# Long-Range Order of the Linear Antiferromagnetic Chain

N. KARAYIANIS,* C. A. MORRISON, AND D. E. WORTMAN*
*The Diamond Ordnance Fuze Laboratories, Washington, D. C.*
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The long-range order in the ground state of the linear antiferromagnetic chain is investigated. A chain of $N$ identical spin-$\frac{1}{2}$ particles coupled by the anisotropic Hamiltonian

$$\mathcal{H}=2J\sum_{j=1}^{N} S_j^z S_{j+1}^z+bJ\sum_{j=1}^{N}(S_j^+S_{j+1}^-+S_j^-S_{j+1}^+)$$

is considered, and the periodic boundary condition, $S_{i+N}\equiv S_i$, is imposed so that the chain may be considered a loop of particles. The operator $\mathfrak{M}$, which rotates the particles in the loop so that each of the particles moves to the next highest site number, is introduced. It is proved that simultaneous eigenstates of $\mathcal{H}$ and $\mathfrak{M}$ have no long-range order as measured by $\sigma^*$ of Kasteleijn. The ground state of the Hamiltonian for $b\neq 0$, $N<\infty$ is shown to be nondegenerate. It is therefore necessarily an eigenstate of $\mathfrak{M}$ and as such has no long-range order.

## INTRODUCTION

THE long-range order of the ground state of the linear antiferromagnetic chain has been the subject of renewed interest in recent years. Several authors have used slightly different definitions of the long-range order, but except for one, the definitions are equivalent. Among the authors who have considered equivalent definitions of order there is disagreement as to the amount of long-range ordering, and although there has been much speculation and conjecture, to our knowledge no rigorous conclusion has been established.

We will limit the discussion to a chain of $N$ identical spin-$\frac{1}{2}$ particles where $N$ is an even integer. The Hamiltonian for the system is

$$\mathcal{H}=2J\sum_{j=1}^{N}(S_j^zS_{j+1}^z-\tfrac{1}{4})+bJ\sum_{j=1}^{N}(S_j^+S_{j+1}^-+S_j^-S_{j+1}^+), \quad (1)$$

where $b$ is an arbitrary parameter. The periodic boundary condition $S_{i+N}\equiv S_i$ is imposed so that, in actuality, the system under consideration is a loop of coupled spins rather than a chain. Letting $b$ take on the values of $+1$ and 0, what will be referred to, respectively, as the isotropic Heisenberg Hamiltonian and Ising Hamiltonian, are obtained from the above anisotropic Hamiltonian.

Kasteleijn[1] defines long-range order by the coefficient $\sigma^*=(2/N)(m_1-m_2)$, where $m_1$ and $m_2$ are, respectively, the number of spins up on the sublattices $A$ and $B$, $A$ being the set of odd-integer sites and $B$ being the set of even-integer sites. A zero value for $\sigma^*$ is interpreted as no long-range order. Kasteleijn found that $\sigma^*$ necessarily vanishes in the ground state of the Hamiltonian for values of $b$ in the range $1>b>0.483$. As $b$ is decreased from 0.483 to 0, Kasteleijn found that the ground state may be chosen so that $\sigma^*$ increases to the value $\sigma^*=1$ for $b=0$, which is the perfect ordering found in one of the ground states of the Ising Hamiltonian (e.g., $\alpha\beta\alpha\beta\alpha\beta\cdots$).

Marshall[2] refers to the number of spins up on a given sublattice, say $A$. The long-range order is characterized by this number which he calls $p$ (equal to $m_1$ of Kasteleijn). In a state having a total of $N/2$ spins up, no long-range order is signified by $p=N/4$, i.e., half of the total number of spins up are in the $A$ sublattice. The state with $x$ spins up has no long-range order if $p=x/2$.

Orbach[3] refers to Kasteleijn's $\sigma^*$ and probably has this quantity in mind when he states that the long-range order question is an unanswered one.

Walker[4] refers to $\sum_i S_i^z$, where the sum is taken over one of the sublattices, as "such measure of long-range order." He states that the above quantity "would lead to a vanishing value" in the exact ground state. In an arbitrary state, no long-range order is indicated if the expectation value of the above quantity in that state is one-half of the total spin projection of the state.

