THE
PHYSICAL REVIEW
A journal of experimental and theoretical physics established by E. L. Nichols in 1893
SECOND SERIES, VOL. 126, NO. 5 JUNE 1, 1962

Cathode Spot

A. W. HULL
General Electric Research Laboratory, Schenectady, New York
(Received January 29, 1962)

The properties and motions of the mercury arc cathode spot are calculated on the basic assumptions: (1) that the current density is 10^5 amp/cm^2, with positive ion component 1/20; (2) that the spot is made up of a number of small "emitting areas," of current density 10^7 amp/cm^2 moving randomly within the spot. It is shown that these assumptions lead to explanations of the mechanism of emission ("field" emission); retrograde motion, both "turning points" and velocity; pressure on the spot; and evaporation from the spot; in good agreement with experimental values.

1. INTRODUCTION

THERE have been many recent attempts to explain the mercury cathode spot, most of them only qualitative. The nearest to a quantitative theory is that of Ecker,[1] which reproduces with great accuracy the experimental curves of retrograde velocity and turning point as functions of magnetic field, pressure, and current, with only a single arbitrary parameter for each curve.

In this paper we give a very simple picture of the cathode spot, which leads to predictions in good quantitative agreement with experimental values.

Only two assumptions are needed. The first is the current density, for which we choose 10^5 amp/cm^2, with a positive-ion component of 1/20. This is not a new assumption, since it agrees with the best experimental value, but it must be specified in order to make quantitative deductions possible.

The second assumption is radical, but not new, since it has often been suggested, though not quantitatively. It is: The cathode spot is made up of a number of small "emitting areas," of random sizes, moving randomly within the area of the spot, each having a current density of 10^7 amp/cm^2. These "areas" are held together within the area of the spot by their mutual magnetic attraction, but frequently wander outside the spot and disappear or form new spots, while new "areas" are formed frequently by splitting of the larger ones. It is to be expected that the shape of the spot will fluctuate rapidly and randomly; when round, its radius a is given by $\pi a^2 = i/I_0$, where i = arc current in amperes, and $I_0 = 10^5$ amp/cm^2 = current density.

2. MECHANISM OF EMISSION FROM "AREAS"

The emission is *field emission*, as may be seen from the following analysis:

The electric field at the surface of an "area," assumed plane, is easily calculated from Langmuir's space-charge equation for positive ions and the current density of positive ions, $I_+ = 10^7/20 = 5\times 10^5$ amp/cm^2. Langmuir's equation is

$$i_p = \frac{2.33\times 10^{-6} V^{\frac{3}{2}}}{(M/m)^{\frac{1}{2}} x^2}, \tag{1}$$

where V = voltage drop; M, m, = masses of ion and electron, respectively; and x = thickness of space-charge sheath. Taking $(M/m)^{\frac{1}{2}} = 605$, we have

$$i_p = 3.87\times 10^{-9} (V^{\frac{3}{2}}/x^2).$$

We assume $V = 10$ v.

Taking the square root of both sides, and differentiating, gives

$$dV/dx = \tfrac{4}{3} V^{\frac{1}{4}} (i_p/3.87\times 10^{-9})^{\frac{1}{2}} = 2.70\times 10^7 \text{ v/cm},$$
$$x = 10/2.7\times 10^7 = 3.7\times 10^{-7} \text{ cm}.$$

This field is just sufficient, with a roughness factor of 2.5, to satisfy the Fowler-Nordheim equation,[2] for the required emission of 10^7 amp/cm^2. The roughness may

[1] G. Ecker and K. G. Müller, Z. Physik **151**, 578 (1958); J. Appl. Phys. **29**, 1606 (1958); G. Ecker, Ergeb. exak. Naturw. **33**, 1–104, (1961).

[2] R. H. Good and E. W. Müller, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. 21, p. 188.

be accounted for by assuming that the electron emission takes place from the edge of the deep channels closely adjacent to the "areas," as suggested in Sec. 9.

However, it is unnecessary to "account" for the roughness, since the thickness of the space-charge layer is only a little over 7 times the atomic diameter, and "molecular roughness" must be an important factor.

3. TEMPERATURE OF SPOT (HEAT BALANCE)

The temperature of the spot as given by the heat balance is easily calculated. If Q w/cm² are delivered to a spot of radius a on a plane surface of thermal conductivity K w/cm-°C and diffusivity k cm²/sec for t sec, the spot will rise in temperature, due to *conduction cooling*, by an amount[3]

$$\Delta T=\frac{2Qa}{K}\left(\frac{kt}{a^2}\right)^{\frac{1}{2}}\left\{\frac{1}{\sqrt{\pi}}-i\ \mathrm{erfc}\frac{1}{2(kt/a^2)^{\frac{1}{2}}}\right\}\ {}^\circ\mathrm{C}$$

$$=\frac{Qa}{K}\ \text{for}\quad t=\infty. \tag{2}$$

Hence the power input necessary to maintain this temperature rise, for conduction cooling is

$$Q(\text{conduction})=K\Delta T/a.$$

For evaporation cooling, we have

$$Q(\text{evaporation})=mL.$$

For combined conduction and evaporation cooling, the power input is

$$Q_{(\text{input})}=Q_{\text{cond}}+Q_{\text{evap}}=K\Delta T/a+mL, \tag{3}$$

where K = thermal conductivity of mercury = 0.064 w/cm-°C; $\Delta T=T-300$; T = temp. of spot in °K; a = radius of spot $=(i/\pi I_0)^{\frac{1}{2}}=5.65\times10^{-3}$ cm for $i=10$ amp; m = mass evaporated in g/cm²-sec $=5.83\times10^{-2}\ p_{\text{mm}}(M/T)^{\frac{1}{2}}$; p_{mm} = pressure of mercury vapor in mm of mercury; L = latent heat of vaporization of mercury = 285 joules/g; and M = molecular weight of mercury = 200.

