Electrostatic Corrections to Nucleon-Nucleon Dispersion Relations*

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(Received January 15, 1962)

Assuming that a nonrelativistic wave-function description of the nucleon-nucleon scattering system is valid at low energy and large distances, we investigate the analytic properties of the nuclear (i.e., total minus Coulomb) partial wave scattering amplitudes as a function of the c.m. momentum q. For singlet states, we find that the essential singularity at $q^2=0$ can be treated exactly in the partial wave dispersion relation. The remaining singularities are branch points located at the same position as for the non-Coulomb case. The discontinuity across the one-pion exchange cut is a simple multiplicative factor (which goes to unity as e^2 goes to zero) times the discontinuity in the absence of Coulomb scattering. From this structure we derive an integral equation for the partial wave scattering amplitude.

We investigate the consequences of these corrections in the $^1S_0$ state. For $q^2<-\mu^2$, there will be additional electrodynamic corrections to the interaction produced by multipion exchange, which we cannot compute. However, these should be at most of the order of the $3\frac{1}{2}\%$ mass difference between charged and neutral pions. Our statement of the charge independence hypothesis is therefore that any phenomenological treatment of the multi-pion exchange should differ by at most $3\frac{1}{2}\%$ between n-p and p-p scattering. Representing this interaction by a single pole, but including excatly single-pion exchange, and the charged-neutral pion mass difference in that exchange, we show that low-energy $^1S_0$ n-p and p-p scattering are consistent with charge independence. This result is firmer than that previously obtained with phenomenological potential models because of our *a priori* estimate of residual charge-dependent effects. Assuming exact charge symmetry, we predict an n-n scattering length of -27 ± 1.4 f. Since a departure from charge symmetry of $\pm5\%$ would vary this prediction between -18 f and -53 f, we conclude that a measurement of this quantity will provide a sensitive quantitative test of charge symmetry.

I. INTRODUCTION

SINCE Mandelstam's[1] proposal of a two-dimensional representation for elastic scattering amplitudes, it has become feasible for dispersion relations to play the role of a dynamical theory for elastic scattering processes. For the nucleon-nucleon problem, several authors have formulated various schemes of calculation.[2,3] So far, electromagnetic interaction between the nucleons has been neglected. However, it is well known that, at low energies, the static Coulomb effect for p-p scattering is quite important. It is therefore highly desirable that one incorporates, at least, the static Coulomb correction into the dispersion theory framework. We shall show that the "outer part" of the Coulomb force can be taken into account exactly, including the Coulomb correction on the one-pion exchange force. Our study of the one-pion force also indicates that the Coulomb effect on the two-pion and shorter-range forces is quite negligible. We can therefore relate the p-p, n-n, and n-p phase shifts if charge independence of the nuclear force holds. Conversely, experimental information can be inserted into the dispersion relation for a test of charge independence. In this paper, only the singlet amplitudes are considered. The present approach should be applicable to the triplet problem with no difficulty in principle.

Although the "nuclear" phase shift is defined as "total minus Coulomb," there is still an essential singularity at zero kinetic energy. However, it is possible to construct a function of the "nuclear" phase shift which is free of singularities in the neighborhood of zero kinetic energy. In Sec. II, we shall arrive at this function of the "nuclear" phase shift by studying the low-energy limit of Coulomb wave functions.