As interpreted above, the three measures of long-range order are equivalent. Specifying the total spin projection operator by

$$S^z=\sum_{j=1}^{N} S_j^z,$$

the quantities are related by

$$\frac{1}{2}\left(\frac{N}{2}\sigma^*+S^z\right)=p-\frac{N}{4}=\sum_{j\ \text{odd}} S_j^z, \quad (2)$$

where the equalities are understood in the sense of expectation values in a given state.

A measure of long-range order not equivalent to the above quantities has been used by Rodriquez.[5] He

* Now at the University of Indiana, Bloomington, Indiana.
[1] P. W. Kasteleijn, Physica **18**, 104 (1952).
[2] W. Marshall, Proc. Roy. Soc. (London) **A232**, 48 (1955).
[3] R. Orbach, Phys. Rev. **112**, 309 (1958).
[4] L. R. Walker, Phys. Rev. **116**, 1089 (1959).
[5] S. Rodriquez, Phys. Rev. **116**, 1474 (1959).

defines order by the operator

$$(4/N)\sum_{j=1}^{N}\mathbf{S}_j\cdot\mathbf{S}_{j+\lambda}$$

so that the order[6] is a function of the parameter $\lambda$. The long-range order is measured by the operator if $\lambda$ is set equal to a large even integer less than or equal to $N/2$. As Ruijgrok[7] points out, this measure of order is difficult to calculate except in special cases, so instead, he estimates the value of

$$(4/N)\sum_{j=1}^{N}S_j^zS_{j+\lambda}^z.$$

In this paper, we shall not concern ourselves with the measure of long-range order as defined by Rodriquez nor with the question of which definition of long-range order is more meaningful for a true antiferromagnetic material. We shall show that the long-range order as measured by $\sigma^*$ vanishes in the ground state of the anisotropic Hamiltonian for $N<\infty$, whenever $b\neq 0$. Moreover, it is demonstrated that there exists a complete set of eigenstates of the anisotropic Hamiltonian, each of which has zero long-range order.

## THE HEISENBERG HAMILTONIAN

For simplicity we shall first consider the Heisenberg Hamiltonian ($b=1$), and discuss some of its properties. This Hamiltonian is usually written

$$H=2J\sum_{j=1}^{N}(\mathbf{S}_j\cdot\mathbf{S}_{j+1}-\tfrac{1}{4}),\quad \text{where } S_{N+1}=S_1. \tag{3}$$

The total spin operator $S^2$, where $\mathbf{S}=\sum_{j=1}^{N}\mathbf{S}_j$, and the total spin projection operator $S^z=\sum_{j=1}^{N}S_j^z$ both commute with each other and with each component of the Hamiltonian (i.e., with each $\mathbf{S}_i\cdot\mathbf{S}_{i+1}$), hence with the Hamiltonian itself. For spin-$\frac{1}{2}$ particles, the Hamiltonian may be expressed in terms of the interchange operator $m_{i,j}$. This operator interchanges the particles at the $i$th and $j$th sites.[8] By the very nature of $m_{i,j}$, it is easily deduced that

$$m_{i,j}=m_{j,i};\quad (m_{i,j})^2=1;\quad m_{ii}=1. \tag{4}$$

Moreover, we have the identity

$$\mathbf{S}_i\cdot\mathbf{S}_j=\tfrac{1}{2}(m_{i,j}-\tfrac{1}{2}+\delta_{ij}). \tag{5}$$

In terms of the interchange operator, the Hamiltonian reduces to

$$H=J\sum_{j=1}^{N}(m_{j,j+1}-1). \tag{6}$$

In this form, one can calculate the effect of the Hamiltonian on a spin state very quickly. If neighboring spins are parallel, the $m_{i,i+1}-1$ yields zero. If neighboring spins are antiparallel, then the result of $J(m_{i,i+1-1})$ is $-J$ times the state plus a different state where the two neighboring antiparallel spins have been interchanged. Therefore the net effect of the Hamiltonian on a given state is $-Jq$ times the state plus $J$ times $q$ different states, where $q$ is the number of antiparallel pairs in the state, and each of the $q$ new states differs from the old state by the interchange of one of the $q$ antiparallel pairs.

