Magnetically Confined Plasma with a Maxwellian Core

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While a magnetically confined plasma cannot have a Maxweillian velocity distribution at each point, it is shown that a high-β Maxwellian plasma containing no magnetic fields can exist when separated by a non-Maxwellian boundary layer from the region of no plasma and homogeneous confining magnetic field.

1. INTRODUCTION

IT is evident that a plasma confined by a static magnetic field cannot be Maxwellian at each point, since then no currents would flow.[1] Can it be Maxwellian except for a boundary layer? In this paper we describe such a solution of the self-consistent plasma equations without collisions. This paper describes a uniform Maxwellian plasma with zero internal magnetic field separated by boundary layers from exterior regions of asymptotically uniform magnetic field and vanishing plasma density. The volume of space within which the distribution is exactly Maxwellian may be as large as desired.

The distribution we describe is also one which may be called a truncated Maxwellian distribution: At every point in phase space, the distribution either has the Maxwellian value appropriate to the electromagnetic potentials at that point, or it is zero.

The existence of such a "high-beta" solution of the self-consistent collisionless plasma equations is of great interest, especially for researchers in controlled thermonuclear energy. Existing solutions, for example, the original self-focusing solutions of Bennett,[2] or the Rosenbluth sheath,[3] describe the confinement of a non-Maxwellian plasma, but it has been an open question whether equilibria of the kind presented here could exist.[4]

2. PLANAR "POSITRONIUM" PLASMA

To begin with a simple problem, consider a plasma of equally massive particles of opposite electric charge, the "positronium" plasma. We seek equilibria with no charge separation ($E=0$), with parallel magnetic field lines ($B=B_z$), and with only one-dimensional coordinate dependence of the field quantities:

$$\partial/\partial t=\partial/\partial y=\partial/\partial z=0,$$
$$B_x=B_y=0.$$

The magnetic vector potential $\mathbf{A}$ will thus retain only one component $A_y\equiv A$. There is to be no plasma at $x\rightarrow+\infty$ and no magnetic field at $x\rightarrow-\infty$, where we may take $A_y=0$.

Since we seek a stationary solution, the distribution function $\mathbf{f}(\mathbf{x},\mathbf{p})$ for either species of particle can be expressed as a function of the constants of motion for individual particles in the self-consistent magnetic field. Three constants of motion are immediate, following from the symmetries we have assumed. The particle energy

$$H=c[(\mathbf{p}-q\mathbf{A})^2+m^2c^2]^{\frac{1}{2}} \tag{1}$$

is conserved because $\partial H/\partial t=0$. The y momentum

$$p_y=\gamma mv_y+qA \tag{2}$$

is conserved because $\partial H/\partial y=0$. (It will be noted that the treatment is relativistic; $\gamma=[1-v^2c^{-2}]^{-\frac{1}{2}}$ is the usual relativistic dilation factor.) The z momentum

$$p_z=\gamma mv_z \tag{3}$$

is conserved because $\partial H/\partial z=0$. Consequently,

$$f(\mathbf{x},\mathbf{p})=f(H,p_y,p_z) \tag{4}$$

automatically satisfies the collisionless Boltzmann equation. The solution must be self-consistent. This provides an equation for the vector potential:

$$\partial^2A/\partial x^2=-\mu_0\sum_{+-}q\int fv_yd^3p. \tag{5}$$

In order to picture the solution for $f(H,p_y,p_z)$, let us look at the distribution function in phase space. Since f is independent of y and z, and p_z does not contribute to the magnetic field, it is sufficient to look at the three-dimensional (x,p_x,p_y) subspace (Fig. 1). The $H=$const and $p_y=$const surfaces cut out curves in this space representing particle trajectories in phase

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[1] E. S. Weibel, Phys. Fluids **2**, 52 (1959).
[2] W. H. Bennett, Phys. Rev. **45**, 890 (1934); **98**, 1584 (1955).
[3] M. N. Rosenbluth, in *Magnetohydrodynamics*, edited by R. Landshoff (Stanford University Press, Stanford, California, 1957).
[4] In the meantime, the following papers have been published on this problem: H. Grad, Phys. Fluids **4**, 1366 (1961); A. I. Morozev and L. S. Solov'ev, J. Exptl. Theoret. Phys. U.S.S.R. **40**, 1316 (1961) [translation: Soviet Phys.—JETP **13**, 927 (1961)]; and J. Hurley, Phys. Rev. **124**, 1307 (1961). See also G. Schmidt and D. Finkelstein, Am. Phys. Soc. **6**, 290 (1961).