To investigate further the analytic structure of the partial wave amplitudes, we consider the example of an Yukawa potential $(-f^2\exp(-\mu r)/r)$ plus the Coulomb potential. The partial wave amplitudes are shown to have branch cuts starting at $q^2=-n^2\mu^2/4$, $n=1, 2, \cdots$, as expected. The discontinuity across the first branch cut is linear in the coupling constant f^2 and is identified as the one-meson exchange term since such an identification is known to be correct in the absence of the Coulomb force. The calculation of the exact discontinuity across the one-meson cut is given in Sec. III. Except near $q^2=-\mu^2/4$, it differs from the no Coulomb case by less than 5% over most of the negative axis (in q^2). Beyond the second branch point $(q^2=-\mu^2)$, this difference is less than 3%, reflecting the fact that the Coulomb effect becomes less significant in the short-range region. In Sec. IV, we assume the branch cut on the negative axis to be known and construct the partial wave amplitude in terms of the solution of an integral equation which also guarantees the correct threshold behavior. In Sec. V we discuss the charge independence of the $^1S_0$ state. If the branch cut along the negative q^2 axis is assumed to be approximately equal for the n-p and p-p singlet amplitudes, then a simple effective-range formula gives the difference of the n-p and p-p scattering lengths and effective ranges in fair agreement with experimental values. A more careful study of the charge

* Work supported in part by the U. S. Atomic Energy Commission.

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[2] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).
[3] D. Amati, E. Leader, and B. Vitale, Nuovo cimento **17**, 68 (1960); **18**, 409, 458 (1960).

independence treats the one-meson cut exactly, and the charge and neutral pion mass-difference is also taken into account. The "far away" cuts (two-pion, three-pion exchanges, etc.) are assumed to be identical for the n-p, p-p, and n-n states. Numerical results are given near the end of Sec. V.

II. ANALYTIC STRUCTURE OF "NUCLEAR" PHASE SHIFTS NEAR $q^2=0$

For sufficiently small kinetic energy in the c.m. system, a wave-function description of the two-nucleon system at large distances can be written as

$$U_l(q^2,r)=N[\cot\delta_l(q^2)U_R(q^2,r)+U_I(q^2,r)], \quad (2.1)$$

where U_l is the wave function for the lth angular momentum state (we only consider uncoupled states in this paper, although a generalization to include coupled states should not be too difficult), U_R and U_I are the regular and irregular Coulomb wave functions, respectively, N is a normalization factor, and δ_l is the "nuclear" phase shift. Asymptotically,

$$\begin{aligned} U_R &\to \sin[qr-\tfrac{1}{2}\pi l-\eta\ln(2qr)+\phi_l], \\ U_I &\to \cos[qr-\tfrac{1}{2}\pi l-\eta\ln(2qr)+\phi_l], \end{aligned} \quad (2.2)$$

where $\eta=(Me^2/2q)$ and ϕ_l is the phase of $\Gamma(l+1+i\eta)$.

The connection between U_R, U_I, and the confluent hypergeometric functions is given by[4]

$$U_R=Ae^{-x/2}x^{c/2}\Phi(a,c;x), \quad (2.3)$$

$$U_I=Ae^{-x/2}x^{c/2}\left[\frac{2\Gamma(2l+2)}{\Gamma(l+1-i\eta)}\times e^{\pi\eta-i\pi(l+\frac{1}{2})}\Psi(a,c;x)-i\Phi(a,c;x)\right], \quad (2.4)$$

where $a=l+1+i\eta$, $c=2l+2$, $x=-2iqr$, and

$$A=\tfrac{1}{2}[\Gamma(l+1-i\eta)/\Gamma(2l+2)]\exp[\tfrac{1}{2}i\pi(l+1)+i\phi_l-\tfrac{1}{2}\pi\eta].$$

From Eqs. (2.3) and (2.4) one can see that U_R is an analytic function of q at $q=0$ whereas U_I has an essential singularity at that point. Since $U_l(q^2,r)$ defined by (2.1) must be reducible to a combination of the zero-energy Coulomb wave functions as $q\to 0$, one sees that $\cot\delta_l(q^2)$ must also have an essential singularity at $q=0$ in order to compensate for the singularity in U_I. It was shown by Breit, Condon, and Present[5] that a more convenient function of the "nuclear" phase shift in the zero-energy limit is given by $[C^2q\cot\delta_l+Q]$ where

$$\begin{aligned} C^2 &= 2\pi\eta/[\exp(2\pi\eta)-1], \\ Q &= Me^2[\tfrac{1}{2}\psi(i\eta)+\tfrac{1}{2}\psi(-i\eta)-\ln\eta], \\ \psi(\pm i\eta) &= \Gamma'(\pm i\eta)/\Gamma(\pm i\eta). \end{aligned} \quad (2.5)$$