Denote by $\mathfrak{M}$ the operator which rotates the ensemble of spins so that each of the particles moves to the next highest site number, the $N$th particle moving to the first site. This unitary operator commutes with the three operators $H$, $S^2$, and $S^z$. Hence there exists a complete set of eigenstates of $H$ which change at most by a phase on rotation.

To demonstrate the effect of $\mathfrak{M}$, consider an arbitrary operator $\mathfrak{R}(i_1,i_2,\cdots i_k)$ which acts on the spins at the sites $i_1, i_2, \cdots i_k$. Let it act on an arbitrary state and then let the resultant be rotated through $\lambda$ sites, i.e., perform the operation $\mathfrak{M}^\lambda\mathfrak{R}$. Now if the state is rotated first through $\lambda$ sites and then operated on by $\mathfrak{R}(i_1+\lambda, i_2+\lambda, \cdots, i_k+\lambda)$, the same result must be obtained. Therefore, one concludes that

$$\mathfrak{M}^\lambda\mathfrak{R}(i_1\cdots i_k)\mathfrak{M}^{-\lambda}=\mathfrak{R}(i_1+\lambda\cdots i_k+\lambda), \tag{7}$$

where the convention $i+N\equiv i$ is understood.

That $H$, $S^2$, $S^z$, and $\mathfrak{M}$ are mutually commuting operators now follows immediately. We may therefore choose a complete set of orthonormal eigenstates such that all four operators are diagonal in that basis. Denote the eigenvalues as follows:

$$\begin{aligned}&H\psi_n=E_n\psi_n,\\ &S^2\psi_n=s_n(s_n+1)\psi_n,\\ &S^z\psi_n=s_n^z\psi_n,\\ &\mathfrak{M}\psi_n=\tau_n\psi_n,\quad \psi_n^\dagger\psi_m=\delta_{nm},\end{aligned} \tag{8}$$

where $\tau_n=e^{i(2\pi\nu_n/N)}$ and we choose $0\leq\nu_n\leq N-1$.

## MATRIX ELEMENTS AND LONG-RANGE ORDER

Useful statements concerning matrix elements in the space of $\psi_n$ may be obtained by using the rotational properties of the $\psi_n$. Letting $\mathfrak{R}(i_1,\cdots i_k)$ be an operator acting on the spins at the sites $i_1\cdots i_k$, we obtain

$$\begin{aligned}&\psi_n^\dagger\mathfrak{R}(i_1\cdots i_k)\psi_m\\ &\quad=\psi_n^\dagger(\mathfrak{M}^\lambda)^\dagger\mathfrak{M}^\lambda\mathfrak{R}(i_1\cdots i_k)\mathfrak{M}^{-\lambda}\mathfrak{M}^\lambda\psi_m\\ &\quad=e^{i(2\pi/N)(\nu_m-\nu_n)\lambda}\psi_n^\dagger\mathfrak{R}(i_1+\lambda\cdots i_k+\lambda)\psi_m.\end{aligned} \tag{9}$$

[6] This measure of full correlation, properly normalized, should appear as $(4/3N)\sum_{j=1}^{N}\mathbf{S}_j\cdot\mathbf{S}_{j+\lambda}$, since $S^2=\frac{3}{4}$. The error in Rodriquez' normalization shows up in the calculation of the short-range order in the $N=2$ ground state, for example. Here the ground state is $\chi^s\equiv(1/\sqrt{2})(\alpha\beta-\beta\alpha)$ and $\chi^{s\dagger}[(4/3N)\sum_{j=1}^{N}\mathbf{S}_j\cdot\mathbf{S}_{j+1}]\chi^s=-1$, which shows the full antiparallel correlation of the neighboring spins.

[7] T. W. Ruijgrok and S. Rodriquez, Phys. Rev. **119**, 596 (1960).

[8] Since the only manifestation of the particles is their spin, this operator is equivalent to the operator used in nuclear physics and attributed to Bartlett. The Bartlett operator interchanges the spins of two particles and leaves unaffected other properties which the particles may possess, such as position and isotopic spin.