The power input is

$$Q_{(\text{in})}=(V_c+V_i-\varphi_0)\times(I_0/20)\ \text{w/cm}^2,$$

V_c = "cathode fall" = 10 v, V_i = ionizing potential = 10.4 v, and φ_0 = work function of mercury = 4.5 v. With these values, Eq. (3) yields $p=8.7\times10^3$ mm Hg (11.5 atm), $T=805$°K, according to the heat balance: $Q_{(\text{in})}=7.9\times10^4=[0.68\times10^4\ (\text{cond})+7.2\times10^4\ (\text{evap})]$ w/cm². It is seen that at this temperature and pressure, conduction cooling is only 10% of evaporation cooling.

However, this pressure of 11.5 atm is never attained. The spot would be unstable at this pressure, as shown in the next section, and would have to move because the ions would be blown away by the mercury blast. Therefore the static heat balance does not apply to the spot on a free mercury surface, which must be in continual motion, in agreement with observation.[4] There is *dynamic* heat balance, the rate of heating being balanced by the *rate of rise* of temperature of the "areas."

4. MAXIMUM TEMPERATURE OF SPOT

The maximum temperature which the spot may reach before moving can be calculated, approximately, as follows:

The spot must move when the ions approaching the surface are blown away by the evaporating atoms; that is, when the velocity $\bar{c}$ of the evaporating atoms is greater than the velocity μE acquired by the ions in the field E v/cm between the edge of the space-charge region and the region of maximum ionization, which may be taken as $2.5\lambda_e$ distant from the space-charge edge. Here λ_e = mean free path of electrons; μ = mobility of ions $=0.815\sqrt{2}(e/M_a)(\lambda_e/\bar{c})$ (Langevin formula, M_a = mass of atom); $\bar{c}$ = average thermal velocity of atoms $=1.45\times10^4\ (T/M)^{\frac{1}{2}}$ (M = molecular weight); and $E=V_0/2.5\lambda_e$.

V_0 is the voltage maximum due to positive space charge, which was estimated by Compton as 0.25 v.[5] It may also be calculated from the average energy of the "residual electrons," as follows:

The energy of the primary electrons is $V_c-\varphi_0=10-4.5=5.5$ v, since they are drawn from the Fermi level, which is 4.5 v positive with respect to the surface. These primary electrons first excite the metastable level at 5.44 v;[6] then additional primaries ionize these metastables, which requires $10.39-5.44=4.95$ v. The average energy of the "residual electrons" is, therefore,[7]

$$V_0=[(5.5-5.44)+(5.5-4.95)]/2=0.305\ \text{v}.$$

The method of averaging is open to some question; $V_0=0.25$ v will be taken as a reasonable value.

With these values,

$$\frac{\mu E}{\bar{c}}=0.815\sqrt{2}\frac{e}{m}\frac{m}{M_a}\frac{\lambda}{\bar{c}}\times\frac{0.25\times10^8}{2.5\lambda\times\bar{c}}$$

$$=0.815\sqrt{2}\frac{e}{m}\frac{m}{M_a}\times\frac{0.25\times10^8 M}{2.5\times(1.45)^2\times10^8 T}$$

$$=523/T.$$

[3] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (Clarendon Press, Oxford, 1947), p. 264.

[4] There is ample evidence that the spot on a mercury surface never stands still. This is not true for a spot stabilized on a molybdenum surface, in which case the heat balance applies. (See Sec. 5.) Even when the spot on a mercury surface appears to stand still, as in the case of a restricted surface [E. Kobel, Phys. Rev. **36**, 1636 (1930)] or at "reversal" in a magnetic field, it is actually in rapid motion about an average position (cf. Sec. 9 and Appendix 2). This is additional proof that the heat balance does not apply, for with heat balance the spot would not have to move.

[5] K. T. Compton, Phys. Rev. **37**, 1089 (1931).

[6] I am indebted to Dr. Karl Kenty (private communication) for the information that this is the level preferentially excited.

[7] I. Langmuir, Z. Physik **46**, 271 (1927) has given evidence that two beams of slow electrons of different velocities average their temperature.

This must equal 1 for the maximum temperature, which gives

$$T_{\max}=523°\mathrm{K}=250°\mathrm{C}, \quad p=75 \text{ mm}.$$

This analysis applies also to the "areas," whose maximum temperature should likewise be 250°C. The question naturally arises whether the moving "areas" can maintain the temperature of the spot at this same value. It would obviously be true if their velocity were infinite. It turns out that their velocity is sufficient. As shown in Appendix 1, the rapid motion of the "areas," at temperature 523°K, maintains the whole surface of the spot within 2.5 degrees of this temperature. Although no great accuracy can be claimed for this calculation, it is believed to be sound, and it turns out that the calculated pressure is in good agreement with experiment (cf. Sec. 9).