In fact, $[C^2q\cot\delta_l+Q]$ is analytic at $q=0$. We shall prove the analyticity as follows. First, we rewrite Eq. (2.1) in the form

$$U_l(q^2,r)=N'[(C^2q\cot\delta_l+Q)U_R(q^2,r)+U_I'(q^2,r)], \quad (2.6)$$

with $N'=N/C^2q$ and $U_I'=C^2qU_I-QU_R$.

From Eqs. (2.3) and (2.4), one obtains the zero-energy limit

$$\begin{aligned} U_R(q^2,r) &\to A_0(i\eta)^{-l-1}yI_{2l+1}(y), \\ U_I'(q^2,r) &\to -A_0(i\eta)^{-l-1}(2Me^2)yK_{2l+1}(y), \end{aligned} \quad (2.7)$$

where $y=2(e^2Mr)^{\frac{1}{2}}$; I_{2l+1} and K_{2l+1} are Bessel functions of the second kind.

It can be shown that $yI_{2l+1}(y)$ and $yK_{2l+1}(y)$ are indeed the regular and irregular zero-energy Coulomb wave functions, respectively. Furthermore the limit given by (2.7) is independent of the path along which q approaches zero. Thus $(C^2q\cot\delta_l+Q)$ must be regular at $q=0$. A close examination of the hypergeometric functions, however, reveals that $U_I(q^2,r)$ contains a sequence of poles at $i\eta=-1, -2, \cdots, -l$. Finally, we are led to consider

$$S_l(q^2)=\prod_{\lambda=1}^{l}(1+\eta^2/\lambda^2)q^{2l}(C^2q\cot\delta_l+Q) \quad (2.8)$$

as the function which possesses the maximal analyticity in the neighborhood of $q=0$. The requirement of time reversal invariance further restricts S_l to be an even function of q. Hence S_l is analytic in q^2 except for singularities arisen from "nuclear" interactions ($q^2\leqslant-\mu^2/4$) and possible zeroes in the phase shift.

III. ANALYTIC STRUCTURE OF SCATTERING AMPLITUDES FROM A COULOMB PLUS A YUKAWA POTENTIAL

We have already made use of the fact that at low energy the asymptotic wave function can be approximated by the nonrelativistic Schrödinger Coulomb wave functions to find what function of the phase shift has no singularities near $q^2=0$. We now carry this analysis further by noting that the singularity due to single-pion exchange is the same (except for relativistic kinematic factors) as that given by the Born approximation for the static Yukawa potential of strength $-f^2$. We therefore feel justified in assuming that if we can calculate the static Coulomb modification of this singularity from the Schrödinger equation, our result will be correct except for v^2/c^2 relativistic corrections and nonstatic radiative corrections of order $e^2/\hbar c$. This belief is strengthened by the result obtained below that this correction is given exactly by the first Born approximation; terms of higher order than f^2 contribute only to that portion of the singularity coming from multiparticle exchange.

The Schrödinger equation for a Coulomb plus a

[4] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1.

[5] G. Breit, E. U. Condon, and R. D. Present, Phys. Rev. **50**, 825 (1936).