An example of the usefulness of this property is found in the calculation of the matrix elements of an arbitrary $m_{i,i+1}$. The matrix elements involving all $\psi_n$ and $\psi_m$ such that $\nu_n=\nu_m$ may be calculated exactly. From the general property of the matrix elements, we have (if $\nu_n=\nu_m$)

$$\psi_n^\dagger m_{i,i+1}\psi_m=\psi_n^\dagger m_{i+\lambda,i+\lambda+1}\psi_m$$
$$=\frac{1}{JN}\psi_n^\dagger J\sum_{j=1}^{N}(m_{j,j+1}-1)\psi_m+\delta_{nm}$$
$$=\left(\frac{E_n}{JN}+1\right)\delta_{nm}. \tag{10}$$

As stated above, the Heisenberg Hamiltonian ($H$) is a special case ($b=1$) of the anisotropic Hamiltonian ($\mathfrak{IC}$). Only for this special case does the Hamiltonian commute with $S^2$, however it commutes with $\mathfrak{M}$ and $S^z$ for all $b$. In general, then, eigenstates of $\mathfrak{IC}$ exist which are simultaneous eigenstates of $\mathfrak{IC}$, $\mathfrak{M}$, and $S^z$ and which go into the set $\psi_n$ as $b\rightarrow 1$. To avoid cumbersome notation, we shall continue to use the symbol $\psi_n$ for the eigenstates of $\mathfrak{IC}$ with the understanding that $\psi_n=\psi_n(b)$. Hence the eigenstates defined in Eq. (11) and used so far are to be understood as $\psi_n(1)$.

In the space of the $\psi_n$, under their more general definition, matrix elements still enjoy the property expressed in Eq. (9) since the $\psi_n$ are eigenstates of $\mathfrak{M}$ for arbitrary $b$. From this general property, we obtain specifically for the diagonal terms

$$\psi_n^\dagger\mathfrak{R}(i_1\cdots i_k)\psi_n=\psi_n^\dagger\mathfrak{R}(i_1+\lambda,\cdots i_k+\lambda)\psi_n. \tag{11}$$

Therefore,

$$\psi_n^\dagger\sum_{j=1}^{\lambda}\mathfrak{R}(i_1+j,\cdots i_k+j)\psi_n$$
$$=\frac{\lambda}{N}\psi_n^\dagger\sum_{j=1}^{N}\mathfrak{R}(i_1+j,\cdots i_k+j)\psi_n. \tag{12}$$

Hence, we obtain

$$\psi_n^\dagger\sum_{j\text{ odd}}S_j^z\psi_n=\frac{N}{2}\psi_n^\dagger S_i^z\psi_n$$
$$=\tfrac{1}{2}s_n^z. \tag{13}$$

Using the appropriate relationship in Eq. (2), which is

$$\psi_n^\dagger\left(\frac{N}{4}\sigma^*+\tfrac{1}{2}S^z\right)\psi_n=\psi_n^\dagger\sum_{j\text{ odd}}S_j^z\psi_n=\tfrac{1}{2}s_n^z, \tag{14}$$

we find

$$\psi_n^\dagger\sigma^*\psi_n=0 \tag{15}$$

for all $\psi_n$.

We conclude that *if an eigenstate of $\mathfrak{IC}$ is simultaneously an eigenstate of $\mathfrak{M}$, the long-range order as measured by $\sigma^*$* (or any of the other two equivalent definitions) *vanishes.*

It has been stated that the $\psi_n$ may be chosen simultaneous eigenstates of $\mathfrak{IC}$, $\mathfrak{M}$, and $S^z$, hence there exists a complete set of $\psi_n$, each of which has no long-range order. However the $\psi_n$, defined as eigenstates of $\mathfrak{IC}$, are necessarily eigenstates of $\mathfrak{M}$ only if they are nondegenerate. We therefore concern ourselves with proving the ground state to be nondegenerate.

## NONDEGENERACY OF THE GROUND STATE

The $\psi_n$ are considered as expansions in the product spin states ($\alpha\beta\beta\alpha\cdots$) which are known to span the space. The complete set of these spin states will be designated by $\{\mu\}$. The spin states in $\{\mu\}$ are also eigenstates of the total spin projection operator $S^z$. Since $\mathfrak{IC}$ commutes with $S^z$, it does not mix $\mu$ of different $s^z$ quantum number. We therefore may characterize a subset $\{\mu(s^z)\}\subset\{\mu\}$ by the total spin projection quantum number $s^z$, and define the subset $\{\mu(s^z)\}$ to be the group of spin states with the property that the operation of $\mathfrak{IC}$ on any $\mu(s^z)\in\{\mu(s^z)\}$ will result in states $\mu'(s^z)\in\{\mu(s^z)\}$. That there are not two disjoint subsets with the same $s^z$ quantum number is proved below in Theorem 1. For an arbitrary element $\mu(s^z)$ in the subset $\{\mu(s^z)\}$ we obtain [see the discussion following Eq. (6)]