5. TEMPERATURE OF SPOT ANCHORED BY MOLYBDENUM

For a spot *anchored* on Mo, for which $K=1.38$ w/cm-°K, with an average current $i=5$ amp, $a=4.0\times10^{-3}$ cm; Eq. (3) yields $T=523°$K, $p=75$ mm, according to the heat balance:

$$Q_{(\mathrm{in})}=7.9\times10^4=7.75\times10^4 \text{ (cond)} +7.6\times10^2 \text{ (evap) w/cm}^2.$$

Hence the anchored spot can be stable, as observed by Tonks.[8]

For larger or smaller currents the maximum temperature of the anchored spot will be larger or smaller, respectively, in proportion to the square root of the current (cf. Sec. 11).

6. RETROGRADE MOTION

A crucial test of any theory of the mercury arc is a satisfactory explanation of the "retrograde" motion of the cathode spot, namely, motion in a direction opposite to that in which electrons from the cathode are deflected.

In the picture given here, this retrograde motion is caused by the drift of slow "residual" electrons in the "correct" direction, parallel to the mercury surface, thus weakening the electric field drawing the +ions to the surface, so that the positive ion current on the "retrograde" side predominates. The energy of these residual electrons is estimated (see Sec. 4 above) to be 0.25 v, corresponding to a temperature $T_e=0.25\times11\,600=2900°$K. Their directions are random.

According to Tonks,[9] these electrons are constrained by a magnetic field, "almost as if by an inclined plane," to drift at an angle α to the direction which they would take under the existing potential and concentration gradients, such that

$$\tan\alpha=\gamma h', \tag{4}$$

where $h'=\omega_H\lambda_e/\bar{c}_e$, $\omega_H=eH/mc=1.78\times10^7H$, λ_e = electron mean free path $=5.04\times10^{-5}\ T(\text{atoms})/p_{\mathrm{mm}}$, $\bar{c}_e$ = mean electron velocity $=6.5\times10^5\sqrt{T}(\text{electrons})$, and γ = correction factor, whose value varies from 0.70 to 1.0.

Gallagher[10] has observed that the ambient[11] mercury vapor pressure at which an arc of 2.2 amp reverses its motion from "retrograde" to "correct," in a magnetic field of 600 gauss, is 10 mm Hg, corresponding to a mercury vapor temperature of 457°K. Below this pressure the motion is "retrograde," above it "correct."

For example, at a lower pressure of 1 mm ($T_a=400°$K), $T_e=2900°$K, $H=600$ gauss, Eq. (4) gives: $h'=6.2$, $\gamma h'=\tan\alpha=0.73\times6.2=4.5$, $\alpha=77.5°$. This condition is represented in Fig. 1(a). It is seen that the drift of residual electrons to the right, nearly parallel to the mercury surface, creates a negative space charge above the surface.[12] This negative space charge reverses the electric field above the surface, so that the flow of ions to the surface on the "retrograde" side predominates, causing the spot to move in the "retrograde" direction.

On the other hand, at a much higher pressure, such as 40 mm, the drift of residual electrons is nearly normal to the surface, and their space charge no longer produces a reverse field near the cathode. The ionization produced farther from the cathode, by the curved paths of the primary electrons, predominates, and the spot moves in the "correct" direction [Fig. 1 (b)].

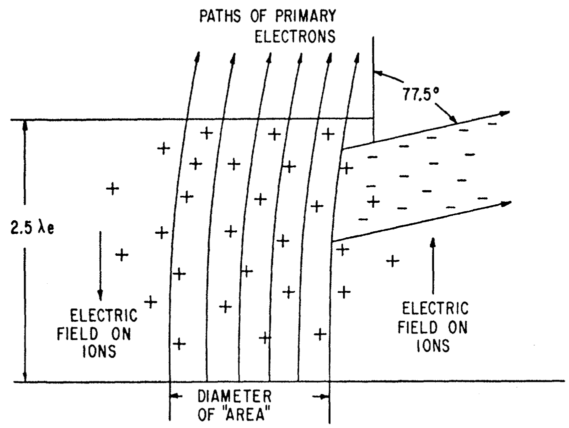
At a pressure of 10 mm ($T_a=457°$K) at which Gallagher observed reversal of a 2.2-amp spot in a magnetic field of 600 gauss, Eq. (4) gives

$$h'=0.706, \quad \gamma h'=\tan\alpha=0.90\times0.706=0.635,$$
$$\alpha=32.5°.$$

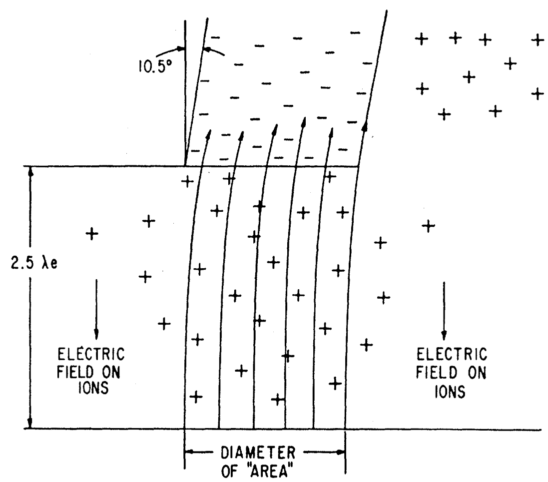
This is illustrated in Fig. 1(c). It is seen that this angle is a "turning point" between the situations illustrated in Figs. 1(a) and 1(b); at larger angles the motion will be "retrograde," at smaller angles "correct."

