Structure-Independent Electrodynamics in the Electric-Dipole Approximation*

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(Received October 5, 1961)

We treat the problem of a nonrelativistic charged particle interacting with its own field, by a method originally devised by Kramers. We show that the ambiguities which arise in Kramers' theory vanish when one utilizes a canonical scheme which explicitly deals with the constraints (relations among the dynamical variables) which arise in the course of mass renormalization. To create a consistent canonical theory in the presence of these constraints, it is convenient to introduce new dynamical variables which no longer satisfy the canonical commutation rules. We show that, if one accounts for the constraints, several different renormalized theories are related to each other by canonical transformation, and that each of these theories may be derived from a given unrenormalized theory by a canonical transformation.

I. INTRODUCTION

IN this article we re-examine and reformulate a problem previously treated by Kramers.[1] Kramers considered a nonrelativistic charged particle interacting with its proper (self) electromagnetic field in addition to external radiation and potential fields. Kramers showed that the interaction of the charge with its own field could be incorporated into the theory by a mass renormalization, if the appropriate choice is made for the self-field. The resulting equations refer only to the observed mass and charge of the electron (Kramers calls this the structure-independent aspect of the theory), and the dynamical equations for the particle contain interaction terms involving only external fields.

Kramers' final equations are arrived at through use of the electric dipole approximation, whose key assumption is that all retardation effects are to be ignored in evaluating the electron's charge distribution. As a result of this assumption, one neglects the magnetic forces acting on the particle, as well as particle recoil.

Kramers placed his structure-independent equations in canonical form, and as a consequence of the mass renormalization his resulting Hamiltonian could be used to predict the Lamb shift.

However, his method suffers from an ambiguous specification of the canonical variables of the Hamiltonian theory; for as we show in this paper, a renormalized canonical theory contains variables which are not all independent of one another. The ambiguities in Kramers' theory arise because he has not formally incorporated the constraints (the relations among the canonical variables) into his Hamiltonian formalism.

In our reformulation of Kramers' problem, we show how the constraints arise from the mass renormalization procedure. Indeed, we show that the structure-dependent (unrenormalized) theory is related to the structure-independent (renormalized) theory through a canonical transformation, when the constraints are explicitly incorporated into the structure-dependent theory. The existence of constraints in the structure-dependent theory had already been suggested in a previous paper, from an analysis of solutions in the problem of a free particle and an oscillator interacting with a radiation field.[2]

In our formulation of this problem we have the option of treating the particle and electromagnetic field variables as classical quantities or as quantum-mechanical operators. In making the transition from the classical theory to quantum mechanics, one simply replaces all Poisson bracket relations by their appropriate commutator counterparts. Anticommutators do not enter the theory because we never second quantize the particle variables.

The constraints that enter our theory have especially simple properties. In the terminology of Dirac, they are all second class, i.e., they fail to commute with one another.[3] Faced only with this type of constraint, we avoid the difficulties introduced by their appearance, by replacing the canonical commutator (Poisson bracket) formalism by a scheme formally equivalent to the Dirac commutator (bracket) formalism.[4]

Kramers' mass renormalization is based on a specific decomposition of the electromagnetic vector potential into an external and a proper field. However, by using a different definition of the external and the proper field, Steinwedel[5] has also accomplished mass renormalization. The two approaches are actually equivalent, since we show that the variables of Kramers and Steinwedel are related by canonical transformation, when we modify Steinwedel's canonical formalism to take explicit account of the constraints.

* Supported in part by the Aeronautical Research Laboratory and the National Science Foundation.
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[2] M. Schwartz, Phys. Rev. **123**, 1903 (1961).
[3] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950).
[4] See reference 3.
[5] H. Steinwedel, Ann. Physik **15**, 207 (1955).

II. STRUCTURE INDEPENDENT EQUATIONS OF MOTION[6]

We consider an electron in an electromagnetic field which everywhere satisfies the field equations ($c=1$):

$$\Box\mathbf{A}=-\mathbf{j}+\nabla\partial\phi/\partial t,\quad \nabla\cdot\mathbf{A}=0,\quad \nabla^2\phi=-\rho. \tag{1}$$

$\mathbf{A}$ and ϕ are the potentials of the electromagnetic field vectors $\mathbf{E}$ and $\mathbf{H}$,

$$\mathbf{E}=-\nabla\phi-\partial\mathbf{A}/\partial t,$$
$$\mathbf{H}=\nabla\times\mathbf{A}.$$

In our nonrelativistic approximation it is valid to assume that the electron is a rigid body of unrenormalized mass m_0 and charge e. The charge distribution of the electron has spherical symmetry and extends over a small region of space of the dimensions of the classical-electron radius. The exact size is unimportant since in all our computations we shall pass to the limit of the point electron.

The electron equations of motion are

$$m_0\ddot{\mathbf{R}}=e\langle\mathbf{E}\rangle+e\dot{\mathbf{R}}\times\langle\mathbf{H}\rangle-\nabla U. \tag{2}$$

$\mathbf{R}$ denotes the position of the electron center and U the potential energy of the electron in an external, static field. The ρ and $\mathbf{j}$ of Eq. (1) refer to the charge and current distribution of our single charge, so that $\mathbf{j}=\rho\dot{\mathbf{R}}$.