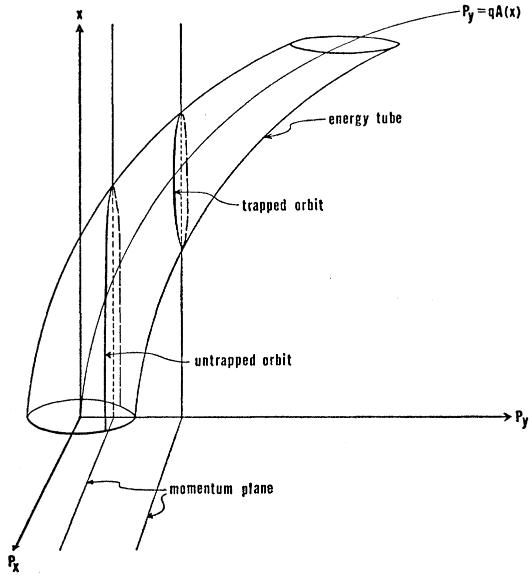

FIG. 1. Trapped and untrapped particles.

space. Equation (4) means that f depends only on the trajectory, or, in other words, that it is constant along such a trajectory.

The form (1) for H shows that the surfaces $H=\text{const}$ are represented by tubes of circular cross section about the central curve,

$$p_y = qA, \quad p_x = 0, \tag{6}$$

shown in Fig. 1. We will suppose that A is monotone, approaches 0 in the interior, and is asymptotically linear in the remote exterior $x \to +\infty$ (uniform external field). The surfaces $p_y=\text{const}$ are planes in this space. Clearly the intersections of the tubes and planes are of two kinds, closed and open, representing trapped and untrapped particles.

The distribution function in the region of phase space occupied by the open trajectories is uniquely determined if the distribution function $x \to -\infty$ is given (in our case it is Maxwellian). Each particle carries with it the density it had in the Maxwellian region. The closed trajectories can be populated with particles at will (keeping in mind, of course, that the density is constant along the same trajectory). In that way, an infinite number of solutions can be devised. We will seek a solution where trapped particles are absent.

The phase space now contains two regions, one populated with phase points, the other empty. Since the Maxwellian distribution depends only on H, Eq. (4) can now be rewritten as

$$f(\mathbf{x},\mathbf{p}) = \epsilon f_0(H), \tag{7}$$

where $\epsilon(\mathbf{x},\mathbf{p})$ is unity in the filled and zero in the empty region. The boundary between the two regions is cut out on the energy tube in the interior of the plasma by the plane asymptotically tangent to the energy tube at $x \to -\infty$. This plane is given by $p_y = P(H,p_z)$, where evidently $P(H,p_z)$ is the radius of the energy tube,