Yukawa potential can be written as

$$U(r)=U_R(r)-f^2M\int_0^\infty G(r|r_0)(e^{-\mu r_0}/r_0)U(r_0)dr_0, \quad (3.1)$$

where the Green's function for the scattering problem is simply

$$G(r|r_0)=\frac{1}{q}\begin{cases} U_I(r)U_R(r_0); & r\geqslant r_0 \\ U_R(r)U_I(r_0); & r<r_0. \end{cases} \quad (3.2)$$

Consequently, we can write the tangent of the phase shift to first order in f^2M as

$$\tan\delta_l{}^B=-\frac{Mf^2}{4q}\frac{\Gamma(l+1+i\eta)\Gamma(l+1-i\eta)}{[\Gamma(2l+2)]^2}\times e^{-\pi\eta-i\pi(l+1)}I_l(q), \quad (3.3)$$

$$I_l(q)=\int_0^\infty (e^{-\mu r_0}/r_0)e^{2iqr_0}(-2iqr_0)^{2l+2}\times\Phi^2(l+1+i\eta;2l+2;-2iqr_0)dr_0.$$

Since we are interested in the singularity in this quantity for $q^2\leqslant-\mu^2/4$, we let $q=ik(i\eta=Me^2/2k)$. It is sufficient to consider only the upper half-plane of q (the physical sheet in q^2). The change of variable $t=2kr_0$ reduces the integral to a form which can be evaluated in terms of the ordinary hypergeometric function F.[4]

$$I_l(q)=\Gamma(2l+2)(\mu/2k)^{-2l-2-2i\eta}(1+\mu/2k)^{2i\eta}\times F(l+1+i\eta;l+1+i\eta;2l+2;4k^2/\mu^2). \quad (3.4)$$

Since F has a singular point at $4k^2/\mu^2=1$, we displace this by means of the identity,[4]

$$\begin{aligned}F(l+1+i\eta;l+1+i\eta;2l+2;4k^2/\mu^2)&\\ \equiv\Gamma(2l+2)\Gamma(-2i\eta)/[\Gamma(l+1-i\eta)]^2&\\ \times F(l+1+i\eta;l+1+i\eta;1+2i\eta;1-4k^2/\mu^2)&\\ +(1-4k^2/\mu^2)^{-2i\eta}\Gamma(2l+2)\Gamma(2i\eta)/[\Gamma(l+1+i\eta)]^2&\\ \times F(l+1-i\eta;l+1-i\eta;1-2i\eta;1-4k^2/\mu^2),&\quad (3.5)\end{aligned}$$

to obtain the result

$$\tan\delta_l{}^B=+\frac{Mf^2}{4q}\left(\frac{4k^2}{\mu^2}\right)^{l+1}\{\exp[-\pi\eta+i\pi(l+1)]\}\times\left[\left(1+\frac{2k}{\mu}\right)^{2i\eta}H(i\eta)+\left(1-\frac{2k}{\mu}\right)^{-2i\eta}H(-i\eta)\right], \quad (3.6)$$

$$H(i\eta)=\frac{\Gamma(l+1+i\eta)\Gamma(-2i\eta)}{\Gamma(l+1-i\eta)}\times F(l+1+i\eta;l+1+i\eta;1+2i\eta;1-4k^2/\mu^2). \quad (3.7)$$

We can now see directly that neither F nor the Γ functions are singular for $q^2\leqslant-\mu^2/4$, and that we must only find the discontinuity in $(1-2k/\mu)^{-2i\eta}$ as we approach the negative real q^2 axis from above. This is given by

$$[(1-2k/\mu)^{-2i\eta}]_{q=ik-\epsilon}^{q=ik+\epsilon}=\left(\frac{2k}{\mu}-1\right)^{-Me^2/k}\left(-2i\sin\frac{\pi Me^2}{k}\right). \quad (3.8)$$