The symbol $\langle Q\rangle$ is a mean value given by

$$e\langle Q\rangle=\int Q(\mathbf{x},t)\rho(\mathbf{x},t)d^3x. \tag{3}$$

These averages provide a means for evaluating field quantities at the position of the charge, and they serve to eliminate from the equations of motion certain infinite self-forces which cannot be removed by mass renormalization. In evaluating an integral of the form (3), we first integrate over the surface of the charge, then along its radius, and finally we pass to the limit of a point charge so that the dimensions of the particle will not appear in our equations. In other words, the charge density ρ must be replaced by a delta function, but in the manner prescribed above.

We now decompose $\mathbf{A}$ into two parts,

$$\mathbf{A}=\mathbf{A}_0+\mathbf{A}_1, \tag{4}$$

where $\mathbf{A}_0$ is the divergence-free solution of the equation

$$-\nabla^2\mathbf{A}_0=\mathfrak{T}\mathbf{j}=\mathbf{j}-\nabla\partial\phi/\partial t. \tag{5}$$

The letter $\mathfrak{T}$ in Eq. (5) signifies the "transverse part of" the vector which follows.

The solutions for ϕ and $\mathbf{A}_0$ are

$$\phi_P=\frac{1}{4\pi}\int\frac{\rho_Q}{r_{PQ}}dV_Q,\quad \mathbf{A}_{0P}=\frac{1}{4\pi}\int\mathfrak{T}\frac{\mathbf{j}_Q}{r_{PQ}}dV_Q, \tag{6}$$

and at large distances from the center of the electron these become

$$\phi(r)\sim e/4\pi r',\quad \mathbf{A}_0(r)\sim\mathfrak{T}e\mathbf{V}/4\pi r', \tag{7}$$

with $\mathbf{V}=\dot{\mathbf{R}}$, and $r'=|\mathbf{x}-\mathbf{R}|$, the distance from the center of the charge to the point $\mathbf{x}$ at which the potentials are evaluated. In accord with the electric dipole approximation, we shall assume that $\partial r'/\partial t=0$. We make this last assumption in all the calculations which follow, and in addition place the charge at the origin of our coordinate system ($r'=r$).

It we substitute the potentials for the field quantities in (2), and make use of the spherical symmetry of our charge distribution, we have

$$\left(m_0+\lim_{r\to0}\frac{2}{3}\frac{1}{4\pi}\frac{e^2}{r}\right)\ddot{\mathbf{R}}$$
$$=-e\frac{d}{dt}\langle\mathbf{A}_1\rangle+e\dot{\mathbf{R}}\times\langle\nabla\times\mathbf{A}_1\rangle-\nabla U. \tag{8}$$

The potential $\mathbf{A}_1$ satisfies the equations

$$\Box\mathbf{A}_1=\partial^2\mathbf{A}_0/\partial t^2,\quad \nabla\cdot\mathbf{A}_1=0. \tag{9}$$

Solutions for $\mathbf{A}_1$ are easily found.[7] For the purposes of this paper the precise form of these solutions is not important. We are only interested in the qualitative behavior of $\mathbf{A}_1$ in the electric dipole approximation, i.e., when the effects of retardation may be neglected in the evaluation of the electron's charge distribution. In this approximation $\mathbf{A}_1$ has finite values at the position of the charge, and at that point it is a function of the time alone. There is thus no magnetic force acting on our charge. If we take this last into account, and introduce the renormalized mass m,

$$m=m_0+\delta m,\quad \delta m\equiv\lim_{r\to0}\left(\frac{2}{3}\frac{1}{4\pi}\frac{e^2}{r}\right), \tag{10}$$

Eq. (7) becomes

$$m\ddot{\mathbf{R}}=-e\frac{d}{dt}\langle\mathbf{A}_1\rangle-\nabla U. \tag{11}$$

Equations (9) and (11) are the structure independent equations of Kramers' theory, in the electric dipole approximation. The only parameters characterizing the particle in these equations are the charge e and the observed mass m.

III. STRUCTURE INDEPENDENT HAMILTONIAN FORMALISM

The electric dipole approximation to Eqs. (1) and (2) may be derived from the following structure-dependent Lagrangian:

$$L_0=\tfrac{1}{2}m_0\mathbf{V}^2+e\langle\mathbf{A}\rangle\cdot\mathbf{V}-U(\mathbf{R})$$
$$+\frac{1}{2}\int d^3x\,\{(\mathfrak{T}\mathbf{E})^2-(\nabla\times\mathbf{A})^2\}-\int d^3x\frac{\partial\phi}{\partial t}\nabla\cdot\mathbf{A}, \tag{12}$$

[6] This section represents a brief summary of the results of Kramers in reference 1.

[7] See reference 1.

where $\mathfrak{T}\mathbf{E}=-\partial\mathbf{A}/\partial t$. We have not included any terms in Eq. (12) which would permit the derivation of the third of Eqs. (1), because in the electric-dipole approximation this equation is not coupled to our other equations. The last term in Eq. (12) takes account of the subsidiary condition $\nabla\cdot\mathbf{A}=0$; $\partial\phi/\partial t$ in the integrand corresponds to a Lagrange multiplier.