$$P \equiv [H^2/c^2 - p_z^2 - m^2c^2]^{\frac{1}{2}}. \tag{8}$$

Now the integral in (5) can be simplified. Each species

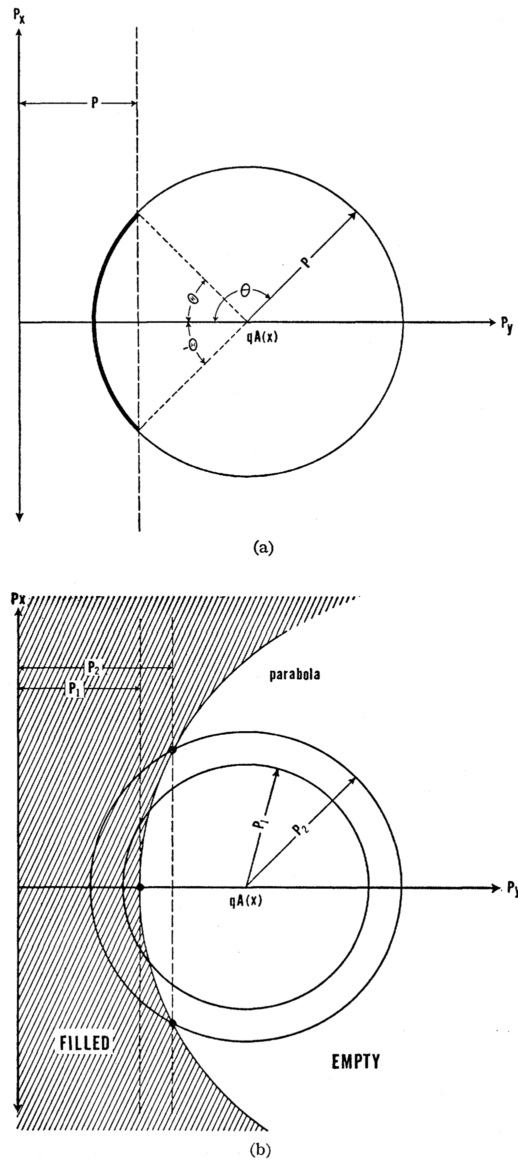

FIG. 2. $x=\text{const}$ section of an energy tube, showing coordinates θ, P. The plane $p_y=P$ is shown dashed. The intersection of this plane with the circle gives the extremes $\pm\Theta$ of the filled sector, shown heavy. By repeating this construction for various values of P, we obtain the filled region shown in (b), separated by a parabola from the empty region of the (p_x,p_y) plane. [*Note added in proof.* The parabola should have been drawn through the intersections of the P_1 line and the P_1 circle, as for P_2.]

contributes to the current density

$$j(x)=q\int f_0(H)v_y(\mathbf{p},\mathbf{x})\epsilon(\mathbf{p},\mathbf{x})d^3p$$

$$=q\int_{|qA|/2}^{\infty}\left[\int_0^{\infty}\left(\int_{-\Theta}^{+\Theta}\frac{f_0(H)P\cos\theta}{\gamma m}d\theta\right)dp_z\right]PdP. \quad (9)$$

Here we have introduced cylindrical coordinates P, θ on the energy tube relative to the central curve (6): (See Fig. 2.)

$$\begin{aligned} p_x&=P\sin\theta,\\ p_y-qA&=-P\cos\theta. \end{aligned} \quad (10)$$

The extremes $\pm\Theta$ of the range of θ have been determined by the requirements $|p_y|\leq P$,

$$\cos\Theta=-(P-|qA|)/P, \quad 0<\Theta<+\pi. \quad (11)$$

The range of the P integration begins at $\Theta=0$, $\cos\Theta=1$, for $qA>0$; at $\Theta=\pi$, $\cos\Theta=-1$, for $qA<0$. Carrying out the θ integration yields

$$j(x)=\frac{|q|}{m}\operatorname{sgn}A\int_{|qA|/2}^{\infty}\left\{\int_{-\infty}^{\infty}\frac{f_0(H)}{\gamma}dp_z\right\}2P^2 \times\left[\frac{2qA}{P}-\left(\frac{qA}{P}\right)^2\right]^{\frac{1}{2}}dP \quad (12)$$

for each species contribution to the current.

It is not possible to carry the integral further for a general internal distribution $f_0(H)$. The p_z integration, in which H is to be substituted from (1) gives a function of P alone which is a kind of "reduced distribution" in P:

$$\varphi(P)=\int_{-\infty}^{\infty}dp_z\, f_0(H)/\gamma.$$

(12) combined with (5) now yields

$$\frac{\partial^2 A}{\partial x^2}=-\mu_0\sum_{-+}\frac{|q|}{m}\operatorname{sgn}A \times\int_{|qA|/2}^{\infty}\varphi(P)2P^2\left[\frac{2qA}{P}-\left(\frac{qA}{P}\right)^2\right]^{\frac{1}{2}}dP. \quad (13)$$

The integral on the right side is usually a difficult one. However, the form of (13) makes it possible to inspect the boundary layer where it joins the interior of the plasma, for the $A\to 0$.

