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Angular Momenta in Relativistic Three-Body Systems

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An elementary but rigorous treatment of the addition of relativistic angular momenta for three-particle systems in their center-of-mass reference frame is given. Canonical variables are defined, in terms of which the usual nonrelativistic formulas hold true. An application is made to the analysis of a $T=0$ three-pion resonance.

THIS paper deals with an elementary but rigorous treatment of a troublesome feature of relativistic quantum mechanics, *viz.* the addition of angular momenta for many-particle systems. For the sake of brevity only a simplified version of the whole treatment is presented here, holding for two particles in any reference frame, or for three particles in their center-of-mass (c.m.) reference frame and in both cases only for spinless particles. The general problem for any number of particles, of any spin, will be discussed elsewhere.[1]

1. TWO NONINTERACTING SPINLESS PARTICLES

In nonrelativistic quantum mechanics the usual way of dealing with angular momenta is the following: if $\mathbf{q}_1,\mathbf{p}_1$ and $\mathbf{q}_2,\mathbf{p}_2$ are position and momentum vectors of the particles, angular momentum is defined by

$$\mathbf{M}=\mathbf{q}_1\times\mathbf{p}_1+\mathbf{q}_2\times\mathbf{p}_2. \tag{1}$$

By introducing the c.m. position vector

$$\mathbf{Q}=(m_1\mathbf{q}_1+m_2\mathbf{q}_2)/(m_1+m_2), \tag{2}$$

the total momentum

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2, \tag{3}$$

and relative position and momentum vectors

$$\mathbf{q}=\mathbf{q}_1-\mathbf{q}_2, \tag{4}$$

$$\mathbf{p}=(m_2\mathbf{p}_1-m_1\mathbf{p}_2)/(m_1+m_2), \tag{5}$$

one gets a second set of canonical variables, in terms of which $\mathbf{M}$ can be written:

$$\mathbf{M}=\mathbf{Q}\times\mathbf{P}+\mathbf{q}\times\mathbf{p}. \tag{6}$$

If the particles are free, both $\mathbf{Q}\times\mathbf{P}$ and $\mathbf{q}\times\mathbf{p}$ are constants of motion commuting with each other.

In relativistic quantum mechanics things are not as easy. A relativistic theory of free particles can be set up along the same lines as nonrelativistic theory,[2] the only difference being that the energy operator for any particle is to be given the usual square root form of relativistic dynamics $[E_i=(m_i^2+p_i^2)^{\frac{1}{2}},\ i=1,2]$. It is easily seen that $\mathbf{Q}\times\mathbf{P}$ and $\mathbf{q}\times\mathbf{p}$ are no longer constants of motion, though Eqs. (2)–(5) still define a canonical transformation.

The question now arises, whether a different transformation will restore the properties of $\mathbf{Q}\times\mathbf{P}$ and $\mathbf{q}\times\mathbf{p}$; and the answer is yes. The transformation to be used, as far as momenta are concerned, is readily understood. One has only to change (5) into the relativistic expression for the momentum of particle 1 in the c.m. reference frame, leaving unaltered the expression (2) for $\mathbf{P}$. A much more complicated task is to find the right forms for $\mathbf{Q},\mathbf{q}$.[3]

These are the following:

$$\mathbf{Q}=\frac{1}{2}\left\{\mathbf{q}_1\frac{E_1}{E}+\mathbf{q}_2\frac{E_2}{E}+\left[(\mathbf{q}_1-\mathbf{q}_2)\times\left(\mathbf{p}-\frac{(\mathbf{p}\cdot\mathbf{P})}{E(E+M)}\mathbf{P}\right)\right]\times\mathbf{P}\frac{1}{(E+M)M}\right\}+\text{H.c.}, \tag{7}$$

$$\mathbf{q}=\frac{1}{2}\left\{\mathbf{q}_1-\mathbf{q}_2+[(\mathbf{q}_1-\mathbf{q}_2)\cdot\mathbf{P}]\left[\frac{1}{M(E+M)}\mathbf{P}-4\frac{(m_1^2-m_2^2)E+M(\mathbf{p}\cdot\mathbf{P})}{E[M^4-(m_1^2-m_2^2)^2]}\mathbf{p}\right]\right\}+\text{H.c.}, \tag{8}$$

where

$$E=E_1+E_2,\quad M=(E^2-P^2)^{\frac{1}{2}}. \tag{9}$$

What is of interest to us is that formula (6) still holds true, with $\mathbf{Q}\times\mathbf{P}$ and $\mathbf{q}\times\mathbf{p}$ commuting constants of motion. All consequences of nonrelativistic theory then become applicable to the relativistic case. For instance, we are allowed to study angular distributions in terms of irreducible tensors built up with $\mathbf{P}$ and $\mathbf{p}$, etc.

2. THREE NONINTERACTING SPINLESS PARTICLES IN c.m. REFERENCE FRAME

This case can be easily reduced to the preceding one. To work in the c.m. reference frame means to restrict oneself to the consideration of the subspace of the eigenvectors of $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3$ belonging to the eigenvalue zero.[4] We can then use the preceding transformation

[1] B. Barsella and E. Fabri (to be published). A group-theoretical discussion of the same problem is given by A. J. MacFarlane, Revs. Modern Phys. **34**, 41 (1962).