If we now drop the last term in Eq. (12), and insert Eqs. (4), (6), and (10) into the remainder, we obtain the structure independent Lagrangian

$$L=\tfrac{1}{2}m\mathbf{V}^2-U(\mathbf{R})+\frac{1}{2}\int d^3x\,(\mathbf{E}_1^2-\mathbf{H}_1^2), \tag{13}$$

where

$$\mathbf{E}_1\equiv\mathfrak{T}\mathbf{E}=-(\partial/\partial t)(\mathbf{A}_1-e\mathfrak{T}\dot{\mathbf{R}}/r), \tag{14}$$

and

$$\mathbf{H}_1=\nabla\times\mathbf{A}_1. \tag{15}$$

We now consider the independent variables to be $\mathbf{A}_1$, $\partial\mathbf{A}_1/\partial t$, $\mathbf{R}$, $\dot{\mathbf{R}}$, and $\ddot{\mathbf{R}}$ [$\ddot{\mathbf{R}}$ appears through $\mathbf{E}_1$ in Eq. (14)]. If we extend the usual variational principle to include the dependence of L on $\ddot{\mathbf{R}}$, the variation of L leads to the following Euler-Lagrange equations:

$$\frac{\delta L}{\delta\mathbf{A}_1}-\frac{\partial}{\partial t}\left(\frac{\delta L}{\delta(\partial\mathbf{A}_1/\partial t)}\right)=0 \tag{16}$$

and

$$\frac{\partial L}{\partial\mathbf{R}}-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\mathbf{R}}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial\ddot{\mathbf{R}}}\right)=0, \tag{17}$$

where δ denotes functional differentiation. Equation (16) corresponds to the first of Eqs. (9), while Eq. (17) corresponds to Eq. (11).

Note that we did not use any subsidiary conditions in deriving Eqs. (16) and (17). In Eq. (12), the last term serves to introduce the longitudinal part of the current density into the field equation for $\mathbf{A}$. This part of the current density has now been absorbed by $\mathbf{A}_0$ through its defining equation, (5).

The same Lagrangian (13) will appear again in Sec. V, when we show that the canonical variables derived from Eq. (12) (the structure dependent theory) are connected with those of Eq. (13) (the structure independent theory) by a canonical transformation.

The appearance of $\ddot{\mathbf{R}}$ in Eq. (13) forces us to use the method of Ostrogradsky for introducing the Hamiltonian formalism.[8] In the Ostrogradsky method we treat $\mathbf{R}$, $\mathbf{V}$, and $\mathbf{A}_1$ as independent coordinate variables, and introduce the momenta canonical to these coordinates, $\mathbf{P}_R$, $\mathbf{P}_V$, and $\mathbf{P}_{A_1}$ through the relations

$$\mathbf{P}_R=\frac{\partial L}{\partial\dot{\mathbf{R}}}-\frac{d}{dt}\left(\frac{\partial L}{\partial(d\mathbf{V}/dt)}\right), \tag{18}$$

$$\mathbf{P}_V=\frac{\partial L}{\partial(d\mathbf{V}/dt)}, \tag{19}$$

$$\mathbf{P}_{A_1}=\delta L/\delta(\partial\mathbf{A}_1/\partial t). \tag{20}$$

The Hamiltonian is defined in the usual way,

$$\mathfrak{IC}=\mathbf{P}_R\cdot\dot{\mathbf{R}}+\mathbf{P}_V\cdot d\mathbf{V}/dt+\int d^3x\,\mathbf{P}_{A_1}\cdot\partial\mathbf{A}_1/\partial t-L. \tag{21}$$

On evaluating Eqs. (18–20), we find

$$\mathbf{P}_R=m\mathbf{V}+e\langle\mathbf{A}_1\rangle, \tag{22}$$

$$\mathbf{P}_V=-e\langle\mathbf{Z}_1\rangle=e\int d^3x\,\mathbf{P}_{A_1}/r, \tag{23}$$

$$\mathbf{P}_{A_1}=-\mathbf{E}_1, \tag{24}$$

where $\mathbf{Z}_1$ is a divergenceless Hertz vector satisfying the relations

$$\nabla^2\mathbf{Z}_1=-\mathbf{E}_1,\quad \partial\mathbf{Z}_1/\partial t=\mathbf{A}_1. \tag{25}$$

We immediately note that Eqs. (22) and (23) are independent equations involving only the canonical variables and not the velocities. Such relations are called constraints. Constraints arise in any theory where the Lagrangian depends on second- and higher-order time derivatives of the coordinate variables. For example, we see from Eqs. (18–20) that, in going over to the Hamiltonian formalism, the elimination of $\dot{\mathbf{R}}$ and $\partial\mathbf{A}_1/\partial t$ must yield at least one constraint. The existence of additional constraints will then depend on the particular functional form for L. In our case L is structured so that there is one additional constraint, so that we are left with two independent vector equations restricting the canonical variables.

We denote the constraints by $\mathbf{C}_1$ and $\mathbf{C}_2$, where

$$\mathbf{C}_1=\mathbf{P}_R-m\mathbf{V}-e\langle\mathbf{A}_1\rangle=0 \tag{26}$$

and

$$\mathbf{C}_2=\mathbf{P}_V+e\langle\mathbf{Z}_1\rangle=0. \tag{27}$$

An appropriate linear combination of these constraints must be added to the Hamiltonian (21) to insure that Hamilton's equation are identical with the Euler-Lagrange equations. The correct Hamiltonian is

$$\mathfrak{IC}=\tfrac{1}{2}m\mathbf{V}^2+U+e\mathbf{V}\cdot\mathbf{A}_1+\frac{1}{2}\int d^3x\,(\mathbf{E}_1^2+\mathbf{H}_1^2)+\mathbf{C}_1\cdot\mathbf{V}+\mathbf{C}_2\cdot d\mathbf{V}/dt, \tag{28}$$

or alternatively,

$$\mathfrak{IC}=\frac{1}{2m}(\mathbf{P}_R^2-e^2\langle\mathbf{A}_1\rangle^2)+U+\frac{1}{2}\int d^3x\,(\mathbf{E}_1^2+\mathbf{H}_1^2)-\frac{e}{m}\mathbf{C}_1\cdot\langle\mathbf{A}_1\rangle+\mathbf{C}_2\cdot d\mathbf{V}/dt. \tag{29}$$

[8] The Ostrogradsky method is discussed in E. T. Whittaker, *Analytical Dynamics* (Dover Publications, New York, 1944), p. 265.