For a 9-amp arc the situation is less satisfactory. Gallagher observed reversal at 4 mm in a field of 600 gauss. In this case the density of residual electrons is much greater than for the smaller current, where inelastic collisions of the primary electrons produce considerable spread and scattering of the narrow beams from the "areas." The author is indebted to Dr. Tonks for the advice that the temperature of the residual electrons will probably be increased, by mutual interactions, by as much as a factor of 4. Langmuir[13] observed

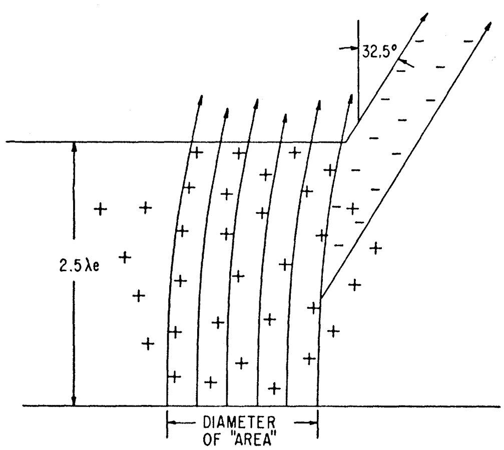
[8] L. Tonks, Phys. Rev. **54**, 634 (1938).
[9] L. Tonks, Phys. Rev. **56**, 361 (1939).
[10] C. J. Gallagher, J. Appl. Phys. **21**, 769 (1950).
[11] It is shown in Appendix 3 that the mercury vapor from the cathode spot has cooled to "ambient" temperature in a distance only slightly above λ_e.
[12] The positive ions drift much more slowly, and new ions are created in only limited numbers in this region by primary electrons that have been scattered by elastic collisions.
[13] I. Langmuir, Phys. Rev. **26**, 585 (1925). [See Appendix 2 for alternative explanation.]



(a)



(b)



(c)

FIG. 1. (a) The drift of "residual" electrons from the cathode spot, at a mercury vapor pressure of 1 mm Hg, in a magnetic field of 600 gauss. The spot moves in a "retrograde" direction. (b) Drift of residual electrons at 40 mm Hg, 600 gauss. These electrons do not weaken the electric field on the "correct" side, and the ions produced by the curved paths of the primary electrons cause the spot to move in the "correct" direction. (c) Drift of electrons in a field of 600 gauss at a pressure of 10 mm Hg. The angle of drift, 32.5°, is a "turning point" between the situations illustrated in Figs. 1(a) and 1(b); at larger angles the motion of the spot is "retrograde," at smaller angles "correct."

such an increase in temperature of the "ultimate" electrons with increase of current in a mercury arc, to as much as 80 000 degrees, in some experiments; and suggested that it might be caused by the joint action of radiation and excited atoms. The density of excited atoms in this case is large.

For the 9-amp arc, with $H=600$ gauss, $p=4$ mm, $T_a=431$°K, and $T_e=4\times0.25\times11\ 600=11\ 600$, Eq. (4) gives

$$h'=0.83,\quad \gamma h'=\tan\alpha=0.735,\quad \alpha=36.3°.$$

This is a reasonable value for the "turning point."

An alternative explanation is suggested by the analysis in Appendix 2, namely, that the vapor from the spot has not cooled sufficiently by the time it reaches the drift level, so that the pressure which has to be added is only half the total pressure. This would lead to the same value, $\alpha=36.3°$.

7. VELOCITY OF RETROGRADE MOTION

The velocity of the "retrograde" motion is also a crucial test of any theory. It is easily calculated.

Since the spot must move when its temperature-increase reaches $\Delta T=T-300=223$°K, its velocity is

$$\text{velocity}=\text{diameter of spot}/\text{time to reach }\Delta T=2a/t.$$

It can be shown that evaporation cooling is negligible compared to conduction except very close to the maximum temperature, and therefore has only a slight effect on "t." Hence the formula for conduction cooling can be used.

For conduction cooling and short times t Eq. (2) yields

$$\Delta T=\frac{2Q}{K}\left(\frac{kt}{\pi}\right)^{\frac{1}{2}},\quad \text{or}\quad t=\frac{\pi}{k}\frac{(\Delta T)^2K^2}{4Q^2}. \tag{5}$$

With $a=(i/\pi I_0)^{\frac{1}{2}}=1.78\times10^{-3}\sqrt{i}$, $K=0.064$ w/cm deg, $k=0.044$ cm²/sec, $Q=(V_c+V_i-\varphi_0)\times(I_0/20)$ w/cm², and $\Delta T=223$°K, we obtain

$$\text{velocity}=\frac{2a}{t}=\frac{8a}{\pi}\,\frac{kQ^2}{(\Delta T)^2K^2}=6.1\times10^3\sqrt{i}\ \text{cm/sec}. \quad (6)$$

This is the true velocity, but is not the value that is measured in the retrograde direction. For the motion is very random, with excursions at right angles to the retrograde direction nearly as frequent as in the retrograde direction. These components normal to the retrograde path cancel each other, on the average. Hence the "measured" retrograde velocity is $\frac{1}{3}$ of the true velocity:

$$\nu_{(\text{measured})}=(6.1\times10^3\sqrt{i})/3=2.03\times10^3\sqrt{i}\ \text{cm/sec}. \quad (7)$$

This is the measured velocity in strong magnetic fields. With weaker fields there are frequent excursions in the "correct" direction, which subtract from the measured value, and these "correct" excursions become more frequent as the field is weakened.[14] Hence the velocity for weak fields should be given by an equation similar to Eq. (7), with smaller constants, the constants becoming progressively smaller as the field grows weaker.

