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Role of Electron-Electron Collisions in Galvanomagnetic Effects

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In the present paper we assume a time of relaxation τ_e which gives the rate of approach of a non-Maxwellian distribution to its equilibrium through electron-electron (*e-e*) collisions alone. We then introduce an additional term in the usual Boltzmann's transport equation to account for the effect of *e-e* collisions on the distribution function. A solution of the Boltzmann's equation leads in the usual manner to the resistivity and Hall coefficient.

In the limit $\tau_e \to \infty$ our formulas reduce to the usual equations found in standard books. The corrections to the well-established results are found to be small. A systematic study of these effects with different carrier densities should lead to valuable information regarding the role of (*e-e*) collisions in these effects. The quantity $b=\tau_0/\tau_e$, where τ_0 is the mean time of collision of electrons with lattice vibrations, is shown to be a measurable quantity. It can be measured most easily by observing the changes in Hall constant with increasing magnetic field.

INTRODUCTION

Recently there has been considerable interest in the study of electron-electron (*e-e*) interaction in solids.[1] We do not make an effort to calculate $(\partial f/\partial t)_e$, the rate of change of the distribution function f through *e-e* interaction. We assume $(\partial f/\partial t)_e = -(f-\bar{f}_0)/\tau_e$, where $\bar{f}_0$ is a displaced Maxwellian distribution, the displacement being in the velocity space in the direction of the current (along x). We then introduce this additional term in the Boltzmann's transport equation and solve for f. Knowing the distribution function we calculate the Hall coefficient and the magnetoresistance in the usual manner.

In the present problem, the *e-e* collisions should play a minor role and only make small corrections to the well-established results, obtained by considering electron-lattice collisions alone. Clearly if *e-e* interaction was very strong there would be vanishingly small magnetoresistance, and the magnetic field dependence of Hall coefficient. Since the usual theories in which *e-e* interaction is neglected have been successful in explaining these effects, any corrections that we introduce should be relatively small, but they will give the valuable information regarding Coulomb interaction.

It should be pointed out that Fröhlich and Paranjape[2] have stressed that *e-e* collisions are important in bringing about a thermal equilibrium among electrons in strong electric fields. The present problem, however, is different. Fröhlich and Paranjape have considered strong electric fields, where *e-e* collisions lead to an energy exchange among electrons. They have shown qualitatively that, above a critical electronic density, energy exchange among electrons is more effective in bringing about a Maxwellian distribution through *e-e* interaction than through the lattice-electron interaction. We are not concerned here with the energy exchange. Since the resistivity and the Hall coefficients are defined in the limit of vanishingly small electric fields, the question of energy exchange among electrons does not arise. A small average momentum produced in the direction of applied electric field is greater for low-energy electrons than that for the high-energy electrons, because their times of relaxation with the lattice are large and small, respectively. The effect of *e-e* collisions, in this case, is to bring about an equality in the directed momentum of all electrons.

CALCULATION

In the presence of an external electric field with components E_x and E_y along the x and y axis and a magnetic field along the z axis, Boltzmann's equation can be written as

$$[eE_x/m+(eH/mc)v_y]\partial f/\partial v_x + [eE_y/m-(eH/mc)v_x]\partial f/\partial v_y = \partial f/\partial t|_{\text{collisions}}. \quad (1)$$

Here, $(eH/mc)=\omega$ is the angular frequency of electrons of effective mass m; $(\partial f/\partial t)_{\text{collisions}}$ is the rate of change of the distribution function f through collisions with the lattice vibrations and through *e-e* collisions. We can split this term into two terms as

$$\left.\frac{\partial f}{\partial t}\right|_{\text{collisions}} = \left.\frac{\partial f}{\partial t}\right|_l + \left.\frac{\partial f}{\partial t}\right|_e, \quad (2)$$

where the suffix l and e stands for the collisions with the lattice and electrons, respectively. The effect of lattice collisions can be written in terms of a relaxation time $\tau_l(\epsilon)$, the mean time between two successive collisions

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[1] Joachim Appel, Phys. Rev. **122**, 1760 (1961).
[2] H. Fröhlich and B. V. Paranjape, Proc. Phys. Soc. (London) **B69**, 21 (1956).

for electrons of energy ϵ. Thus,

$$\left.\frac{\partial f}{\partial t}\right|_l = -\frac{f-f_0}{\tau_l(\epsilon)}, \tag{3}$$

where f_0 is the equilibrium distribution function.