Motivated by the result obtained in Sec. II, we now consider the function

$$A_l(q^2)=\frac{e^{i\delta_l}\sin\delta_l}{\Pi_lC^2q}=\frac{\tan\delta_l/\Pi_lC^2q}{1-i\tan\delta_l}, \quad (3.9)$$

with

$$\Pi_l\equiv\prod_{\lambda=1}^{l}(1+\eta^2/\lambda^2).$$

We note that to first order in f^2M, the discontinuity in this function for $q^2\leqslant-\mu^2/4$ is the same as that in $\tan\delta_l{}^B/\Pi_lC^2q$. Since in this range $i\eta=Me^2/2k<0.05$, we are justified in dropping terms of order η^2 to an accuracy of better than 1%. Making this expansion, we find that

$$\left[\frac{\tan\delta_l{}^B}{\Pi_lC^2q}\right]_-^+=-2\pi i\left(\frac{f^2M}{4k^2}\right)\left(\frac{2k}{\mu}-1\right)^{-Me^2/k}\times\left[P_l\left(1-\frac{\mu^2}{2k^2}\right)+\frac{Me^2}{k}\mathcal{P}_l\left(1-\frac{\mu^2}{2k^2}\right)\right], \quad (3.10)$$

where $\mathcal{P}_l$ is the polynomial given by

$$\mathcal{P}_l(z)=\left[\tfrac{1}{2}P_l(z)\ln\left(\frac{z+1}{z-1}\right)-Q_l(z)\right].$$

P_l and Q_l are Legendre functions of the first and the second kind, respectively. One sees that Eq. (3.10) reduces to the familiar expression for the one-meson cut in the zero-charge limit.

If we now look at the second Born approximation for $\tan\delta_l$, we can locate where singularities may occur for $q^2\leqslant-\mu^2/4$ by looking at the exponential factors in the double integral when the iterated wave function is inserted into the integral equation (3.1). The only such term for $-\mu^2<q^2$ turns out to be proportional to the square of $\tan\delta_l{}^B$. If we expand the denominator in (3.9) we find that the coefficient is such that this singularity is precisely canceled by the $(f^2M)^2$ term in the expansion; in fact, we can see that this cancellation persists to all orders in f^2M, just as occurs in the absence of Coulomb forces. We conclude, therefore, that we have succeeded in computing the entire modification of the single-particle exchange discontinuity in the amplitude (3.9) due to electrostatic interaction. We further note that this modification is numerically less than 3.5% for $q^2\leqslant-\mu^2$. Since there is no hope at present of computing the two-pion exchange discontinuity to this

accuracy, and since such a calculation would in any case be subject to charge-dependent corrections of the order of a few percent due to the $\pi^{\pm}-\pi^0$ mass and, possibly, coupling constant differences, we feel that this accuracy is more than adequate. We conclude that at this level, the problem of charge independence can be stated as that of showing that the phenomenological or theoretical discontinuity due to multi-pion exchange which reproduces experiment differs by less than 3.5% among neutron-proton, proton-proton, and neutron-neutron scattering in the same state.

So far our treatment applies only to singlet scattering. Since the static approximation for triplet scattering gives a $1/r^3$ singularity, only the first Born approximation converges, and the above treatment can be carried through only to first order in f^2M. If a cutoff is introduced,[6] the cancellation of higher-order terms would presumably go through as before. For the uncoupled states, the Coulomb modification is certainly the same as in the singlet case. In order to handle the coupled states it will be necessary to construct the analog of the singlet case in terms of a 2×2 matrix. Since $^3P_0$ and $^3P_1$ are uncoupled, and present data do not determine the $^3P_2-{}^3F_2$ parameters at low energy, we defer this treatment until experimental accuracy warrants it.

IV. INTEGRAL EQUATION FOR PROTON-PROTON SCATTERING

In the last two sections, we have seen that $S_l(q^2)$ is analytic in the neighborhood of $q^2=0$ while $A_l(q^2)$ is singular at $q^2=0$ but has branch cuts directly related to exchange forces. Our present task is to construct the phase shift assuming the branch cut for $A_l(q^2)$ on the negative q^2 axis is given. If we approximate the left-hand branch cut of $A_l(q^2)$ by a sequence of poles, then $S_l(q^2)$ is a rational function of q^2 with coefficients determined by the discontinuity across the branch cut. As the number of poles tends to infinity and the spacing between poles approaches zero, we obtain an integral equation similar to the N/D equation for the no-Coulomb problem.[7] We shall give explicit formulas only for the S wave. The generalization to higher angular momentum is quite straightforward.

