# Effective-Field Approach to the Problem of Interacting Polarized Particles

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The potential interaction energy of a cubic lattice of aligned polarized spheroids, of a random assembly of aligned polarized spheroids, and of a random assembly of arbitrary particles is calculated making use of the "effective-field approach." In particular the dependence of the energy on the packing density of the particles is considered. Only systems polarized to saturation are treated.

## INTRODUCTION

IN electrostatics and magnetostatics it is often desired to know the total amount of potential energy stored in the interacting fields of a system of polarized particles. This problem has been treated by Néel,[1] who stated that for a magnetically saturated system of particles of arbitrary shape and random distribution the magnetostatic potential energy is a symmetrical function in $\delta$ and $1-\delta$, where $\delta$ is the partial volume occupied by the particles. In fact Néel supposed the potential energy to be proportional to $\delta(1-\delta)$. This will certainly hold for very dilute or very dense systems, but in general the possible occurrence of higher-order terms cannot be excluded *a priori*.

The present paper gives a more detailed treatment of a similar problem of polarized spheroidal and other particles and proves Néel's relation to be valid for all densities. The problem of interacting spheroids, polarized to saturation, has already been treated by Wohlfarth,[2] who started by replacing the field due to surrounding spheroids acting upon one particular spheroid by the field due to a surrounding medium with homogeneous magnetization equal to the average magnetization of the system. The spheroid in question is assumed to have the same potential energy in this field as it actually has in the fields of the surrounding spheroids. However, this approach, which is called the "effective-field approach," is not correct when applied in this simple form, since the averaging is only valid at large distances from the spheroid in question: The probability of finding one spheroid at a given short distance from another will obviously be influenced by the presence of the latter spheroid. Wohlfarth realized this difficulty in the course of his calculation and rejected the effective-field approach in favor of one from first principles. Another possible approach, which will be adopted in the present paper, is to refine the effective-field method by assuming the magnetization of the imaginary homogeneous medium to be different near the spheroid. This will be applied to systems polarized to saturation (zero susceptibility).

[1] L. Néel, Compt. rend. **224**, 1550 (1947). See also: Z. Hashin and S. Shtrikman, J. Franklin Inst. **271**, 423 (1961).
[2] E. P. Wohlfarth, Proc. Roy. Soc. (London) **A232**, 208 (1955).

## A RANDOM SYSTEM OF ALIGNED SPHEROIDS

Let us consider a random system of conformal prolate spheroids with their axes of revolution parallel to each other. They are uniformly polarized with a moment $\mathbf{I}_s$ per unit volume of the spheroids, and the partial volume occupied by them is $\delta$. The whole system has the shape of a long slender bar with longitudinal polarization (or of a toroid with circular polarization), so that there is no field from poles of the system itself. It is further supposed that all polarization vectors are parallel, which is always the case when the system is polarized by a saturating external field, and directed along the axis of revolution of the spheroids.

Now the total energy $E_{\text{tot}}$ of the system due to the fields of the spheroids is given by

$$E_{\text{tot}}=\frac{1}{8\pi}\int_{\infty}H^2dV,$$

where $H$ is the field strength due to the particles and $dV$ is a volume element. The integration is performed over all space. For such a finite system of polarized particles

$$\int_{\infty}\mathbf{H}\cdot\mathbf{B}dV=0,$$

where $\mathbf{B}$ is the induction, or the sum of polarization and field strength [$\mathbf{B}=\mathbf{H}+4\pi\mathbf{I}_s$ (nonrationalized units)]. From this it can be shown that

$$E_{\text{tot}}=-\tfrac{1}{2}\int_{V_1}\mathbf{H}\cdot\mathbf{I}_s dV, \tag{1}$$

where $V_1$ is the partial volume occupied by the spheroids.

The field strength $\mathbf{H}$ inside a spheroid consists of two components. The first is the depolarizing field $\mathbf{H}_1$ of the spheroid due to its own poles:

$$\mathbf{H}_1=-N\mathbf{I}_s, \tag{2}$$

with $N$ the depolarizing factor which is a scalar independent of coordinates inside the spheroid. The other spheroids provide the other component $\mathbf{H}_2$ of the internal field of the spheroid under consideration. This component $\mathbf{H}_2$ is calculated as follows:

The average polarization $\mathbf{I}$ of the system is given by

$$\mathbf{I}=\mathbf{I}_s\delta.$$

In the neighborhood of the spheroid under consideration the probability of finding some point of another spheroid will be less than $\delta$ and is in fact given by a distribution function $\rho(\mathbf{r})$. This function $\rho(\mathbf{r})$ is equal to $\delta$ at a large distance from the spheroid under consideration and is equal to zero on the surface of this spheroid. If we consider the system of conformal spheroids as an affine transformation of the corresponding random system of spheres it will be clear that the function $\rho(\mathbf{r})$ has a constant value on a spheroidal surface, which is homothetic and concentric with the considered spheroid (and thus has the same axial ratio and orientation). Now a thin shell bounded by two of the homothetic and concentric spheroidal surfaces will be considered. Inside this layer $\rho(\mathbf{r})$ has a constant value. Thus the statistical average of the polarization has also a constant value throughout the shell. Since a uniformly polarized shell bounded by two homothetic spheroidal surfaces gives no contribution to the field inside the inner boundary, the exact form of $\rho(\mathbf{r})$ is of no interest. It is thus sufficient to consider the spheroid in question as situated inside a relatively large hole in a medium of average polarization $\mathbf{I}$, which has the shape of a long bar. The hole has the shape of a spheroid which is homothetic and concentric with the spheroid under consideration. Inside this hole, which has the same depolarizing factor $N$ as the spheroid itself, the field component $\mathbf{H}_2$ is given by

$$\mathbf{H}_2 = N\mathbf{I} = N\delta\mathbf{I}_s. \tag{3}$$

If $\mathbf{H}_1$ and $\mathbf{H}_2$ are substituted into Eq. (1) we find

$$E_{\text{tot}} = \tfrac{1}{2}N\mathbf{I}_s^2(1-\delta)V_1,$$

and, per unit volume of the system,

$$E = \tfrac{1}{2}N\mathbf{I}_s^2(1-\delta)\delta, \tag{4}$$

which is the potential energy due to the dipolar fields of the spheroids per unit volume of the randomly packed system.

## ORDERED SYSTEMS OF ALIGNED SPHEROIDS

When the spheroids are situated on the lattice points of a cubic lattice the spheroid under consideration may be thought to be inside a hole with cubic symmetry, provided that $\rho(\mathbf{r})$ has cubic symmetry. This will be so when the spheroids are small, or in the special case where they are spheres. Then the field component $\mathbf{H}_2$ at the center of the hole is given by

$$\mathbf{H}_2 = \tfrac{4}{3}\pi\mathbf{I}_s\delta \tag{5}$$

and we find by substitution of $\mathbf{H}_1$ of Eq. (2) and $\mathbf{H}_2$ of Eq. (5) into Eq. (1):

$$E = \tfrac{1}{2}N\mathbf{I}_s^2(1-\tfrac{4}{3}\pi\delta/N)\delta. \tag{6}$$

When the polarized particles are spheres, $N = \frac{4}{3}\pi$ and Eq. (6) reduces to Eq. (4), in agreement with Néel's assumption and with the simple effective field method as used by Wohlfarth.[2] In general, however, Eq. (4) is not applicable to ordered systems.

When the ordered array is not cubic, the effective-field approach does not apply at all. The field component $\mathbf{H}_2$ can then be calculated with the help of lattice sums, as has been done by Mueller[3] and Nijboer and de Wette.[4]

## A RANDOM SYSTEM OF ARBITRARILY SHAPED PARTICLES

Let us consider a random distribution of arbitrarily shaped particles. The depolarizing field $\mathbf{H}_1$ at any point inside a particle of arbitrary shape can again be written as

$$\mathbf{H}_1 = -N_1\mathbf{I}_s. \tag{2'}$$

Now, however, $N_1$ is a tensor dependent on the coordinates of the point in question. The tensor $N_1$ averaged over all orientations of the particle with respect to the fixed direction of magnetization is equal to $\frac{4}{3}\pi$ times the unit tensor.

Similarly, the field component $\mathbf{H}_2$ at the same point, for a fixed orientation of the particle, averaged over all distributions of the other particles, is again given by

$$\mathbf{H}_2 = N_2\mathbf{I}_s\delta. \tag{3'}$$

The value of the tensor $N_2$ averaged over all orientations of the particle under consideration is also equal to $\frac{4}{3}\pi$, since the function $\rho(\mathbf{r})$ becomes isotropic owing to this averaging. Substitution of the average values of $\mathbf{H}_1$ and $\mathbf{H}_2$ into Eq. (1) gives

$$E = \tfrac{2}{3}\pi\mathbf{I}_s^2(1-\delta)\delta,$$

which shows that the result already obtained by Néel holds exactly for all values of $\delta$.

[3] H. Mueller, Phys. Rev. **47**, 947 (1935); **50**, 547 (1936).
[4] B. R. A. Nijboer and F. W. de Wette, Physica **24**, 422 (1958).