It is not possible without some justification, to write a similar equation for the effect of *e-e* collisions. It is precisely this difficulty that has limited our understanding of the effect of *e-e* collisions. We make no effort to study this term from the point of view of the Coulomb interaction. We note that, if in a sample carrying a current along x, we could switch off all lattice-electron interactions and external forces acting on electrons, which have at the switch-off time a disturbed Maxwellian distribution, the *e-e* collisions will eventually make it $\bar{f}_0$, which is a Maxwellian distribution displaced in the momentum space along x axis. The displacement in the momentum space will be such that the total current carried by the electrons at the switch-off time will remain unchanged by *e-e* collisions.

We assume that the rate of approach to $\bar{f}_0$ is

$$\left.\frac{\partial f}{\partial t}\right|_e = -\frac{f-\bar{f}_0}{\tau_e}. \tag{4}$$

Clearly $1/\tau_e$ will increase if *e-e* interactions are stronger. Thus, if we could determine τ_e experimentally, we would get some idea of the strength of Coulomb interaction. Of course, τ_e should be a function of energy, but our present knowledge is not adequate to justify any energy dependence of τ_e. We have therefore treated it as a constant, and we believe that the experimental results will give us orders of magnitude of this quantity τ_e. The equilibrium distribution is given by

$$f_0 = n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\epsilon/kT} \equiv A e^{-\alpha v^2}, \tag{5}$$

where n is the number density of electrons.

$$\bar{f}_0 = A\exp\{-\alpha[(v_x-u)^2+v_y^2+v_z^2]\} = f_0\exp(2\alpha u v_x - \alpha u^2). \tag{6}$$

Here, u is related to the electric current:

$$I_x = ne\bar{v}_x = neu. \tag{7}$$

Since we are interested in the limit $u\to 0$, we can expand the exponential in Eq. (6) and neglect all terms except those linear in u. Thus,

$$\bar{f}_0 \approx f_0(1+2\alpha u v_x). \tag{8}$$

Define

$$1/\tau = 1/\tau_l + 1/\tau_e; \tag{9}$$

and rearrange terms in Eq. (1), which then becomes

$$\left(\frac{eE_x}{m}+\omega v_y\right)\frac{\partial f}{\partial v_x} + \left(\frac{eE_y}{m}-\omega v_x\right)\frac{\partial f}{\partial v_y} = -\frac{f-f_0}{\tau} + \frac{2\alpha u v_x f_0}{\tau_e}, \tag{10}$$

where ω is the angular frequency. If we assume

$$f = f_0 + v_x f_1 + v_y f_2, \tag{11}$$

and equate the coefficients of v_x and v_y, we can then solve for f_1 and f_2 and obtain

$$f_1 = \frac{\tau f_0}{kT}\left(\frac{1}{1+(\omega\tau)^2}\right)\left[eE_x + \frac{mu}{\tau_e} + \omega\tau eE_y\right]. \tag{12}$$

$$f_2 = \frac{\tau f_0}{kT}\left(\frac{1}{1+(\omega\tau)^2}\right)\left[eE_y - \omega\tau\left(eE_x + \frac{mu}{\tau_e}\right)\right]. \tag{13}$$

ELECTRICAL RESISTIVITY AND HALL COEFFICIENT

Having solved for f_1 and f_2, we can get all quantities of interest. Both in electrical resistivity and in Hall coefficient measurements, we have the condition in which there is no net electric current along the y direction.