$$\mathfrak{IC}\mu(s^z)=-Jq_\mu\mu(s^z)+bJ\sum_{\mu'}\mu'(s^z), \tag{16}$$

where each $\mu'(s^z)$ lies in the subset $\{\mu(s^z)\}$ by definition. Thus, the $\psi_n$ may be formed by linear combinations of $\mu(s^z)\in\{\mu(s^z)\}$, and hence are themselves eigenstates of $S^z$.[9] For such a choice of the $\psi_n$ it is evident that a *$\psi_n$ is necessarily an eigenstate of $\mathfrak{M}$ if it is nondegenerate with respect to the group of $\psi_n$ formed from the same subset $\{\mu(s^z)\}$*. This result follows because $\mathfrak{M}$ does not mix states with dissimilar $s^z$ eigenvalue, and therefore it is immaterial if $\psi_n$, nondegenerate in the above sense, is degenerate with an eigenstate of dissimilar $s^z$ eigenvalue.

We define a *ground substate* as the eigenstate of the anisotropic Hamiltonian which has the lowest energy eigenvalue among the group of eigenstates characterized by a given total spin projection quantum number, $s^z$. In all, there are at least $N+1$ ground substates. Which of these is the ground state of the system is immaterial, since we shall prove that each is nondegenerate in the sense defined above, hence is an eigenstate of $\mathfrak{M}$ and, as such, has vanishing long-range order.

In the course of proving the above statement, the following theorem is needed:

*Theorem 1. There is a unique subset of spin states, $\{\mu(s^z)\}$, for a given $s^z$.*

*Proof.* Assume that $\{\mu(s^z)\}$ and $\{\lambda(s^z)\}$ are two disjoint subsets of $\{\mu\}$. From the definition of a subset,

[9] It is convenient to form the $\psi_n$ in this way since they will remain eigenstates of the total Hamiltonian on the addition of an external magnetic field term, $-gBS^z$.

it follows that the result of $K$ operations by the anisotropic Hamiltonian on an arbitrary $\mu(s^z)\in\{\mu(s^z)\}$, where $K$ is a positive integer however large, is a group of spin states all of which lie in the subset $\{\mu(s^z)\}$. Hence, since $\{\mu(s^z)\}$ and $\{\lambda(s^z)\}$ are disjoint,

$$\lambda^\dagger(s^z)\mathfrak{IC}^K\mu(s^z)=0 \tag{17}$$

for any

$$\lambda(s^z)\in\{\lambda(s^z)\},\quad \mu(s^z)\in\{\mu(s^z)\}$$

and any integer $K\geq 0$.

Express the anisotropic Hamiltonian in the following form:

$$\mathfrak{IC}=2(1-b)J\sum_{j=1}^{N}(S_j{}^zS_{j+1}{}^z-\tfrac{1}{4}) +bJ\sum_{j=1}^{N}(m_{j,j+1}-1). \tag{18}$$

The part of the Hamiltonian which intercouples states is $bJ\sum_{j=1}^{N}(m_{j,j+1}-1)$. The assumption that there are two disjoint subsets now may be phrased: No products of $K$ operators of the form $m_{i,i+1}$ can carry a state $\mu(s^z)$ into a state in $\{\lambda(s^z)\}$.