We see from Eqs. (28) and (29) that the Hamiltonian depends only on the canonical variables and is independent of accelerations, when the constraints are satisfied.

If the canonical theory is to be consistent, the constraints must continue to vanish in the course of time. That they are indeed constants of the motion is readily checked, since they have vanishing commutators with the Hamiltonian,

$$[\mathbf{C}_1,\mathcal{H}]=[\mathbf{C}_2,\mathcal{H}]=0. \tag{30}$$

In calculating commutators such as those in (30), we assume that the particle variables satisfy the standard-commutation relations. We also require that the divergence-free field variables $\mathbf{A}_1$ and $\mathbf{P}_{A_1}$, satisfy the commutation relations for transverse fields,

$$[\mathbf{A}_1(\mathbf{x},t),\mathbf{P}_{A_1}(\mathbf{x}',t)]=i\left[\delta(\mathbf{x}-\mathbf{x}')-\frac{1}{4\pi}\nabla\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]. \tag{31}$$

On the other hand, the commutator

$$[\mathbf{C}_1,\mathbf{C}_2]=-im_0 \tag{32}$$

does not vanish, and so our C's are, in Dirac's terminology, second-class constraints. A first-class constraint is one whose commutator with all other constraints vanishes.

The appearance of constraints in a classical theory means that the possible dynamical motions are confined to a hypersurface in the full phase space. The dimensions of this hypersurface are determined by the number of constraints. If the constraints are second class, then motions may be generated off the hypersurface simply by permitting one of the constraints to act as the generator of a canonical transformation. A similar problem arises in the quantum theory, where the constraints restrict the dimensions of the Hilbert space of state vectors. If the constraints are second class, a unitary transformation generated by one of the constraints will introduce into the Hilbert space state vectors that are not permitted in the theory. If we wish to insure that the dynamical motions in both classical and quantum mechanics are restricted to their respective hypersurfaces in phase space and in Hilbert space, then it is necessary that all constraints be made first class.

Dirac[3] was the first to find a solution to this problem by requiring that all commutator relations appearing in the theory be replaced by new relations called Dirac commutators. These new commutators have the desired property that all dynamical variables automatically commute with the second class constraints.

In the presence of only second class constraints the same result can be achieved by adding appropriate linear combinations of constraints to each dynamical variable.[9,10]

Let A be a dynamical variable and G^{ik} the commutator

$$G^{ik}=[C^i,C^k]. \tag{33}$$

If the constraints are all second class, then the inverse G_{ik}^{-1} exists,

$$G^{ik}G_{kl}^{-1}=G^{ki}G_{lk}^{-1}=\delta_l{}^i, \tag{34}$$

and the new variables A^*,

$$A^*=A+G_{lk}^{-1}[C^l,A]C^k, \tag{35}$$

all commute with the constraints,

$$[A^*,C^i]=0. \tag{36}$$

If we now replace all our canonical variables by their starred counterparts, our constraints become first class. These modified variables satisfy the same equations of motion as our original variables, and they equal these original variables when the constraints are satisfied.

In our theory the modified variables are

$$\mathbf{R}^*=\mathbf{R}-(1/m_0)\mathbf{C}_2,\quad \mathbf{P}_R{}^*=\mathbf{P}_R; \tag{37}$$

$$\mathbf{V}^*=\mathbf{V}+(1/m_0)\mathbf{C}_1,\quad \mathbf{P}_V{}^*=\mathbf{P}_V-(m/m_0)\mathbf{C}_2; \tag{38}$$

$$\begin{aligned}\mathbf{A}_1{}^*&=\mathbf{A}_1-\mathfrak{g}(\mathbf{x},t)\frac{1}{m_0}\cdot\mathbf{C}_2,\\ \mathbf{P}_{A_1}{}^*&=\mathbf{P}_{A_1}-\frac{e}{m_0}\left(\delta(\mathbf{x})+\frac{1}{4\pi}\nabla\nabla\frac{1}{r}\right)\cdot\mathbf{C}_1;\end{aligned} \tag{39}$$

with the dyadic $\mathfrak{g}(\mathbf{x},t)=(e^2/4\pi)\mathfrak{T}\mathfrak{F}/r$, where $\mathfrak{F}$ is the unit dyadic. The dots in Eqs. (39) imply contraction of the preceding dyadic with the vector which follows.