In Fig. 2 Gallagher's measurements of velocity as a function of arc current are reproduced, with the observed points joined by light lines. The heavy line is a plot of Eq. (7). The agreement is fairly good for strong fields; for weaker fields the velocity would be represented by similar curves with a smaller constant.

In zero field, as in the transition between "retrograde" and "correct" motion, the true velocity is still the same, but is observed as a random motion about a quasi-stationary point.

8. RETROGRADE VELOCITY IN STRONG FIELDS

St. John and Winans[15] have observed that in very strong magnetic fields ($H\geq 11\,000$ gauss) the lines of Hg II appear in the spectrum of the glow above the spot, and simultaneously the speed of retrograde motion doubles. Referring to Eq. (6), the velocity should be proportional to $Q^2/I_p^2=(V_c+V_i-\varphi_0)^2=(15.9)^2=253$, for the first ionization potential $V_i(\text{I})=10.4$ v. For the second ionization potential $V_i(\text{II})=18.65$ v, $Q^2/I_p^2=(10+18.65-4.5)^2=583$, which is about twice the value for Hg I. Hence the velocity should be doubled, as observed.

9. DEPRESSION OF SPOT

It has been assumed, in using Eq. (2), that the spot is a plane area. This will now be proved as follows:

[14] Gallagher informs me that these "correct" motions were a regular feature of the paths in weak fields. In fact, Gallagher states (reference 10, p. 768): "The method (of measurement) was not effective below about 20 cps because of random fluctuations in the position of the path of the spot over the mercury surface."

[15] R. M. St. John and J. G. Winans, Phys. Rev. **98**, 1664 (1955).

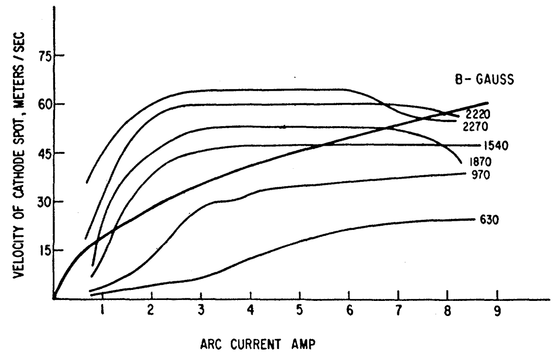


FIG. 2. Gallagher's measurements (reference 10) of the velocity of the cathode spot at various magnetic fields, as a function of arc current (thin lines). The heavy line is a plot of Eq. (7).

If a force of F dynes acts for t sec on a mass m, it will be moved (depressed) a distance

$$d=\tfrac{1}{2}(F/m)t^2,$$

where $m(\text{eff})=3m_0=3\rho\times\frac{2}{3}\pi a^3$ g, $\rho_{\text{Hg}}=13.6$ g/cm³, $a_{10\ \text{amp}}=5.6\times10^{-3}$ cm, and $\Delta T=223$°K.

The force F has been measured by Kobel.[16] He found that a 30-amp spot near the mouth of a small glass tube depressed the mercury column 6.5 cm, which is a pressure of $13.6\times980\times6.5=8.7\times10^4$ dynes/cm². This is the average pressure on the cross section (1.61×10^{-2} cm²) of the glass tube, which was 34 times the area of the 30-amp spot ($30/10^5$ cm²). However, it is shown in Appendix 1 that the spot was able, by its rapid motion, to heat the whole cross section to within 3.75°C of its own temperature, so that the pressure of 8.7×10^4 dynes/cm² represents the actual pressure of the "spot." This is very close to the value 10^5 dynes/cm² (75 mm) calculated in Sec. 4, and is a remarkable confirmation of that calculation.

Kobel's calculation of the actual force on the spot, 1400 dynes, is in error, being based on the assumption that the spot occupied the whole cross section of the glass tube, which was certainly wrong. The correct area of the spot, for a current density of 10^5 amp/cm², which is our basic assumption, is $i/10^5=3\times10^{-4}$ cm² for a 30-amp spot. Hence the pressure on 30-amp spot is $10^5\times3\times10^{-4}=30$ dynes.

With this value for F, the depression of the spot is

$$d=(F/4\pi a^3\rho)\times(\pi^2(\Delta T)^4K^4/16k^2Q^4)=10^{-7}\ \text{cm}.$$

This is negligible compared to the diameter 1.13×10^{-2} cm of the spot, and justifies the assumption that the spot is a plane surface. For the "areas" the depression is still smaller, because of the 100 times greater rate of heating, and the resulting shorter time t for the force to act.

However, as the "areas" race randomly over the surface of the spot, at a speed of 0.5×10^7 cm/sec, maintaining its temperature at an average value of

[16] E. Kobel, Phys. Rev. **36**, 1636 (1930).

