m(y)]}. \tag{52}$$

The Fourier transform of the response function is

$$K_{lm}(q) = \frac{ie^2}{4mc}\int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\{(1+\tau_3)(q+2k)_l G(k+q)\Gamma_m(k,q)G(k)e^{ik^0\epsilon} - (1-\tau_3)(q+2k)_l G(k+q)\Gamma_m(k,q)G(k)e^{-ik^0\epsilon}\} + \frac{ie^2}{2mc}\delta_{lm}\int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\{(1+\tau_3)G(k)e^{ik^0\epsilon}+(1-\tau_3)G(k)e^{-ik^0\epsilon}\}. \tag{53}$$

We will keep the convergence factors in both terms since they are required. To find the equation obeyed by $\Gamma_m(k,q)$ we need

$$\gamma_m(k,q) = \int d^4(x-x')d^4(x-y)\, e^{-ik\cdot(x-x')-iq\cdot(x-y)} \left.\frac{\delta G_0{}^{-1}(x,x')}{\delta[(e/c)A_m(y)]}\right|_{A=0}$$

$$= (1/2m)(q+2k)_m\tau_3. \tag{54}$$

The response to a static, longitudinal vector potential,

$$A_m(\mathbf{q}) = q_m f(\mathbf{q}^2), \tag{55}$$

is obtained by considering

$$\sum_{m=1}^{3} \Gamma_m(k,q)q_m = \Gamma_L(k,q), \tag{56}$$

which function satisfies the integral Eq. (35) with

$$\gamma_L(p,q)=(1/2m)(\mathbf{q}^2+2\mathbf{k}\cdot\mathbf{q})\tau_3. \tag{57}$$

However, since $q^0=0$ we have the exact solution

$$\Gamma_L(k,q)=\tau_3G^{-1}(k)-G^{-1}(k+q)\tau_3. \tag{58}$$

Hence,

$$\sum_m K_{lm}(q)q_m=\frac{ie^2}{4mc}\int\frac{d^4k}{(2\pi)^4}(q+2k)_l\,\mathrm{Tr}\{(1+\tau_3)[G(k+q)-G(k)]e^{ik^0\epsilon}+(1-\tau_3)[G(k+q)-G(k)]e^{-ik^0\epsilon}\}$$
$$+\frac{ie^2}{2mc}q_l\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\{(1+\tau_3)G(k)e^{ik^0\epsilon}+(1-\tau_3)G(k)e^{-ik^0\epsilon}\}. \tag{59}$$

In the terms with $G(k+q)$ we change the variable of integration to $k'=k+q$.

$$\int d^4k\;k_lG(k)=0,$$

since the system is isotropic. The other terms cancel to give

$$\textstyle\sum_m K_{lm}(\mathbf{q})q_m=0. \tag{60}$$

For $q^0=0$ the transverse part of the first term of Eq. (53) is zero due to the nonzero value of energy $2(\phi^2+\theta^2)^{\frac{1}{2}}$, required to excite electrons from the Fermi sea; whereas, the second—diamagnetic—term contributes its full value, $-ne^2/mc$. This result then leads to a gauge-invariant Meissner effect.

## V. CORRECTIONS TO THE SELF-ENERGY

The corrections to the self-energy contained in the last two terms of Eq. (19) may be rewritten as

$$\begin{aligned}\Delta\Sigma(x,x')=-\frac{i}{2}\int d^4x_1d^4x_2\;V(x-x_1)\times\{&G(x,x_2)(1+\vartheta)[\Gamma_{U\uparrow}(x_2,x';x_1)+\Gamma_{U\downarrow}(x_2,x';x_1)\\&-(\gamma_{U\uparrow}(x_2,x';x_1)+\gamma_{U\downarrow}(x_2,x';x_1))]+(1-\vartheta)[\tau_3G(x,x_2)(\Gamma_{U\uparrow}(x_2,x';x_1)\\&-\Gamma_{U\downarrow}(x_2,x';x_1)-(\gamma_{U\uparrow}(x_2,x';x_1)-\gamma_{U\downarrow}(x_2,x';x_1))]+\tau_1G(x_1,x_2)[\Gamma_{A^*}(x_2,x';x_1,x)-\Gamma_A(x_2,x';x,x_1)\\&-(\gamma_{A^*}(x_2,x';x_1,x)-\gamma_A(x_2,x';x,x_1))]-i\tau_2G(x_1,x_2)[\Gamma_{A^*}(x_2,x';x_1,x)+\Gamma_A(x_2,x';x,x_1)\\&-(\gamma_{A^*}(x_2,x';x_1,x)+\gamma_A(x_2,x';x,x_1))]\}.\end{aligned} \tag{61}$$