The equal-time commutators between the starred variables are

$$\begin{aligned}&[\mathbf{R}^*,\mathbf{P}_R{}^*]=i, && [\mathbf{R}^*,\mathbf{V}^*]=i/m_0,\\ &[\mathbf{R}^*,\mathbf{A}_1{}^*]=-(i/m_0)\mathfrak{g}, && [\mathbf{A}_1{}^*,\mathbf{P}_V{}^*]=(im/m_0)\mathfrak{g},\\ &[\mathbf{A}_1{}^*(\mathbf{x},t),\mathbf{P}_{A_1}{}^*(\mathbf{x}',t)]=i\left(\delta(\mathbf{x}-\mathbf{x}')-\frac{1}{4\pi}\nabla\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\\ &\qquad\qquad+\frac{ie}{m_0}\mathfrak{g}(\mathbf{x})\cdot\left(\delta(\mathbf{x}')+\frac{1}{4\pi}\nabla'\nabla'\frac{1}{r'}\right),\\ &[\mathbf{V}^*,\mathbf{P}_{A_1}{}^*]=-\frac{ei}{m_0}\left(\delta(\mathbf{x})+\frac{1}{4\pi}\nabla\nabla\frac{1}{r}\right), && [\mathbf{V}^*,\mathbf{P}_V{}^*]=-i\frac{\delta m}{m_0},\end{aligned} \tag{40}$$

and all remaining commutators vanish. From (40), we see that the modified variables are not canonical, although our original set was. The requirement that the dynamical motion be confined to the constraint hyper-

[9] P. G. Bergmann and I. Goldberg, Phys. Rev. **98**, 531 (1955).
[10] P. G. Bergmann and A. Komar, Phys. Rev. Letters, **4**, 432 (1960).

surface will in general destroy the canonical character of our phase-space variables. In performing calculations on the constraint hypersurface the modified commutators (40) must be used, and all specific reference to the constraints is thereby eliminated from the theory.

One notes that the modified commutators, (40), reintroduce into the theory the unrenormalized mass m_0. However, the dynamical equations involve only the renormalized mass.

IV. KRAMERS' CANONICAL SCHEME

In terms of the original unstarred variables, the Hamiltonian theory of the preceding section is invariant under canonical transformations. In the course of such a transformation the constraints will also be transformed, so that appropriate combinations of constraints must be added to the new Hamiltonian to insure that the new equations of motion are equivalent to the old.

In this section we shall carry out a canonical transformation which leads from our canonical variables of Sec. III to another set originally introduced by Kramers.[1] In this way we justify the consistency of Kramers' canonical scheme, for Kramers in his analysis did not deal explicitly with the problem of constraints. Since Kramers' canonical variables are important for other considerations, it is of some value to carry out the transformation in detail.

Let the canonical pairs in Sec. II be denoted by q_i and p_i, and consider the transformation

$$(q_i,p_i)\rightarrow(Q_i,\gamma_i),$$

where $i=1, 2, 3$. As the generating function for the transformation we choose

$$G(p_i,Q_i)=-\mathbf{Q}_1\cdot\mathbf{P}_R-\frac{1}{m}(\mathbf{Q}_2-e\langle\mathbf{Q}_3\rangle)\cdot\mathbf{P}_V+\mathbf{Q}_1\cdot\mathbf{Q}_2-\int d^3x\,\mathbf{Q}_3\cdot\mathbf{P}_{A_1}. \quad (41)$$

On solving the transformation equations

$$q_i=-\partial G/\partial p_i,\quad \gamma_i=-\partial G/\partial Q_i, \quad (42)$$

we find

$$\mathbf{Q}_1=\mathbf{R},\quad \gamma_R=0; \quad (43)$$

$$\mathbf{Q}_2=\mathbf{P}_R,\quad \gamma_P=-\mathbf{R}-(e/m)\langle\mathbf{Z}_1\rangle; \quad (44)$$

$$\mathbf{Q}_3=\mathbf{A}_1,\quad \gamma_{A_1}=-\mathbf{E}_1+(e^2/m)\delta(\mathbf{x})\langle\mathbf{Z}_1\rangle. \quad (45)$$

The new constraint equations are

$$\mathbf{C}_1{}^K=\gamma_R=0, \quad (46)$$

$$\mathbf{C}_2{}^K=m_0(\gamma_P+\mathbf{R})-e\langle\mathbf{Z}_2\rangle=0 \quad (47)$$

where

$$\langle\mathbf{Z}_2\rangle=\frac{1}{4\pi}\int d^3x\,\mathfrak{T}\gamma_{A_1}/r,$$

and the superscript K refers to Kramers.

The new Lagrangian is

$$M=-\textstyle\sum_i q_i\dot{p}_i-H-dG(p_i,Q_i)/dt,$$

which reduces to

$$M=-\frac{1}{2m}(\mathbf{P}^2-e^2\langle\mathbf{A}_1\rangle^2)-U(\mathbf{R})-\mathbf{R}\cdot\dot{\mathbf{P}}+\frac{1}{2}\int d^3x\,(\mathbf{E}_1{}^2-\mathbf{H}_1{}^2). \quad (48)$$

If one uses Eq. (48), the relation $\ddot{\mathbf{R}}=m^{-1}(d/dt)\times(\mathbf{P}-e\langle\mathbf{A}_1\rangle)$ must be inserted into E_1 and one need not then use the Ostrogradsky method to identify canonical variables. The Hamiltonian is

$$\mathfrak{H}=\frac{1}{2m}(\mathbf{P}^2-e^2\langle\mathbf{A}_1\rangle^2)+u(\mathbf{R})+\frac{1}{2}\int d^3x\,(\mathbf{E}_1{}^2+\mathbf{H}_1{}^2)+\mathbf{C}_1{}^K\cdot\mathbf{V}+\frac{1}{m_0}\mathbf{C}_2{}^K\cdot d\mathbf{P}/dt. \quad (49)$$

As expected, Eq. (49) differs from Eq. (29) only in the constraint terms.