For convenience we introduce the variable $x=-\mu_0^2/4q^2$ (μ_0=mass of the neutral pion); hence $i\eta=Me^2x^{\frac{1}{2}}$. In terms of this notation, the scattering amplitude becomes

$$A_{pp}(x)\equiv\frac{e^{i\delta_{pp}}\sin\delta_{pp}}{C^2q}=1/[S_{pp}(x)+\tfrac{1}{2}x^{-\frac{1}{2}}f(x)], \quad (4.1)$$

with

$$f(x)=2Me^2x^{\frac{1}{2}}\left[\frac{\pi}{2}\cot(\pi Me^2x^{\frac{1}{2}})+\gamma+\ln(Me^2x^{\frac{1}{2}}) -\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{n}{n^2-M^2e^4x}\right)\right]. \quad (4.2)$$

Here,

$$\gamma=0.57721\cdots,\quad Me^2=0.0507\quad(\mu_0c^2=135\text{ Mev}). \quad (4.3)$$

We have introduced this notation so that we can compare directly with n-p scattering by taking $f(x)=1$ ($Me^2=0$). For the same reason we define

$$(d/dx)[\tfrac{1}{2}x^{-\frac{1}{2}}f(x)]=-\tfrac{1}{4}x^{-\frac{3}{2}}g(x), \quad (4.4)$$

$$g(x)=Me^2\left[\frac{\pi^2Me^2x}{\sin^2(\pi Me^2x^{\frac{1}{2}})}-2x^{\frac{1}{2}}-\sum_{n=1}^{\infty}\frac{4M^2e^4x^{\frac{3}{2}}}{(n^2-M^2e^4x)^2}\right], \quad (4.5)$$

and the Coulomb modification of the cut

$$h(x)=(x^{-\frac{1}{2}}-1)^{2Me^2x^{\frac{1}{2}}}. \quad (4.6)$$

Our requirement on the function is that if the n-p amplitude has the discontinuity $2\pi\rho(x)$, then the p-p amplitude has the discontinuity $2\pi\rho(x)h(x)$ for $0\leqslant x\leqslant 1$. This Coulomb modification is exact for the one-pion cut and is less than 3% in the multipion region.

We now recall from Sec. II that $S_{pp}(x)$ has no singularities in the neighborhood of $q^2=0$. Therefore the essential singularity and the unitarity cut starting at $q^2=0$ must come entirely from the term $\frac{1}{2}x^{-\frac{1}{2}}f(x)$, and we are free to choose $S_{pp}(x)$ so that it will force $A_{pp}(x)$ to have the known discontinuity computed for $0\leqslant x\leqslant 1$, provided this introduces no additional singularities elsewhere. But the discontinuity across a branch cut can be represented to arbitrary precision by a finite set of poles distributed along the cut. Therefore S_{pp} can be represented to arbitrary precision by the rational function

$$S_{pp}(x)=[1+\textstyle\sum_i\alpha_i/(x_i-x)]/[-x\sum_i\beta_i/(x_i-x)]. \quad (4.7)$$

To insure that $A_{pp}(x)$ has the cut in the correct place, we must take $0\leqslant x_i\leqslant 1$ for all i, and require

$$\alpha_i=\tfrac{1}{2}x_i^{\frac{1}{2}}f(x_i)\beta_i. \quad (4.8)$$

Since the residues of these poles must be $\rho(x_k)h(x_k)\Delta x_k$ if Δx_k is the distance between the poles, we must also require

$$\left.\frac{dS_{pp}}{dx}\right|_{x=x_k}-\tfrac{1}{4}x_k^{-\frac{3}{2}}g(x_k)=[\rho(x_k)h(x_k)\Delta x_k]^{-1}. \quad (4.9)$$