When $A\ll P_{\mathrm{av}}$, the second term under the square root is negligible in comparison to the first and

$$\mu_0 j(x)\to aA^{\frac{1}{2}}(x), \quad (14)$$

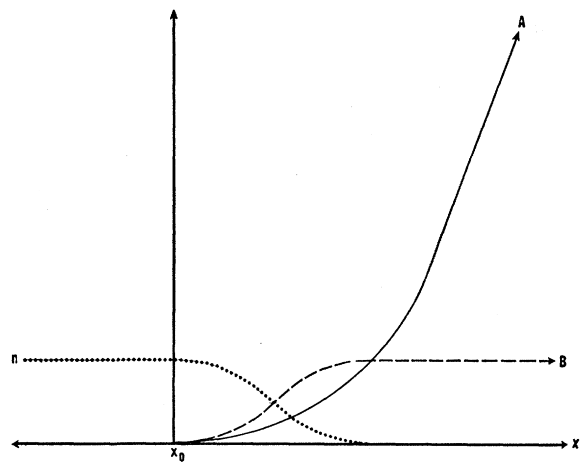

FIG. 3. Confined Maxwellian plasma and boundary layer.

where a is the constant

$$a=\sum \mu_0|q|/m\operatorname{sgn}A\int_0^{\infty}2^{\frac{3}{2}}P^{\frac{3}{2}}\varphi(P)dP. \quad (15)$$

Thus, the vector potential near the interior region obeys

$$\partial^2 A/\partial x^2=aA^{\frac{1}{2}}. \quad (16)$$

Near a zero x_0 of A and $\partial A/\partial x$, the solution of this equation has the form

$$A=(a/12)^2(x-x_0)^4. \quad (17)$$

Clearly this solution does not approach zero asymptotically for $x\to-\infty$.[5] It is possible, however, to fit this function to another solution, namely, $A=0$ for $x<x_0$, since both A and $\partial A/\partial x$ are continuous around $x=x_0$. The point x_0 is a weak "wandering singularity" of the kind typical of nonlinear differential equations. We are indebted to Grad[6] for pointing out the possibility of pasting together the two solutions in this manner.

Hence, as shown in Fig. 3, in the region $x<x_0$ the magnetic field is strictly zero and the plasma Maxwellian. Particles reaching $x=x_0$ from the left enter the magnetic field and, after reaching a turning point, return on a symmetrical path into the plasma, generating in the $x>x_0$ region currents to produce this self-consistent field configuration. In a similar manner, other configurations like the ones shown in Fig. 4 can be constructed.

The particle density for a species in the boundary

[5] Nevertheless, it leads to a valid solution of the differential equations of the problem, though not of the boundary conditions we have imposed. It describes a plane symmetric "plasmoid" confined on two faces by reversed magnetic fields, and Maxwellian only on the central plane $x=0$. This is depicted in Fig 4(b).

[6] H. Grad (private communication, 1961).

layer can also be computed.

$$n(x)=\int f_0(H)\epsilon d^3p=\int_{|qA|/2}^{\infty}\int_{-\infty}^{+\infty}\left[\int_{-\Theta}^{+\Theta} f_0(H)d\theta\right]dp_z\,PdP$$

$$=2\int_{|qA|/2}^{\infty}\left[\int_{-\infty}^{+\infty} f_0(H)dp_z\right]\Theta PdP=2\int_{|qA|/2}^{\infty} f_\perp(P)P\cos^{-1}\left(\frac{|qA|-P}{P}\right)dP, \quad (18)$$

where

$$f_\perp(P)=\int_{-\infty}^{+\infty} f_0(H)dp_z.$$

At the beginning of the boundary layer ($x=x_0$, $A=0$), we find

$$n(0)=2\pi\int Pf_\perp(P)dP, \quad (19)$$

or just the particle density inside the Maxwellian plasma. It is easy to see that $(\partial n/\partial x)_{x=x_0}$ is zero but the second derivative is not.