One easily verifies that the constraints $\mathbf{C}_1{}^K$ and $\mathbf{C}_2{}^K$ are second class. In fact, we have

$$[\mathbf{C}_1{}^K,\mathbf{C}_2{}^K]=-im_0. \quad (50)$$

The modified variables in the Dirac formalism are

$$\mathbf{R}^*=\mathbf{R}-(1/m_0)\mathbf{C}_2{}^K,\qquad \gamma_R{}^*=\gamma_R-\mathbf{C}_1{}^K=0; \quad (51)$$

$$\mathbf{P}^*=\mathbf{P}+\mathbf{C}_1{}^K,\qquad \gamma_P{}^*=\gamma_P; \quad (52)$$

$$\mathbf{A}_1{}^*=\mathbf{A}_1-(1/m_0)\mathfrak{g}\cdot\mathbf{C}_1{}^K,\quad \gamma_{A_1}{}^*=\gamma_{A_1}. \quad (53)$$

The equal-time commutators between the modified variables are:

$$\begin{aligned}&[\mathbf{R}^*,\gamma_R{}^*]=0,\qquad [\mathbf{R}^*,\mathbf{P}^*]=i,\\ &[\mathbf{R}^*,\mathbf{A}_1{}^*]=-(i/m_0)\mathfrak{g},\quad [\mathbf{P}^*,\gamma_P{}^*]=i,\\ &[\mathbf{A}_1{}^*,\gamma_{A_1}{}^*]=i\left[\delta(\mathbf{x}-\mathbf{x}')-\frac{1}{4\pi}\nabla\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right].\end{aligned} \quad (54)$$

All the other equal-time commutators between modified-canonical variables which do not appear in Eqs. (54) are zero.

There are some distinct advantages in using Kramers' canonical variables in place of those we introduced in Sec. II. Since he has managed to transform one of the constraints into a dynamical variable, he has fewer equations to deal with. In addition, his variables lend themselves most naturally to the usual perturbation theory expansions in powers of e.

However, Kramers' original derivation of Eq. (48) is somewhat ambiguous in the identification of canonical variables and constraints. Kramers starts with the

structure-dependent Lagrangian (12), and forms

$$M=L_0-(d/dt)(\mathbf{R}\cdot\mathbf{P}).$$

He assumes that L_0 is a function only of $\mathbf{R}$, $\dot{\mathbf{R}}$, $\mathbf{A}$, and $(\partial\mathbf{A}_1/\partial t)$ and in addition makes use of the relation $\mathbf{P}=m_0\mathbf{V}+e\langle\mathbf{A}\rangle$. But it is not at all clear that there is a canonical transformation in going from L_0 to M with Kramers' identification of independent variables. In the absence of such a demonstration, the role of the relation $\mathbf{P}=m\mathbf{V}+e\langle\mathbf{A}_1\rangle$ is uncertain.

Furthermore, instead of Eq. (48), Kramers could have used the relation $\mathbf{P}=m\mathbf{V}+e\langle\mathbf{A}_1\rangle$ to write the alternative Lagrangian,

$$M=-\tfrac{1}{2}m\mathbf{V}^2-u(\mathbf{R})-e\mathbf{V}\cdot\langle\mathbf{A}_1\rangle-\mathbf{R}\cdot(md\mathbf{V}/dt+ed\langle\mathbf{A}_1\rangle/dt)+\frac{1}{2}\int d^3x\,[\mathbf{E}_1^2-\mathbf{H}_1^2]. \quad (55)$$

In Eq. (55), $M=M(\mathbf{R},\mathbf{V},d\mathbf{V}/dt,\mathbf{A}_1,\partial\mathbf{A}_1/\partial t)$, so that the Ostrogradsky method is called for. The canonical pairs one then obtains from Eq. (55) are

$$\mathbf{R},\quad \gamma_R'=0; \quad (56)$$

$$\mathbf{V},\quad \gamma_V'=-m\mathbf{R}-e\langle\mathbf{Z}_1\rangle; \quad (57)$$

$$\mathbf{A}_1,\quad \gamma_{A_1}'=-E_1-e\delta(\mathbf{x})\mathbf{R}. \quad (58)$$

The constraints are

$$\mathbf{C}_1'^K=\gamma_R'=0, \quad (59)$$

$$\mathbf{C}_2'^K=m_0\mathbf{R}+\gamma_V'-e\langle\mathbf{Z}_2'\rangle=0, \quad (60)$$

where

$$\langle\mathbf{Z}_2'\rangle=\frac{1}{4\pi}\int d^3x\,\mathfrak{T}\frac{\gamma_{A_1}'}{r}. \quad (61)$$

The Hamiltonian is

$$\mathfrak{IC}=\tfrac{1}{2}m\mathbf{V}^2+u(\mathbf{R})+e\mathbf{V}\cdot\langle\mathbf{A}_1\rangle+\frac{1}{2}\int d^3x\,[\mathbf{E}_1^2+\mathbf{H}_1^2]+\mathbf{C}_1'^K\cdot\mathbf{V}+\mathbf{C}_2'^K\cdot d\mathbf{V}/dt, \quad (62)$$

and the equal-time commutator between the constraints is

$$[\mathbf{C}_1'^K,\mathbf{C}_2'^K]=-im_0. \quad (63)$$

Thus the modified variables are

$$\mathbf{R}^*=\mathbf{R}-(1/m_0)\mathbf{C}_2'^K,\qquad \gamma_R'^*=\gamma_R'-\mathbf{C}_1'^K=0; \quad (64)$$

$$\mathbf{V}^*=\mathbf{V}+(1/m_0)\mathbf{C}_1'^K,\qquad \gamma_V'^*=\gamma_V'; \quad (65)$$

$$\mathbf{A}_1^*=\mathbf{A}_1-(1/m_0)\mathfrak{g}\cdot\mathbf{C}_1'^K,\qquad \gamma_{A_1}'^*=\gamma_{A_1}'. \quad (66)$$