Taking the limit as $x\rightarrow x_k$ in the derivative requires a little care, but can be shown to give

$$\left.\frac{dS_{pp}}{dx}\right|_{x=x_k}=\left[\alpha_k/x_k+1+\sum_{i\neq k}\frac{1}{x_i-x_k}\times(\alpha_i-\alpha_k\beta_i/\beta_k)\right]\Big/x_k\beta_k. \quad (4.10)$$

We therefore have derived the equation

$$\frac{x_k\beta_k}{\rho_kh_k\Delta x_k}=1+\tfrac{1}{2}\sum_{i\neq k}\frac{\beta_i}{x_i-x_k}[x_i^{\frac{1}{2}}f_i-x_k^{\frac{1}{2}}f_k]+\frac{\beta_k}{4x_k^{\frac{1}{2}}}[2f_k-g_k]. \quad (4.11)$$

[6] H. P. Noyes and S. P. Pandya, Phys. Rev. **102**, 269 (1956).
[7] H. P. Noyes and D. Y. Wong, Phys. Rev. Letters **3**, 191 (1959).

Since

$$\lim_{x_i\to x_k}[(x_i^{\frac{1}{2}}f_i-x_k^{\frac{1}{2}}f_k)]/(x_i-x_k)=(2f_k-g_k)/2x_k^{\frac{1}{2}}, \quad (4.12)$$

the term involving $(2f-g)$ is just the missing term in the sum. Therefore we can define a new function $E(x)=x\beta(x)/\rho(x)h(x)\Delta x$ and pass to the limit $\Delta x\to 0$, giving the integral equation

$$E(x)=1+\frac{1}{2}\int_0^1 dy\frac{\rho(y)h(y)[y^{\frac{1}{2}}f(y)-x^{\frac{1}{2}}f(x)]}{y(y-x)}E(y)dy. \quad (4.13)$$

Substituting this back into our expression for S and changing back to the physically interesting variable q^2, we have finally that

$$(C^2q\cot\delta_{pp}+Q)=\frac{1+2q^2\int_0^1 dy\frac{\rho(y)h(y)f(y)E(y)}{y(1+4q^2y)}}{\int_0^1 dy\frac{\rho(y)h(y)E(y)}{y(1+4q^2y)}}. \quad (4.14)$$

This is identical with the expression for $(q\cot\delta_{np})$ previously derived by us,[7] if we let $f=g=h=1$. If we wish to supplement a theoretical calculation of $\rho(x)$ by empirical information on the phase shift, we can add a series of delta functions and adjust their residues[8] (which is equivalent to adding phenomenological poles to the cut). We can use this trick also to insure that S_l goes as q^{-2l} when $q^2\to 0$.

V. CHARGE INDEPENDENCE OF THE $^1S_0$ NUCLEON-NUCLEON STATE

It was first shown by Breit, Condon, and Present[5] that the same phenomenological potential would approximately account for *n-p* and *p-p* scattering in the $^1S_0$ state. This is not immediately apparent if we compare the shape-independent approximation to the effective-range expansion for *n-p* scattering

$$[q\cot\delta_{np}=-(1/a_{np})+\tfrac{1}{2}r_{np}q^2]$$

to that for *p-p* scattering

$$[C^2q\cot\delta_{pp}+Q=-(1/a_{pp})+\tfrac{1}{2}r_{pp}q^2],$$

since experimentally $a_{np}=-23.74$ f while $a_{pp}=-7.8$ f. If, however, a particular phenomenological potential is adjusted to fit the *p-p* scattering length and effective range with e^2/r included in the Schrödinger equation, this same model predicts $a\sim-17$ f when the e^2/r term is dropped.[9]