Each of the vertex functions in Eq. (61) satisfies an integral equation identical in form with Eq. (32). The free-particle vertices are given by

$$\begin{aligned}\gamma_{U\uparrow}(x_2,x';x_1)+\gamma_{U\downarrow}(x_2,x';x_1)&=-\delta(x_2-x')\delta(x_2-x_1),\\ \gamma_{U\uparrow}(x_2,x';x_1)-\gamma_{U\downarrow}(x_2,x';x_1)&=-\tau_3\delta(x_2-x')\delta(x_2-x_1),\\ \gamma_{A^*}(x_2,x';x_1,x)-\gamma_A(x_2,x';x,x_1)&=\tau_1\delta(x-x')\delta(x_2-x_1),\\ \gamma_{A^*}(x_2,x';x_1,x)+\gamma_A(x_2,x';x,x_1)&=i\tau_2\delta(x-x')\delta(x_2-x_1).\end{aligned} \tag{62}$$

When the Fourier transform of $\Gamma_A$ is defined by

$$\Gamma_A(x,x';z,z')=\int\frac{d^4pd^4qd^4k}{(2\pi)^{12}}e^{ip\cdot(x-x')+iq\cdot(x-z)+ik\cdot(z-z')}\Gamma_A(p,q,k), \tag{63}$$

it satisfies the equation

$$\Gamma_A(p,q,k)=\gamma_A(p,q,k)-iV(q)\int\frac{d^4\lambda}{(2\pi)^4}\,\mathrm{Tr}G(\lambda+q)\Gamma_A(\lambda,q,k)G(\lambda)+i\int\frac{d^4\lambda}{(2\pi)^4}V(p-\lambda)G(\lambda+q)\Gamma_A(\lambda,q,k)G(\lambda). \tag{64}$$

The Fourier transforms of the free-particle vertices given in Eq. (62) are

$$\begin{aligned}\gamma_{U_\uparrow+U_\downarrow}(p,q)&=-1\equiv\gamma_0(p,q),\\ \gamma_{U_\uparrow-U_\downarrow}(p,q)&=-\tau_3,\\ \gamma_{A^*-A}(p,q,k)&=\tau_1\delta(p+k),\\ \gamma_{A^*+A}(p,q,k)&=i\tau_2\delta(p+k).\end{aligned}\tag{65}$$

The Fourier transform of Eq. (61) is

$$\begin{aligned}\Delta\Sigma(p)=&-\frac{i}{2}\int\frac{d^4q}{(2\pi)^4}V(-q)\{(1+\vartheta)G(p+q)[\Gamma_{U_\uparrow+U_\downarrow}(p,q)-\gamma_{U_\uparrow+U_\downarrow}(p,q)]\\ &+(1-\vartheta)\tau_3G(p+q)[\Gamma_{U_\uparrow-U_\downarrow}(p,q)-\gamma_{U_\uparrow-U_\downarrow}(p,q)]\}-\frac{i}{2}\int\frac{d^4qd^4k}{(2\pi)^8}V(p+k)(1-\vartheta)\{\tau_1G(p+q)\\ &\times[\Gamma_{A^*-A}(p,q,k)-\gamma_{A^*-A}(p,q,k)]-i\tau_2G(p+q)[\Gamma_{A^*+A}(p,q,k)-\gamma_{A^*+A}(p,q,k)]\}.\end{aligned}\tag{66}$$

To obtain an estimate of the correction to the self-energy we choose $\vartheta=1$, and note that $\Gamma_{U_\uparrow+U_\downarrow}(p,q)$ is identical with $\Gamma_0(p,q)$, which was determined previously, Eq. (41), for small $|\mathbf{q}|$. At long wavelengths the Coulomb interaction is predominant:

$$V(q)\sim e^2/\mathbf{q}^2.$$

Equation (48) then leads to the well-known plasmon dispersion relation for the poles of $\Gamma_0$:

$$1-V(q)X_0(q)\approx1-\frac{ne^2}{m(q^{02}-\alpha^2\mathbf{q}^2)}\equiv1-\frac{\omega_0^2}{q^{02}-\alpha^2\mathbf{q}^2}.\tag{67}$$

The vertex $\Gamma_0$ is written

$$\Gamma_0(p,q)=\left(-1+\frac{2q^0(\tau_2\theta-\tau_1\phi)}{q^{02}-\alpha^2\mathbf{q}^2+i\delta}\right)\frac{q^{02}-\alpha^2\mathbf{q}^2}{q^{02}-\omega_0^2-\alpha^2\mathbf{q}^2+i\eta},\quad\delta,\eta\to0^+.\tag{68}$$

As noted previously, the vertex functions are related to various two-particle functions,

$$\begin{aligned}\frac{\delta G(x,x')}{\delta F(z)}&=-\int d^4x_1d^4x_2\,G(x,x_1)\Gamma_F(x_1,x_2;z)G(x_2,x')\\ &=-\int\frac{d^4kd^4q}{(2\pi)^8}e^{ik\cdot(x-x')+iq\cdot(x-z)}G(k+q)\Gamma_F(k,q)G(k).\end{aligned}\tag{69}$$