The modified commutators are

$$[\mathbf{R}^*,\gamma_R'^*]=0,\qquad [\mathbf{R}^*,\mathbf{V}^*]=i/m_0,$$
$$[\mathbf{R}^*,\mathbf{A}_1^*]=-(i/m_0)\mathfrak{g},\qquad [\mathbf{V}^*,\gamma_V'^*]=i,$$
$$[\mathbf{A}_1^*,\gamma_{A_1}'^*]=i\left[\delta(\mathbf{x}-\mathbf{x}')-\frac{1}{4\pi}\nabla\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]. \quad (67)$$

All the other equal-time commutators between modified-canonical variables vanish.

We see that the ambiguities in Kramers' original theory disappear when one takes explicit account of the presence of constraints.

V. CANONICAL EQUIVALENCE OF STRUCTURE-DEPENDENT AND STRUCTURE-INDEPENDENT HAMILTONIAN THEORIES

The equivalence, via canonical transformations, of various structure-independent Hamiltonian schemes has been exhibited in earlier sections in this paper.

We should now like to show that all of these structure-independent theories may be generated by canonical transformations from the structure-dependent theory. In order to find these transformations, we must first generalize the original structure-dependent Hamiltonian theory to include the velocity $\mathbf{V}$ and its conjugate momentum $\mathbf{P}_V$ as canonical variables.

We formally apply the Ostrogradsky procedure to the Lagrangian (12), noting of course that L_0 does not depend on $d\mathbf{V}/dt$. We find the following canonical pairs

$$\mathbf{R},\quad \mathbf{P}_R=m_0\mathbf{V}+e\langle\mathbf{A}\rangle; \quad (68)$$

$$\mathbf{V},\quad \mathbf{P}_V'=0; \quad (69)$$

$$\mathbf{A},\quad \mathbf{P}_A=\partial\mathbf{A}/\partial t. \quad (70)$$

The constraints are

$$\mathbf{C}_1^0=\mathbf{P}_R-m_0\mathbf{V}-e\langle\mathbf{A}\rangle=0, \quad (71)$$

$$\mathbf{C}_2^0=\mathbf{P}_V'=0. \quad (72)$$

The Hamiltonian is

$$\mathfrak{IC}=\tfrac{1}{2}m_0\mathbf{V}^2+U(\mathbf{R})+\frac{1}{2}\int d^3x\,[\mathbf{P}_A^2+(\nabla\times\mathbf{A})^2]+\mathbf{C}_1^0\cdot\mathbf{V}+\mathbf{C}_2^0\cdot d\mathbf{V}/dt, \quad (73)$$

or alternatively

$$\mathfrak{IC}=\frac{1}{2m_0}(\mathbf{P}-e\langle\mathbf{A}\rangle)^2+U(\mathbf{R})+\frac{1}{2}\int d^3x\,[\mathbf{P}_A^2+(\nabla\times\mathbf{A})^2]+\mathbf{C}_2^0\cdot d\mathbf{V}/dt. \quad (74)$$

The equal-time commutator between the constraints is

$$[\mathbf{C}_1^0,\mathbf{C}_2^0]=-im_0. \quad (75)$$

A generating function for a canonical transformation from the structure-independent variables of Sec. III to the present structure-dependent variables is

$$G(p_i,Q_i)=-\mathbf{P}_R\cdot\mathbf{Q}_1-\mathbf{P}_V\cdot\mathbf{Q}_2-\int d^3x\,\mathbf{P}_{A_1}\cdot\left[\mathbf{Q}_3(\mathbf{x})-\frac{e}{4\pi}\mathfrak{T}\frac{\mathbf{Q}_2}{r}\right]. \quad (76)$$

One easily determines the transformation equations

$$\mathbf{Q}_1=\mathbf{R}, \qquad \mathbf{P}_1=\mathbf{P}_R; \tag{77}$$

$$\mathbf{Q}_2=\mathbf{V}, \qquad \mathbf{P}_2=\mathbf{P}_V'=\mathbf{P}_V+e\langle\mathbf{Z}_1\rangle=0; \tag{78}$$

$$\mathbf{Q}_3=\mathbf{A}=\mathbf{A}_1+\mathbf{A}_0, \quad \mathbf{P}_3=\mathbf{P}_A=\mathbf{P}_{A_1}. \tag{79}$$

A generating function for a canonical transformation from the present structure-dependent variables to Kramers' structure-independent variables is

$$G(p_i,Q_i)=-\mathbf{P}_R\cdot\mathbf{Q}_1-\mathbf{P}_V'\cdot(\mathbf{Q}_2-e\langle\mathbf{Q}_3\rangle)/m+\mathbf{Q}_1\cdot\mathbf{Q}_2 - \int d^3x\,\mathbf{P}_A\cdot\left[\mathbf{Q}_3(\mathbf{x})+\frac{e}{4\pi m}\mathfrak{T}\left(\frac{\mathbf{Q}_2-e\langle\mathbf{Q}_3\rangle}{r}\right)\right]. \tag{80}$$