[2] L. L. Foldy, Phys. Rev. **102**, 569 (1956); E. Fabri and L. Picasso (to be published).

[3] The expression for $\mathbf{Q}$ was given in another connection by M. H. L. Pryce, Proc. Roy. Soc. (London) **A150**, 166 (1935).

[4] For a discussion of this point see E. Fabri, Nuovo cimento, **11** 479 (1954) (footnote 4).

TABLE I. Matrix elements for the ω decay ($T=0, J\leqslant 1$).

Meson Spin	Parity	Final state l	L	Matrix element
0	−	1	1	$\mathbf{p}_{12}\cdot\mathbf{p}_3+\mathbf{p}_{23}\cdot\mathbf{p}_1+\mathbf{p}_{31}\cdot\mathbf{p}_2=-2W^2\frac{(E_1-E_2)(E_2-E_3)(E_3-E_1)}{(M_{23}+M_{31})(M_{31}+M_{12})(M_{12}+M_{23})}$
		3	3	$(\mathbf{p}_{12}\cdot\mathbf{p}_3)[(\mathbf{p}_{12}\cdot\mathbf{p}_3)^2-\frac{3}{5}\mathbf{p}_{12}{}^2\mathbf{p}_3{}^2]+\text{cycl.}$
1	+	0	1	$\mathbf{p}_{12}+\mathbf{p}_{23}+\mathbf{p}_{31}=\left(\frac{1}{2}\frac{E_1-E_2}{E_1+E_2+M_{12}}\mathbf{p}_3\right)+\text{cycl.}$
		2	1	$[(\mathbf{p}_{12}\cdot\mathbf{p}_3)\mathbf{p}_3-\frac{1}{3}\mathbf{p}_3{}^2\mathbf{p}_{12}]+\text{cycl.}=[-\frac{1}{3}(E_1-E_2)M_{12}\mathbf{p}_3]+\text{cycl.}$
1	−	1	1	$\mathbf{p}_{12}\times\mathbf{p}_3+\mathbf{p}_{23}\times\mathbf{p}_1+\mathbf{p}_{31}\times\mathbf{p}_2=3\mathbf{p}_1\times\mathbf{p}_2$
		3	3	$\{[(\mathbf{p}_{12}\cdot\mathbf{p}_3)^2-\frac{1}{5}\mathbf{p}_{12}{}^2\mathbf{p}_3{}^2]\mathbf{p}_{12}\times\mathbf{p}_3\}+\text{cycl.}$

for particles 1 and 2, substituting $-\mathbf{p}_3$ for $\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2$. The total angular momentum for the three particles becomes

$$\mathbf{q}_{12}\times\mathbf{p}_{12}+(\mathbf{q}_3-\mathbf{Q}_{12})\times\mathbf{p}_3, \tag{10}$$

where the subscript 12 has been added in order to make manifest that the corresponding variables refer to the pair 1,2. Thus we have effected the decomposition of angular momentum into two parts, one giving the "inner" angular momentum of the pair 1,2, the other expressing the "relative" angular momentum of particle 3 with respect to that same pair.

An angular distribution can therefore be analyzed in terms of $\mathbf{p}_{12}$ and $\mathbf{p}_3$, as in the nonrelativistic case—the only difference arising from the different expression for $\mathbf{p}_{12}$, which only for low momenta will be approximated by Eq. (5).

3. APPLICATION TO THE ω MESON

As an example of the application of our results, we will sketch an analysis of the ω-meson decay.[5] Since we have a three-pion system, we are allowed to apply our analysis in the c.m. reference frame. We will denote by W the rest mass of the ω, by m the rest mass of the pion, and by M_{12} the rest of the pair 1,2 etc; we have then the relations

$$W=E_1+E_2+E_3, \tag{11}$$

$$M_{12}{}^2=W^2+m^2-2WE_3, \tag{12}$$

$$\mathbf{p}_{12}=\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)+\frac{E_1-E_2}{2(E_1+E_2+M_{12})}\mathbf{p}_3, \tag{13}$$

$$\mathbf{p}_{12}{}^2=\tfrac{1}{4}M_{12}{}^2-m^2, \tag{14}$$

$$\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0. \tag{15}$$

The matrix element for the decay is easily evaluated, taking into account that it must be antisymmetric in the momentum variables ($T=0$). The results for the simplest cases are reported in Table I. An inspection of this table reveals some differences with respect to the analysis of Alvarez and co-workers.[5] While we agree in the 1− case, in the other two cases our results show that their matrix elements come from having taken for the final state a superposition of many eigenstates (perhaps an infinite number) of the "good" angular momenta. Their assignment of spin and parity to the ω meson should, however, not be disturbed, since it depends mainly on such qualitative features of matrix elements as their vanishing on particular lines of the Dalitz plot, and these features are also shared by our matrix elements.

In spite of the fortunate situation we have found in this case, and apart for the theoretical interest of the problem, its practical importance should not be underestimated. In the widespread application of angular momentum analyses to high-energy processes, relativity should be properly taken into account, even if only qualitatively correct results are required.

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[5] L. W. Alvarez, B. C. Maglić, A. H. Rosenfeld, M. L. Stevenson, Phys. Rev. Letters **7**, 178 (1961); Phys. Rev. **125**, 687, 2208(E) (1962).