3. PLANAR HYDROGENIC PLASMA

We now consider the case where both electrons and one species of heavier ions enter the magnetic field from the remote interior of the plasma. Due to the difference in Larmor radii and the depths of penetration, an electrostatic potential $\phi(x)$ may develop. We take $\phi(-\infty)=0$. While the conserved momenta are still given by Eqs. (2) and (3), the one-particle energy is now

$$H=c[(p-qA)^2+m^2c^2]^{\frac{1}{2}}+q\phi(x) \quad (20)$$

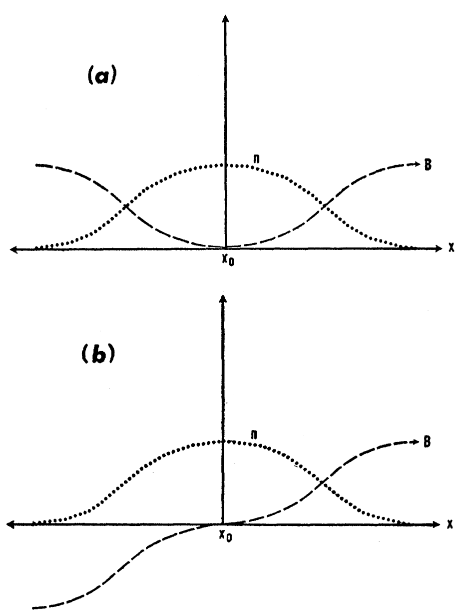

FIG. 4. (a) Plasmoid in a magnetic field; (b) plasmoid in reversed fields.

instead of (1). The three constants of motion (20), (2), and (3) again establish the relation between the phase-space density at a finite x and the density at $x\rightarrow-\infty$. The energy tube of Fig. 1 now has a radius that depends on x as well as on H and p_z (like the surface of a tornado) and its central curve is again given by (6). Designating this radius by $P(x,H,p_z)$, we find

$$cP(x,H,p_z)=\{[H-q\phi(x)]^2-m^2c^4-p_z^2c^2\}^{\frac{1}{2}}, \quad (21)$$

which should be compared to (8).

The other surface that cuts out the path in (p_x,p_y,x) space is still given by conservation of p_y and is the plane $p_y=\text{const}$. We make the simplifying assumption that this plane intersects the energy tube (20) in a connected curve which is closed for sufficiently large p_y and open for sufficiently small p_y (for a fixed energy tube).

There could now be two limiting paths given by

$$p_y=\pm P(-\infty,H,p_z)\equiv\pm P_{-\infty}, \quad (22)$$

where $P_{-\infty}$ can be found from (20):

$$c[P^2+p_z^2+m^2c^2]^{\frac{1}{2}}+q\phi=c[P_{-\infty}^2+p_z^2+m^2c^2]^{\frac{1}{2}},$$
$$P_{-\infty}^2=P^2+q^2\phi^2/c^2+(2/c)q\phi[P^2+p_z^2+m^2c^2]^{\frac{1}{2}}. \quad (23)$$

The current density now becomes

$$j=\sum_{+-}2q\int_{-\infty}^{+\infty}\left\{\int_{qA-P_{-\infty}}^{\infty}\left[\int_{\Theta_1}^{\Theta_2} f_0(H)v_y d\theta\right]PdP\right\}dp_z, \quad (24)$$

where we again introduce cylindrical coordinates, and

$$\cos\Theta_1=(qA+P_{-\infty})/P, \quad (25)$$

$$\cos\Theta_2=(qA-P_{-\infty})/P. \quad (26)$$

The lower limit of the integration in the variable P is again determined by setting $\Theta_2={}_{\pi}{}^{0}$ for $qA\gtrless 0$. Note that P is a function of p_z. With

$$v_y=(p_y-qA)/\gamma m=-P\cos\theta/\gamma m, \quad (27)$$

the integration over θ can again be performed and one obtains

$$j=\sum_{+-}2|q|\ \text{sgn}A\int_{-\infty}^{+\infty}\left\{\int\frac{P^2}{\gamma m}f_0(H)\right.$$
$$\left.\times[(1-\cos^2\Theta_2)^{\frac{1}{2}}-(1-\cos^2\Theta_1)^{\frac{1}{2}}]dP\right\}dp_z, \quad (28)$$

where $\cos\Theta_1$, $\cos\Theta_2$, and $P_{-\infty}$ are given by (25), (26), and (23), respectively. The integration over P has to be carried out by integrating over regions where the integrand is real.