We thus have

$$I_x = e\int v_x^2 f_1 d^3v = neu, \tag{14}$$

$$I_y = e\int v_y^2 f_2 d^3v = 0. \tag{15}$$

Assuming that the magnetic fields are not too large so that $(\omega\tau)^2 < 1$, we can neglect $(\omega\tau)^4$ in the expansion of the denominator in Eqs. (12) and (13). Using further the known energy dependence of

$$\tau_l = \tau_0(\epsilon/kT)^{-\frac{1}{2}} \equiv \tau_0 x^{-\frac{1}{2}},$$

denoting

$$(\tau_0/\tau_e) = b,$$

and substituting in Eqs. (14) and (15), we have

$$\frac{3\pi^{\frac{1}{2}}m}{4ne\tau_0}I_x = L_1 eE_x + \omega\tau_0 K_2 eE_y + bL_1(mu/\tau_0) = \frac{3\pi^{\frac{1}{2}}m}{4ne\tau_0}neu, \tag{16}$$

$$\frac{3\pi^{\frac{1}{2}}m}{4ne\tau_0}I_y = L_1 eE_y - \omega\tau_0 L_2 eE_x - \omega\tau_0 bL_2\frac{mu}{\tau_0} = 0, \tag{17}$$

where

$$K_p = \int_0^\infty \frac{x^{\frac{3}{2}}e^{-x}dx}{(b+x^{\frac{1}{2}})^p} \tag{18}$$

and

$$L_p = K_p - (\omega\tau_0)^2 K_{p+2}. \tag{19}$$

Substituting $neu = I_x$, we calculate the resistivity in the presence of a magnetic field by eliminating E_y from (16) and (17):

$$\rho = \frac{E_x}{I_x} = \frac{m}{ne^2\tau_0}\frac{[(3\pi^{\frac{1}{2}}/4) - bL_1]L_1 - (\omega\tau_0)^2 bK_2L_2}{L_1^2 + (\omega\tau_0)^2 K_2 L_2}. \tag{20}$$

In the case when $H=0$,

$$\rho_0=\frac{m}{ne^2\tau_0}\frac{(3\pi^{\frac{1}{2}}/4)-bK_1}{K_1}, \tag{21}$$

$$\frac{\Delta\rho}{\rho_0}=\frac{\rho-\rho_0}{\rho_0}$$

$$=(\omega\tau_0)^2\left[\frac{K_3}{K_1}-\frac{K_2^2}{K_1^2}+b\frac{K_1K_3-K_2^2}{[(3\pi^{\frac{1}{2}}/4)-bK_1]K_1}\right]. \tag{22}$$

The Hall coefficient, $R=E_y/HI_x$, is obtained by eliminating E_x from (16) and (17). Thus,

$$R=\frac{1}{nec}\frac{(3\pi^{\frac{1}{2}}/4)L_2}{L_1^2+(\omega\tau_0)^2K_2L_2}. \tag{23}$$

Define R_0 as R in the limit $H\to 0$, i.e., independent of H:

$$R_0=\frac{(3\pi^{\frac{1}{2}}/4)}{nec}\frac{K_2}{K_1^2}. \tag{24}$$

We have

$$\frac{\Delta R}{R_0}=\frac{R-R_0}{R_0}=(\omega\tau_0)^2\left[\frac{2K_3}{K_1}-\frac{K_2^2}{K_1^2}-\frac{K_4}{K_2}\right]. \tag{25}$$

Equations (21), (22), (24), and (25) all contain $b=\tau_0/\tau_e$ and in the limit $b\to 0$, i.e., $\tau_e\to\infty$ these formulas reduce to the usual equations found in standard books on electron theories. Actually, the experiments are in reasonable agreement with the usual theory. We therefore expect that our formulas should give a small correction due to *e-e* collisions.

DISCUSSION

The integrals K_p are worked out in the Appendix. We notice that K_4 depends more strongly on b than any of the other K's. For very small values of $b\sim 10^{-2}$, $K_4\sim\pi^{\frac{1}{2}}+8b\ln b$. Since K_4 occurs only in $\Delta R/R_0$; we should expect the deviations from the usual theory to be easily noticeable in the measurement of this quantity.