Suppose further that the set $\{\mu(s^z)\}$ is not null. Let a state $\mu(s^z)\in\{\mu(s^z)\}$ be characterized by the indices $(i_1,i_2,\cdots i_k)$, where these are the sites of $\mu(s^z)$ with spin up, all other sites having spin down. Order the $i_n$ so that $i_n<i_{n+1}$. Since the total number of particles is $N$, and the total spin projection is $s^z$, $k=(N/2)+s^z$. An arbitrary state $\nu(s^z)$ with total spin projection equal to that of $\mu(s^z)$ may be characterized in the same manner. Let this characterization be $(j_1,j_2,\cdots j_k)$, where $j_n$ are ordered such that $j_n<j_{n+1}$. From the set of numbers $i$ and $j$, discard all the $i$ and $j$ which are equal. Now relabel the remaining numbers $(i_1',i_2',\cdots i_r')$ and $(j_1',j_2',\cdots j_r')$, keeping the same order. In these sets, no two numbers are equal. Now, a transformation $T$ which carries $\mu(s^z)$ into $\nu(s^z)$ is

$$T=m_{i_1',j_1'}m_{i_2',j_2'}\cdots m_{i_r',j_r'}. \tag{19}$$

From the properties of the $m_{i,j}$ it is readily deduced that

$$m_{i,j}=m_{i,i+1}m_{i+1,i+2}\cdots m_{j-1,j}m_{j-1,j-2}\cdots \times m_{i+1,i};\quad j>i. \tag{20}$$

Therefore $T$ is expressible as a product of operators of the form $m_{i,i+1}$. The total number $\mathfrak{N}$ of such operators is

$$\mathfrak{N}=2\sum_{n=1}^{r}|i_n'-j_n'|-r. \tag{21}$$

If now we set $K\geq\mathfrak{N}$, we obtain

$$\nu^\dagger(s^z)\mathfrak{IC}^k\mu(s^z)\neq 0 \tag{22}$$

so that $\nu(s^z)\in\{\mu(s^z)\}$ by definition. But $\nu(s^z)$ is an arbitrary state, hence $\{\lambda(s^z)\}$ must be a null subset, and the proof is complete.

Following Marshall[2] and the proof of Peierls included therein, let the expansion of a normalized eigenstate of $\mathfrak{IC}$ and $S^z$ be

$$\psi_n=\sum_\mu c_\mu\mu(s^z), \tag{23}$$

where $S^z\psi_n=s_n{}^z\psi_n$, and $\mu(s^z)\in\{\mu(s^z)\}$. The expectation value of the Hamiltonian is

$$\psi_n{}^\dagger\mathfrak{IC}\psi_n=-J\sum_\mu c_\mu{}^*c_\mu q_\mu+bJ\sum_{\mu''}c_{\mu''}{}^*[\mu''(s^z)]^\dagger \times\sum_\mu c_\mu(\sum_{\mu'}\mu'(s^z)), \tag{24}$$

where $q_\mu$ is the number of antiparallel pairs in $\mu(s^z)$, and each $\mu'(s^z)$ is a state which differs from $\mu(s^z)$ by the interchange of an antiparallel pair of neighboring spins. The $\mu'(s^z)$ are $q_\mu$ in number.

Consider $b>0$ and let

$$c_\mu=(-1)^{p_\mu}a_\mu, \tag{25}$$

where $p_\mu$ is the number of spins up on the $A$ sublattice of $\mu(s^z)$. The states $\mu'(s^z)$ to which $\mu(s^z)$ is coupled have $p_{\mu'}=p_\mu\pm 1$ since they differ from $\mu(s^z)$ by the interchange of a neighboring antiparallel pair. Hence the expectation value of the Hamiltonian becomes

$$\psi_n{}^\dagger\mathfrak{IC}\psi_n=-J\sum_\mu a_\mu{}^*\sum_{\mu'}(a_\mu+ba_{\mu'}). \tag{26}$$

In the ground substate, the $a_\mu$ must be positive definite numbers (excepting an over-all phase factor for the state). If they were not, one could form the state

$$\phi=\sum_\mu(-1)^{p_\mu}|a_\mu|\mu(s^z), \tag{27}$$

using the $a_\mu$ of the ground substate, and this state would yield a lower expectation value for the energy, which is a contradiction.

For $b<0$ let $b=-b'$ so that the expectation value of the Hamiltonian is

$$\psi_n{}^\dagger\mathfrak{IC}\psi_n=-J\sum_\mu c_\mu{}^*\sum_{\mu'}(c_\mu+b'c_{\mu'}). \tag{28}$$

By similar arguments, we conclude that the ground substate in this case has all positive real $c_\mu$.