NONRELATIVISTIC LIMIT

In the nonrelativistic limit these equations are somewhat simplified. When $\gamma\rightarrow 1$, (23) becomes

$$P_{-\infty}^2=P^2+2qm\phi. \tag{29}$$

$P_{-\infty}$ is now not a function of p_z, and we can integrate out for p_z in (28). This results in

$$\int_{-\infty}^{+\infty} f_0(P^2/2m+p_z^2/2m+q\phi)dp_z = f_\perp(P^2/2m+q\phi)=f_\perp(H_\perp). \tag{30}$$

Equation (28) becomes now with help of (25), (26), (29), and (30):

$$j=\sum_{+-} 2|q| \operatorname{sgn}A\int_\alpha^\infty f_\perp(H_\perp)\mathbf{P}/m[2qA(P^2+2qm\phi)^{\frac{1}{2}}-q^2A^2-2qm\phi]^{\frac{1}{2}}dP -\sum_{+-} 2|q| \operatorname{sgn}A\int_0^\beta f_\perp(H_\perp)[-2|qA|(P^2+2qm\phi)^{\frac{1}{2}}-q^2A^2-2qm\phi]^{\frac{1}{2}}dP. \tag{31}$$

Since

$$dH_\perp=(P/m)dP, \tag{32}$$

one obtains

$$\partial^2A/\partial x^2=-\mu_0\sum_{+-} 2|q| \operatorname{sgn}A\int_\alpha^\infty f_\perp(H_\perp)[2|qA|(2mH_\perp)^{\frac{1}{2}}-q^2A^2-2qm\phi]^{\frac{1}{2}}dH_\perp +\mu_0\sum_{+-} 2|q| \operatorname{sgn}A\int_0^\beta f_\perp(H_\perp)[-2|qA|(2mH_\perp)^{\frac{1}{2}}-q^2A^2-2qm\phi]^{\frac{1}{2}}dH_\perp, \tag{33}$$

where

$$\begin{aligned}\alpha&=1/2m(|qA|/2+qm\phi/|qA|)^2 \quad \text{for } q^2A^2+2qm\phi>0\\ &=0 \quad \text{for } q^2A^2+2qm\phi<0,\end{aligned} \tag{34}$$

and

$$\begin{aligned}\beta&=0 \quad \text{for } q^2A^2+2qm\phi>0\\ &=1/2m(|qA|/2+qm\phi/|qA|)^2 \quad \text{for } q^2A^2+2qm\phi<0.\end{aligned} \tag{35}$$

Equation (33) is now the self-consistent equation for the vector potential, instead of (13). A similar equation is needed for the scalar potential:

$$\frac{\partial^2\phi}{\partial x^2}=-\frac{1}{\epsilon_0}\sum_{+-} 2q/m\int_\alpha^\infty f_\perp(H_\perp)\sin^{-1}\left[\frac{2|qA|(2mH_\perp)^{\frac{1}{2}}-q^2A^2-2qm\phi}{2mH_\perp-2qm\phi}\right]^{\frac{1}{2}}dH_\perp +\frac{1}{\epsilon_0}\sum_{+-}\frac{2q}{m}\int_0^\beta f_\perp(H_\perp)\sin^{-1}\left[\frac{-2|qA|(2mH_\perp)^{\frac{1}{2}}-q^2A^2-2qm\phi}{2mH_\perp-2qm\phi}\right]^{\frac{1}{2}}dH_\perp. \tag{36}$$

Equations (33) and (36) can be integrated numerically for a given electron and ion distribution functions to lead to the functions $\phi(x)$ and $A(x)$, as well as to the distribution of particle density in the boundary layer. They are consistent with an asymptotic behavior

$$A,\ \phi\sim x^4,\quad x\rightarrow 0+,$$

similar to that of the positronium plasma.