250°C, there will be portions of the spot, closely adjacent to the area, which, for short times, are free from "areas," and are heated, more slowly, only by the diffuse ions produced by the "area" electron emissions. It is shown in Appendix (3) that if the current density of these ions is $\frac{1}{10}$ the average for the spot, a "spotlet" the size of an average "area," with diameter 3.56×10^{-4} cm, will be depressed 3.3×10^{-2} cm, which is 100 times its diameter. Long before this happens the walls of the "hole" will collapse, and particles of mercury, comparable in diameter to the "spotlet," will be projected (see next section). The edge of this "hole" provides the roughness factor of 2.5 needed for field emission of the moving "area."

10. HEIGHT TO WHICH MERCURY DROPS MAY BE THROWN

It is common observation that a shower of droplets rises from the cathode spot of a mercury arc, to a height of about 20 cm, and rectifiers are provided with baffles to prevent these droplets from striking the hot anode and raising the pressure. In a sodium pool arc, these droplets can be seen exploding as a beautiful display of yellow rockets.

Assuming that these droplets come from "spotlets" between the emitting "areas," as suggested in the last section, this height can be calculated, as follows:

Let p=pressure of mercury vapor above a "spotlet" in dynes/cm², h=vertical distance, H=maximum height, R=radius of droplet, r_0=radius of the "spotlet" (assumed a hemispherical boss), and r=radius of the expanding vapor.

The work done on the droplet by the pressure p is $\int_{r_0}^{\infty} p_r(\pi R^2)dh$, where $p_r=p_{r_0}(r_0/r)^2$, $h=r-r_0$, $dh=dr$. The work done by the pressure on the droplet is, therefore,

$$\text{work}=p_0\pi R^2\int_{r_0}^{\infty}(r_0/r)^2dr=\pi R^2p_0r_0.$$

This must equal mgH, the work necessary to raise the particle H cm against gravity. Hence

$$H=\frac{\pi R^2p_0r_0}{\frac{4}{3}\pi R^3\rho g}=\frac{3p_0}{4\rho g}\left(\frac{r_0}{R}\right).$$

With $p_0=75$ mm$=10^5$ dynes/cm², one has $H=5.6r_0/R=20$ cm if R is about $\frac{1}{3}r_0$.

If the droplet is projected before the pressure reaches 75 mm, the height H will be smaller for a droplet of radius $R=\frac{1}{3}r_0$, but still may be 20 cm for a smaller droplet.

11. RATE OF EVAPORATION

Tonks[17] observed a rate of evaporation of 2×10^{-3} g/sec for a 10-amp arc anchored on molybdenum. Dushman[18] gives the rate of evaporation at temperature T°K and pressure p mm, as

$$G=5.83\times10^{-2}p_{\text{mm}}(M/T)^{\frac{1}{2}}\text{ g/cm}^2\text{ sec.}$$

For a 10-amp spot anchored on Mo, Eq. (3), Sec. 3 gives $T=615$°K, $p=563$ mm. This yields $G=18.8$ g/cm² sec. Hence for a spot of area 10^{-4} cm² (10 amp), the rate of evaporation is 1.88×10^{-3} g/sec, in good agreement with Tonk's measurement.

Kobel[16] found the rate of evaporation from a 10-amp arc on a mercury surface in a deep glass well to be 1.7×10^{-4} g/sec. This arc spot, since it was on free mercury, must be considered to have been moving rapidly over the restricted area, rather than stabilized. Hence its temperature should be 523°K and pressure 75 mm, for which the equation for G, above, and area $=10^{-4}$ cm², gives the evaporation rate

$$2.7\times10^{-4}\text{ g/cm}^2.$$

This is nearly twice the value measured by Kobel, which is well accounted for by the interception of evaporated atoms by the glass walls of Kobel's cavity.

12. SUMMARY

Two assumptions are made: (1) that the average current density of the "spot" is 10^5 amp/cm², with a positive ion component of 1/20; (2) that the spot is made up of a number of small "emitting areas," each with current density 10^7 amp/cm².

It is shown that these assumptions lead to a quantitative explanation of the following phenomena, in good agreement with experiment as follows:

(1) field emission, in agreement with the Fowler-Nordheim equation.

(2) temperature of spot=250°C, pressure 75 mm Hg, in agreement with Kobel's measurement of the pressure on the spot.

(3) retrograde motion of spot, with "reversing pressure" in good agreement with Gallagher's measurement.

(4) velocity of retrograde motion, in agreement with Gallagher's measurements.

(5) height to which mercury droplets are thrown, in agreement with observation.

(6) rate of evaporation of mercury, as observed by Tonks and by Kobel.

APPENDIX 1

A. Uniformity of Cathode-Spot Temperature

The cathode spot is heated by the "areas," which race randomly over its surface, at their constant temperature of 250°C. The sum of the "areas" within a spot must

[17] L. Tonks, Phys. Rev. **50**, 226 (1936).

[18] S. Dushman, *Scientific Foundations of Vacuum Technique* (John Wiley & Sons, Inc., New York, 1962), p. 17.

equal 1/100 of the area of the spot, since their electron emission is 100 times the average emission of the spot. Hence the sum of their heating, by positive ions, namely 100 times the average for the spot over 1/100 of its area, is just the amount needed to maintain the spot at their temperature, 250°C, which it obviously could do if their velocities were infinite. The problem is to determine the actual temperature over the surface of the spot.