The transformation equations are

$$\mathbf{Q}_1=\mathbf{R}, \qquad \gamma_R=\mathbf{P}-\mathbf{Q}_2=0; \tag{81}$$

$$\mathbf{Q}_2=\mathbf{P}, \qquad \gamma_P=-\mathbf{R}-(e/m)\langle\mathbf{Z}_1\rangle; \tag{82}$$

$$\mathbf{Q}_3=\mathbf{A}_1=\mathbf{A}-\mathbf{A}_0, \quad \gamma_{A_1}=-\mathbf{E}_1+(e^2/m)\delta(\mathbf{x})\langle\mathbf{Z}_1\rangle. \tag{83}$$

VI. ALTERNATIVE DEFINITIONS OF EXTERNAL AND PROPER FIELDS

The decomposition of the vector potential $\mathbf{A}$ into an external and a proper field is unique only with respect to the structure-independent equations of motion for the particle. That is to say, it is the external field $\mathbf{A}_1$ as defined by Kramers which enters the equations of motion. However, the above-mentioned decomposition is not unique with respect to the canonical formalism we have presented. One can base canonical schemes on decompositions of $\mathbf{A}$ other than Kramers', as long as one imposes the requirement that such decompositions lead to the same structure-independent particle equations of motion. For example, van Kampen[11] and Steinwedel[5] have used decompositions of $\mathbf{A}$ which are different from Kramers,[1] and which satisfy the above criterion. However, neither van Kampen nor Steinwedel have taken the constraints into account.

As an illustration of an alternative decomposition, we will complete Steinwedel's canonical transformation for a free particle by extending his formalism to account for the constraints. For a free particle, the Steinwedel external field $\mathbf{A}_1'$ is defined by

$$\mathbf{A}_1'=\mathbf{A}-\mathbf{A}_0', \tag{84}$$

where

$$\mathbf{A}_0'=(e/4\pi m)\mathfrak{T}\mathbf{P}/r. \tag{85}$$

The relationship between $\mathbf{A}_1$ and $\mathbf{A}_1'$ is determined from $\mathbf{A}_1+\mathbf{A}_0=\mathbf{A}_1'+\mathbf{A}_0'$. The generating function of a canonical transformation from the structure-dependent canonical frame of Sec. V to the extended (to include constraints) Steinwedel frame may be written

$$G(p_i,Q_i)=-\mathbf{P}_R\cdot\mathbf{Q}_1-\mathbf{P}_V'\cdot[m^{-\frac{1}{2}}\mathbf{Q}_2-(e/m_0)\langle\mathbf{Q}_3\rangle] +m^{\frac{1}{2}}\mathbf{Q}_1\cdot\mathbf{Q}_2-\int d^3x\,\mathbf{P}_A \cdot[\mathbf{Q}_3(\mathbf{x})+m^{-\frac{1}{2}}(e/4\pi)\mathfrak{T}\mathbf{Q}_2/r]. \tag{86}$$

The transformation equations are

$$\mathbf{Q}_1=\mathbf{R}, \qquad \gamma_R=\mathbf{P}_R-m^{\frac{1}{2}}\mathbf{Q}_2=0; \tag{87}$$

$$\mathbf{Q}_2=\mathbf{P}'=\mathbf{P}_R/m^{\frac{1}{2}}, \quad \gamma_{P'}=-m^{\frac{1}{2}}\mathbf{R}-(e/m^{\frac{1}{2}})\langle\mathbf{Z}_1\rangle; \tag{88}$$

$$\mathbf{Q}_3=\mathbf{A}_1', \qquad \gamma_{A_1'}=\mathbf{P}_A. \tag{89}$$

Equations (88–89) above are equivalent to Steinwedel's Eqs. (16). Equations (87) are obtained in addition to the others because of the constraints. The constraints in the Steinwedel frame are

$$\mathbf{C}_1{}^S=\gamma_R=0, \tag{90}$$

$$\mathbf{C}_2{}^S=\gamma_{P'}+m^{\frac{1}{2}}\mathbf{R}+(e/m^{\frac{1}{2}})\langle\mathbf{Z}_1\rangle=0, \tag{91}$$

where the superscript S refers to Steinwedel.

The equal-time commutator between the constraints is

$$[\mathbf{C}_1{}^S,\mathbf{C}_2{}^S]=-im^{\frac{1}{2}}. \tag{92}$$

The Hamiltonian is

$$\mathfrak{H}=\tfrac{1}{2}\mathbf{P}'^2+\frac{e^2}{2m_0}\langle\mathbf{A}_1'\rangle^2+\frac{1}{2}\int d^3x\,[\gamma_{A_1'}{}^2+(\nabla\times\mathbf{A}_1')^2] +\mathbf{C}_1{}^S\cdot\mathbf{V}+\mathbf{C}_2{}^S\cdot d\mathbf{P}'/dt. \tag{93}$$

VII. CONCLUSION

In this paper we have developed a general Hamiltonian theory in electrodynamics for a set of structure-independent equations originally introduced by Kramers. The problem still remains of discussing the present formalism in relation to solutions which arise when one specifies the potential function $U(R)$. The latter problem has been partially discussed in a previous publication by one of the authors.[2] It will be more fully discussed in a future publication. In addition, the theory must be extended to the relativistic domain, and we must examine the implications of the present work for a fully quantized field theory.

ACKNOWLEDGMENTS

We wish to thank the participants in the Theoretical Physics Seminar at Syracuse University for their helpful comments on the subject matter of this paper. We are especially indebted to Professor Peter G. Bergmann for discussions on the proper formulation of canonical theories in the presence of constraints.

[11] N. G. van Kampen, Kgl. Danske Videnskab Selskab, Mat.-fys. Medd. **26**, No. 15 (1951).