In either case, i.e., $b>0$ or $b<0$, the coefficients of the states $\mu'(s^z)$ to which $\mu(s^z)$ is coupled and which couple to $\mu(s^z)$ all have the same relative sign. Therefore none of the $\mu(s^z)$ may appear in the ground substate with zero coefficient. If a $c_\mu$ were zero for a particular $\mu$, then the ground substate being an eigenstate of $\mathfrak{IC}$ requires $\sum_{\mu'}c_{\mu'}$ to vanish. But since all of the $c_{\mu'}$ are of the same relative sign, the only way the sum can vanish is if each $c_{\mu'}$ is zero. The argument proceeds in this manner to the coefficients of the totality of spin states which are intercoupled by the Hamiltonian, the $\{\mu(s^z)\}$. However, we have proved that the subset $\{\mu(s^z)\}$ contains all the spin states with spin projection quantum number $s^z$. Hence, each ground substate of the anisotropic Hamiltonian for $b\neq 0$ contains all of the spin states in $\{\mu(s^z)\}$ where $s^z$ characterizes the particular ground substate. The arguments break down for $b=0$ because there is no intercoupling of the spin states, and each $\mu(s^z)$ separately is an eigenstate of the Hamiltonian and of $S^z$.

*Theorem 2. Ground substates of the anisotropic Hamiltonian for $b\neq 0$ are nondegenerate in the sense that there are no two ground substates with the same energy and total spin projection quantum numbers.*

*Proof.* Assume the contrary. With no loss of generality, one may assume the degenerate states to be orthonormal. Let $\psi_n$ and $\psi_m$ be two of these orthonormal, degenerate ground substates, and let their expansions in terms of the $\mu(s^z)$ be

$$\psi_n=\sum_\mu c_\mu{}^n\mu(s^z), \quad \psi_m=\sum_\mu c_\mu{}^m\mu(s^z). \tag{29}$$

For the sake of argument, assume $b>0$. The argument for $b<0$ follows in exactly the same manner. The inner product of the orthogonal states is

$$\psi_n{}^\dagger\psi_m=0=\sum_\mu a_\mu{}^n a_\mu{}^m. \tag{30}$$

The $a_\mu$ it is recalled, are positive definite numbers, where $c_\mu=(-)^{p_\mu}a_\mu$. We invoke the theorem regarding the uniqueness of the subset $\{\mu(s^z)\}$ to preclude the possibility of the sets $\mu(s^z)$ in the expansion of $\psi_n$ and $\psi_m$ from being disjoint. Hence, each pair $a_\mu{}^n a_\mu{}^m=0$, and if $a_\mu{}^m=0$ for a particular $\mu$, then all the $a_\mu{}^m$ must vanish. There can be, therefore, no degeneracy of the type considered in the ground substate and the theorem is proved. We have shown above that the type of nondegeneracy proved in Theorem 2 is sufficient to ensure that the ground substates are eigenstates of $\mathfrak{M}$. Thus it follows that the ground substates of the anisotropic Hamiltonian, for $b\neq 0$, necessarily have no long-range order. With the establishing of this fact, the goals of this paper have been achieved.

## CONCLUSIONS

The purpose of this paper has been to determine rigorously the long-range order in the ground state of the anisotropic Hamiltonian for $N<\infty$ as measured by $\sigma^*$. We have slightly exceeded this goal and have proved that each of the set of $N+1$ ground substates, of which the ground state is a member, has no long-range order, i.e., $\sigma^*=0$ in each of these states.

Our results do not hold rigorously in the limit $N=\infty$ as each of the coefficients $(c_\mu)$ in the expansion of the ground substates must approach zero in this limit. Thus the requirement $\sum_{\mu'} c_{\mu'}=0$ is automatically satisfied and yields no condition on the inclusion of the states $\{\mu(s^z)\}$ in the ground substate. The above expresses the well-known result[10] that the energy difference between the ground state and certain excited states approaches zero as $1/N$ for $N\rightarrow\infty$.

The result of Kasteleijn is expected to be an approximation in the limit $N=\infty$ and, as such, cannot be compared with the results of this paper. However, even in the limit $N=\infty$, i.e., when the ground state is no longer nondegenerate, one may choose the ground state to have no long-range order—a conclusion which is not incompatible with Kasteleijn's result.

The results obtained in this paper can be extended with little modification to higher dimensions.

[10] L. Hulthén, Proc. Acad. Sci. Amsterdam **39**, 190 (1936).