Physically, one has good reasons to expect this. The flow of an electric current in velocity space (*xz* plane) is represented in Fig. 1. Electrons of nearly the mean energy $\bar{\epsilon}$ have a mean displacement in the velocity space OM. Electrons with $\epsilon>\bar{\epsilon}$ are displaced by OH ($OH<OM$), and electrons of energy $\epsilon<\bar{\epsilon}$ are displaced by $OL>OM$ (mobility of electrons increases as their energy decreases). Since in *e-e* collisions energy and momentum are conserved, the net current carried by all the electrons is not affected directly. Electron-electron collisions produce a greater uniformity in the mean current per particle in the entire energy range, making it behave like a jelly in the velocity space. Thus, the low-energy electrons which have a mean velocity $OL>OM$ effectively try to push all other electrons; and the high-energy electrons which have a mean velocity

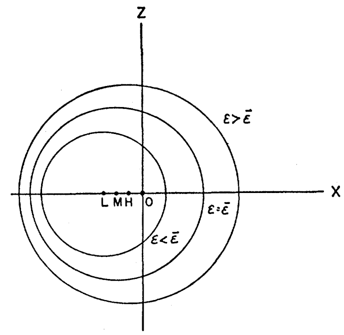

FIG. 1. Qualitative diagram of the *xz* plane in the velocity space of the electrons, when an external electric field is present in the x direction. The three circles with their centers at L, M, and H are the cross sections of equal-energy surfaces in the absence of an external field. Their displacements OL, OM, and OH indicate, qualitatively, their contributions to the electric current.

$OH<OM$ tend to slow down all other electrons. The net result, which is obtained by appropriate averaging, gives a resistivity which differs from the standard value ρ_s ($\rho_s=\lim_{b\to 0}\rho$). Using (21) and (A9), we have

$$\rho-\rho_s\approx\rho_s b\left(\frac{\pi^{\frac{1}{2}}}{2}-\frac{4}{3\pi^{\frac{1}{2}}}\right)\approx 0.07\rho_s b. \tag{26}$$

$\rho-\rho_s$ is in the same direction as predicted by Appel.[1]

The changes in R_0 and $\Delta\rho/\rho_0$ can be understood from the nature of the averaging that has to be done in obtaining these results.

The *yz* plane in velocity space (see Fig. 2) looks completely different from the *xz* plane. In the presence of a magnetic field along the z axis, electrons of energy ϵ are subject to a Lorentz force which depends on the velocity of the electrons. Its average value is $e\langle\mathbf{v}(\epsilon)\rangle_{\mathrm{av}}\times\mathbf{H}/c$. In addition to the Lorentz force there is the Hall field E_y which produces a force eE_y independent of the velocity. Since the Lorentz force depends on the mean current carried by the electrons, it is stronger for electrons of energy $\epsilon<\bar{\epsilon}$ and weaker for those with $\epsilon>\bar{\epsilon}$ than it is for electron of energy $\bar{\epsilon}$. The force produced by the Hall field is equal to the Lorentz force on an electron of roughly the energy $\bar{\epsilon}$. Thus, electrons of high energy are moving in a direction opposite to the direction in which the low-energy electrons move in the y direction. The experimental conditions further require that there should be no current along the y direction. Thus, in the *yz* plane, electrons of high energy and low energy are displaced in the velocity space in opposite directions. Actually, E_y is fixed by requiring that there is no current along y, and the statement that $neE_y=\mathbf{I}_x\times\mathbf{H}/c$ is only correct in the first order. Thus, the second-order corrections to E_y and, hence, to the Hall constant depend on the displacements OL and OH in the *yz* plane. Electron-electron collisions play a very important role in reducing both OL and OH to zero, because here the particles are moving in opposite directions. We, thus, expect that a

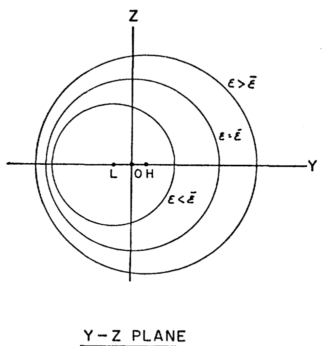

FIG. 2. Qualitative diagram of the yz plane in the velocity space of the electrons, when E_x, E_y, and the magnetic field are simultaneously present. The three circles with their centers at L, O, and H are the cross sections of equal energy surfaces in the absence of external fields. The relative displacements OL and OH are in opposite directions.

study of the second-order correction to the Hall constant (i.e., $\Delta R/R_0$) should give crude estimates of b.

In semiconductors the number density of electrons can be easily varied by orders of magnitude, which should change τ_e but not τ_l. Thus, a systematic study of $\Delta R/R_0$ with varying concentration of carriers should give considerable information about b.