Consider any portion of the surface the size of an "area." It will be heated to 250°C whenever a moving area passes over it, and will cool by conduction between passages.

The probability that the center of an area will pass over or "collide" with the given surface element, in a single crossing of the spot, is just the ratio of diameters of area and spot $=2r/2a$. The average number of crossings per second is

$$(\text{velocity of area})/(\tfrac{2}{3}\text{ diam of spot})=(\text{vel})/(\tfrac{2}{3}\times 2a);$$

(the average transit distance is $\frac{2}{3}$ of the spot diameter). Hence the number of collisions per second is $(r/a)\times(\text{vel})/(\frac{4}{3}a)$ for a single moving "area," and for n "areas" is $(nr\times\text{vel})/(\frac{4}{3}a^2)$. The time τ between "collisions" is $1/(\text{collisions/sec})=(\frac{4}{3}a^2)/(nr\times\text{vel})$. This is the time available for cooling of the selected area. The actual cooling will be this time multiplied by the rate of cooling $=(dT/dt)\times\tau$.

The rate of cooling of the hemispherical volume under an area, of radius r, is $-dT/dt=4rK\Delta T/(\rho c\times\frac{2}{3}\pi r^3)=6k\Delta T/\pi r^2$, where K is the thermal conductivity and k the diffusivity of mercury, and $4rK$ is the thermal conductance of a small plane area of radius r.

Hence the total cooling between "collisions" is $-(dT/dt)\times\tau=(6k\Delta T/\pi r^2)\times(\frac{4}{3}a^2/nr\times\text{vel})$. Insertion of the velocity of an "area" [Eq. (6), Sec. 7], $\text{vel}=(8r/\pi)\times[kQ^2/K^2(\Delta T)^2]$, gives the average fall in temperature below 250°C of any portion of the spot:

$$-\frac{dT}{dt}\times\tau=\frac{6K\Delta T}{\pi r^2}\times\frac{\frac{4}{3}a^2}{nr}\times\frac{\pi K^2(\Delta T)^2}{8rkQ^2}$$

$$=\frac{a^2K^2(\Delta T)^3}{nr^4Q^2}=\frac{10^4nK^2(\Delta T)^3}{a^2Q^2},$$

since $n\times\pi r^2=\pi a^2/100$, and hence $nr^4=a^4/10^4n$.

Taking $n\leq 10$ areas per spot, $K^2=(0.064)^2=41\times10^{-4}$, $(\Delta T)^3=(250-30)^3=11.1\times10^6$, $a^2=(5.35\times10^{-3})^2=28.7\times10^{-6}$ (for $i=9$ amp), and $Q^2=[(V_c+V_i-\varphi)^2/(20)^2]\times10^{14}=0.635\times10^{14}$, one obtains

$$-\frac{dT}{dt}\times\tau\leq\frac{10^4\times10^1\times41\times10^{-4}\times11.1\times10^6}{28.7\times10^{-6}\times0.635\times10^{14}}\leq 2.5°\text{C}.$$

If one makes the reasonable assumption that the density of areas within the spot is constant for different currents, then $n/a^2=\text{constant}$, and the maximum cooling of any part of the spot, for *any* value of current, is 2.5°C.

B. Pressure on "Spot" in Kobel's Test

Kobel observed a pressure of 6.5 cm Hg $=8.7\times10^4$ dynes/cm², with a current of 30 amp to a mercury surface in a glass tube. On the assumption that his 30-amp "spot" occupied the total cross section of the glass tube, Kobel calculated the thrust of his spot to be 1400 dynes. Hence 1400 dynes $=8.7\times10^4$ (dynes/cm²) $\times$tube area, and his tube area was $1400/8.7\times10^4=1.61\times10^{-2}$ cm².

The actual area of his spot, according to our assumption of 10^5 amp/cm², was 30 amp/(10^5 amp/cm²) $=3\times10^{-4}$ cm²; and the actual thrust on the spot was $8.7\times10^4\times3\times10^{-4}=26$ dynes.

Consider a portion of the surface of the glass tube the size of a 30-amp spot. The probability of a head-on "collision" with this area by a moving spot is

$$(\text{diameter of spot})/(\text{diameter of tube})=2r_{\text{spot}}/2r_{\text{tube}}.$$

Proceeding as in A above, with one moving "spot" instead of n moving "areas," and with $Q^2=0.635\times10^{10}$, and $\pi r^2_{\text{spot}}/\pi r^2_{\text{tube}}=3\times10^{-4}/1.61\times10^{-2}=1/54$, yields

$$\frac{dT}{dt}\times\tau=\frac{r_t^2K^2(\Delta T)^3}{r_s^4Q^2}=\frac{54K^2(\Delta T)^3}{r_s^2Q^2}=3.75°\text{C}.$$

Hence the temperature, and pressure, were practically constant over the cross section of Kobel's tube, and the pressure of 8.7×10^4 dynes/cm² on the mercury surface was the same as the pressure on the "spot."

C. Ratio of Ion Current to Atoms Evaporated

A simple calculation shows that the current of ions striking an "area" per second is much greater than the number of evaporated atoms leaving, and raises questions of the supply of atoms for ionization, and how the ions can be blown away by evaporation atoms at the maximum temperature calculated in Sec. 4.