To illustrate the connection between the two-particle functions and the vertex function, we calculate two of the components of the matrix related to $\Gamma_0$ by the equation above.

$$\begin{aligned}\frac{\delta G_{11}(x,x')}{\delta U_\uparrow(z)}+\frac{\delta G_{11}(x,x')}{\delta U_\downarrow(z)}=&-\langle T(\psi_\uparrow^\dagger(z)\psi_\uparrow(z)\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle-\langle T(\psi_\downarrow^\dagger(z)\psi_\downarrow(z)\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle\\ &+\langle\psi_\uparrow^\dagger(z)\psi_\uparrow(z)\rangle\langle T(\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle+\langle\psi_\downarrow^\dagger(z)\psi_\downarrow(z)\rangle\langle T(\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle\\ =&-\langle T(n(z)\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle+\langle n\rangle\langle T(\psi_\uparrow(x)\psi_\uparrow^\dagger(x'))\rangle,\\ \frac{\delta G_{21}(x,x')}{\delta U_\uparrow(z)}+\frac{\delta G_{21}(x,x')}{\delta U_\downarrow(z)}=&-\langle T(n(z)\psi_\downarrow^\dagger(x)\psi_\uparrow^\dagger(x'))\rangle+\langle n\rangle\langle T(\psi_\downarrow^\dagger(x)\psi_\uparrow^\dagger(x'))\rangle.\end{aligned}\tag{70}$$

By picking out the appropriate matrix elements of the Fourier transform of Eq. (69), we may identify the collective intermediate states occurring in the two-particle functions with the poles of $G(k+q)\Gamma(k,q)G(k)$. As pointed out by Nambu,[3] $\bar{X}_0(q)$ may be identified with the density-density correlation function. The low-lying pole $(q^0)^2=\alpha^2\mathbf{q}^2$ does not appear in this or any other physically measurable function.

Let us examine $\Delta\Sigma$, evaluated on the energy shell, $p^0=E(p)$, when $\chi$ and $\zeta$ are neglected.

$$\Delta\Sigma(p)\approx-i\int\frac{d^4q}{(2\pi)^4}V(-q)\left[\frac{(p^0+q^0)\tau_3+\epsilon(p+q)-i\tau_1\theta-i\tau_2\phi}{(p^0+q^0)^2-E^2(p+q)+i\epsilon}\right]\left[-\frac{\omega_0^2}{q^{02}-\omega_0^2-\alpha^2q^2+i\eta}+\frac{2q^0(\tau_2\theta-\tau_1\phi)}{q^{02}-\omega_0^2-\alpha^2q^2+i\eta}\right].\tag{71}$$

The first term of the integrand in the equation above is familiar in the electron gas problem. It corresponds to the replacement of the Coulomb potential by the propagator for the Coulomb field. The second term contains the factor characteristic of a superconductor. Performing the $q^0$ integration assuming $V$, $\theta$, and $\phi$ to be independent of energy, and neglecting $\alpha^2\mathbf{q}^2$ compared to $\omega_0^2$, we obtain

$$\Delta\Sigma(\mathbf{p},E(\mathbf{p}))\approx-\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}V(-\mathbf{q})\left\{\left[\frac{E(\mathbf{p}+\mathbf{q})\tau_3+\epsilon(\mathbf{p}+\mathbf{q})-i\tau_1\theta-i\tau_2\phi}{E(\mathbf{p}+\mathbf{q})}\right]\left[\frac{-\omega_0^2+2[E(\mathbf{p}+\mathbf{q})-E(\mathbf{p})](\tau_2\theta-\tau_1\phi)}{[E(\mathbf{p}+\mathbf{q})-E(p)]^2-\omega_0^2+i\eta}\right]\right.$$
$$\left.+\left[\frac{[E(\mathbf{p})+\omega_0]\tau_3+\epsilon(p+q)-i\tau_1\theta-i\tau_2\phi}{[E(\mathbf{p})+\omega_0]^2-E^2(\mathbf{p}+\mathbf{q})+i\epsilon}\right][-\omega_0+2(\tau_2\theta-\tau_1\phi)]\right\}. \quad (72)$$

The quantity $\omega_0$, a large energy compared to the others involved, appears in each of the denominators of interest. The integrand is thereby prevented from having poles and does not contribute significantly to the self-energy.[8] However, before explicit calculations of this order correction to $\Sigma$ is justified, reliable techniques for solving the lowest order integral equation (21), must be developed.

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[8] Prior to this calculation, Professor J. Bardeen and Professor J. R. Schrieffer suggested to the author that the correction to $\Sigma$ would be suppressed in this way.