The scattering of electrons by impurities is not considered in the present paper. Therefore, the results are applicable only in the region of temperatures where electrons interact with the lattice more strongly than they do with the impurities. The calculation can, of course, be easily extended to include the case of impurity scattering.

The calculation can easily be extended to include metals, where the number density of electrons remains constant, and hence so does τ_e. At sufficiently low temperatures τ_0 can be altered, because the impurity scattering becomes predominant at low temperatures. The impurity scattering can be changed by taking different samples of varying impurities. The present lack of experimental data does not permit even a crude estimate of b.

Note added in proof. The authors wish to thank the referee for pointing out a paper by Robert W. Keyes, J. Phys. Chem. Solids **6**, 1 (1958), where a similar calculation is made.

APPENDIX

$$K_1=\int_0^\infty \frac{x^{\frac{3}{2}}e^{-x}}{b+x^{\frac{1}{2}}}dx \quad \text{which} \quad =\int_0^\infty \frac{2y^4e^{-y^2}}{b+y}dy.$$

K_2, K_3, and K_4 are related to K_1 by the relations

$$K_2=-\partial K_1/\partial b, \quad K_3=-\tfrac{1}{2}(\partial K_2/\partial b),$$

and

$$K_4=-\tfrac{1}{3}(\partial K_3/\partial b).$$

TABLE I. The functions K_1, K_2, K_3, and K_4, evaluated numerically for values of $b\sim 10^{-2}$ with the help of the IBM 1620.

b	0.02	0.04	0.06	0.08
K_1	0.98	0.97	0.95	0.93
K_2	0.85	0.81	0.78	0.75
K_3	0.91	0.83	0.77	0.72
K_4	1.35	1.13	0.97	0.85

Thus, it is adequate to work with the K_1 integral:

$$K_1=2b^4\int_0^\infty \frac{e^{-b^2x^2}}{1+x}x^4dx. \tag{A1}$$

Integrating by parts, we obtain

$$K_1=2b^4\int_0^\infty [\tfrac{1}{4}x^4-\tfrac{1}{3}x^3+\tfrac{1}{2}x^2-x+\ln(1+x)] \times 2b^2xe^{-b^2x^2}dx \tag{A2}$$

$$=1-\tfrac{1}{2}\pi^{\frac{1}{2}}b+b^2-\pi^{\frac{1}{2}}b^3+4b^6\int_0^\infty \ln(1+x)e^{-b^2x^2}xdx. \tag{A3}$$

We need the integral

$$I=\int_0^\infty \ln(1+x)e^{-b^2x^2}xdx. \tag{A4}$$

Partial integration gives us

$$I=2b^2\int_0^\infty xe^{-b^2x^2}[\tfrac{1}{2}(x^2-1)\ln(1+x)+1+\tfrac{1}{2}x-\tfrac{1}{4}x^2]dx \tag{A5}$$

Rearrangement of terms results in

$$\frac{\partial I}{\partial b^2}+I\left(1+\frac{1}{b^2}\right)=\frac{1}{b^2}+\frac{\pi^{\frac{1}{2}}}{4b^3}-\frac{1}{4b^4}. \tag{A6}$$

The integrating factor is

$$e^{b^2}b^2=\exp\left\{-\left[\int^{b^2}\left(1+\frac{1}{b^2}\right)db^2\right]\right\}.$$

Thus,

$$Ie^{b^2}b^2=\int^{b^2}\left[\frac{1}{b^2}+\frac{\pi^{\frac{1}{2}}}{4b^3}-\frac{1}{4b^4}\right]e^{b^2}b^2db^2+C, \tag{A7}$$

where C is the constant of integration. For small values of b, the exponential can be replaced by unity, and the result is

$$I\approx(1/b^2)[C-\tfrac{1}{2}\ln b]. \tag{A8}$$

Thus,

$$K_1=1-\tfrac{1}{2}\pi^{\frac{1}{2}}b+b^2-\pi^{\frac{1}{2}}b^3-4b^4[C-\tfrac{1}{2}\ln b]. \tag{A9}$$

The constant C can be determined by numerically evaluating K_4. Its value is found to be ~ -0.4.

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