The number of ions $=(\text{coul/cm}^2\text{ sec})/(\text{coul/ion})=(10^7/20)/(1.59\times10^{-19})=3.15\times10^{24}/\text{cm}^2$ for an "area," and $(10^5/20)/(1.59\times10^{-19})=3.15\times10^{22}/\text{cm}^2$ for a "spot."

The number of atoms evaporating per cm² at 523°K and 75 mm Hg is $3.51\times10^{22}\,p_{\text{mm}}/\sqrt{(MT)}=8.15\times10^{21}$.

The first question, the sufficiency of atoms for ionization, is easily answered. There is obviously an ample supply of atoms to be ionized, since the whole surface of the spot, and a considerable area around the spot, is evaporating at the same rate as the "area."

In the case of Kobel's tube there was an ample supply also. The area of the tube was 54 times the area of the

30-amp spot; hence

(evap. atoms)/ions

$$=(54\times 8.13\times 10^{21})/(3.15\times 10^{22})=13.7.$$

This shows that there were ample atoms in the tube, but that Kobel's measured evaporation was reduced by 7% by the return of ions.

The answer to the second question, the ability of the evaporating atoms to blow away the approaching ions, requires a more detailed examination of the geometry of an "area."

It was assumed in Sec. 2 that the "area" is a hemispherical boss; and it was suggested in Sec. 9 that this boss might be furnished by the roughness created by "channeling" closely adjacent to an area.

The thickness of the space-charge sheath that accelerates the electrons is (Sec. 2):

$$X=3.7\times 10^{-7}\text{ cm.}$$

The mean free path of electrons, at $p=75$ mm Hg and $T=523$°K, is $\lambda=4/\pi n\delta^2$, where $n=$atoms/cc$=9.65\times 10^{18}\ p_{\text{mm}}/T$, and $\pi\delta^2=$cross section of a mercury atom$=8.2\times 10^{-15}$ cm^2. This gives $\lambda=3.5\times 10^{-4}$ cm, and $\lambda/X=1000$. This means that the electrons are shot out at right angles to the surface of the "boss," in a highly divergent stream; and at the distance of 2.5λ the density of ions, which converge toward the "area," is less than 1/200 of their density at the area. Alternatively, without the boss, the ions produced by electrons which have been deflected by elastic collisions with mercury atoms form a very divergent beam. Hence it is possible that these slow-moving ions can be blown away by the atoms evaporating from the surface surrounding the "area."

APPENDIX 2

An exact calculation of the cooling of the mercury vapor from the cathode spot is difficult, but an approximate calculation may be made as follows:

Vapor from a spot of radius $a=5.3\times 10^{-3}$ cm (9 amp) expanding from a volume V_1 at temperature $T_1=523$°K, pressure $p_1=75$ mm, to a volume V_2 at temperature T_2, pressure p_2, cools according to the chemical formula:

$$c_v \ln(T_2/T_1)=-R\ln(V_2/V_1),$$

or

$$\log_{10}T_2=-\tfrac{2}{3}\log_{10}(V_2/V_1)+\log_{10}T_1.$$

V_1 may be considered a hemispherical volume $V_1=\frac{2}{3}\pi a^3$. In expanding, a distance of 1.5 mean free paths$=1.5\times 3.5\times 10^{-4}$ cm, the volume is increased, approximately, to $V_2=\frac{2}{3}\pi(a+1.5\lambda)^3$. Hence $V_2/V_1=(a+1.5\lambda)^3/a^3=(5.82/5.30)^3=1.33$, and $\log_{10}T_2=-\frac{2}{3}\log_{10}(1.33)+\log_{10}523=2.6359$. This gives $T_2=433$°K, $p_2=4.25$ mm Hg.

For a 2.2-amp arc, $a=2.64\times 10^{-3}$ cm, the cooling at one mean free path distance $\lambda=3.5\times 10^{-4}$ is $V_2/V_1=[(a+\lambda)/a]^3=1.50$, $\log_{10}T_2=2.6011$, $T_2=409$°K, $p_2=1.4$ mm.

Hence, for the 2.2-amp arc, it may be assumed that the residual electrons drift in "ambient" vapor; for the 9-amp arc it is probable that there is some pressure from the spot at the distance of drift, which adds to the observed pressure, and may account for the observed added pressure being too low (cf. Sec. 6).

APPENDIX 3

The depression of a "spotlet" of size equal to an average "emitting area" $r=1.78\times 10^{-4}$ cm, in a field where the rate of heating is $\frac{1}{10}$ the average for spots, viz. $Q=\frac{1}{10}\times(15.9/20)\times 10^5$ w/cm^2, is easily calculated as follows:

$$d=\frac{1}{2}\frac{F}{m}t^2=\frac{1\times 10^5\times \pi r^2}{2\times 2\rho\pi r^3}t^2$$
$$=\frac{10^5}{4\rho r}\times\frac{\pi^2(\Delta T)^4K^4}{k^2\times 16Q^4}$$
$$=3.3\times 10^{-2}\text{ cm.}$$

This is a depth 100 times the diameter $2r=3.56\times 10^{-4}$ cm of the spotlet. Hence the walls of the *hole* will collapse long before the "spotlet" reaches the maximum temperature $T=523$°K and pressure $p=75$ mm.