Schwinger[10] attempted to explain this residual discrepancy by noting that the *n-p* and *p-p* systems also differ because of the magnetostatic interaction between their magnetic moments. For singular potentials such as the Yukawa potential, he showed that inclusion of this effect could explain the residual discrepancy; since the magnetostatic interaction varies like $1/r^3$ as $r\to 0$, most other phenomenological models do not produce sufficient differences to explain the discrepancy although a few have been found.[11,12] "Hard core" potential models exclude enough of the important $1/r^3$ region to make them incompatible with charge independence.[13] The situation was rendered unambiguous by Riazuddin,[14] who pointed out that the extended magnetic moment distribution measured in electron-nucleon scattering[15] shows that the $1/r^3$ singularity is absent, and hence that the magnetostatic correction is negligible regardless of the nature of the nuclear force. He[16,17] and Sugie[18] noted that the $\pi^{\pm}-\pi^0$ mass difference gives a difference between *n-p* and *p-p* scattering which is an order of magnitude larger than the magnetostatic effect, but could not compute this correction with any precision. We will see that our present approach allows a more precise statement to be made.

Before discussing the results of more elaborate calculations, it is interesting to find that we can explain 90% of the difference between the *n-p* and *p-p* scattering lengths simply in terms of the known average range of nuclear forces. As already noted,[7] the usual "shape-independent" effective range approximation is equivalent to assuming that the dynamical discontinuity in the scattering amplitude due to particle exchanges can be replaced by a simple pole. This pole falls in the region of two-pion exchange, so it is reasonable to assume it has the same strength and position for both *n-p* and *p-p* scattering. We therefore assume that $\rho(x)h(x)=\Gamma\delta(x-z)$, giving

$$E_{pp}(z)=1+\frac{\Gamma[2f(z)-g(z)]}{4z^{\frac{3}{2}}}E_{pp}(z). \quad (5.1)$$

Further,

$$C^2q\cot\delta_{pp}+Q=-1/a_{pp}+\tfrac{1}{2}r_{pp}q^2, \quad (5.2)$$

with

$$-1/a_{pp}=z/\Gamma-[2f(z)-g(z)]/4z^{\frac{1}{2}} \quad (5.3)$$

and

$$r_{pp}=4z^{\frac{1}{2}}f(z)-8z/a_{pp}. \quad (5.4)$$

Since, as already noted, the corresponding results for *n-p* scattering are obtained by putting $f=g=1$, we obtain the remarkably simple result that

$$-(1/a_{pp}-1/a_{np})=\tfrac{1}{4}z^{-\frac{1}{2}}[1+g(z)-2f(z)], \quad (5.5)$$
$$r_{pp}-r_{np}=2z^{\frac{1}{2}}[g(z)-1].$$

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If we fit the p-p phase shifts at 1.397 and 2.425 Mev,[19] $z=0.226$, corresponding to an average range of $\bar{r}=z^{\frac{1}{2}}\hbar/\mu_0 c=0.4754$ neutral pion Compton wavelengths. In these units we therefore predict a value of 0.1373 for $-(1/a_{pp}-1/a_{np})$. as compared to the measured value of 0.1250; we also expect the n-p effective range r_{np} to be greater than the p-p value by 0.051, a $2\frac{1}{2}\%$ effect which is within the uncertainty of our calculation.

We now investigate whether this result can be improved by taking explicit account of single pion exchange, and whether it is sensitive to the part of the interaction coming from multiparticle exchange. To start with we note that because of electromagnetic effects and in particular the fact that these produce a mass difference between charged and neutral pions, there are four physically distinguishable coupling constants for pseudoscalar pions which are involved. These are $G_{\pm p}$, $G_{\pm n}$, G_{0p}, and G_{0n}, where the first subscript refers to the pion and the second to the nucleon, (we write $\pm$ for the charged pion since by crossing symmetry these constants are equal; similarly, if we replace nucleons by antinucleons, the G must not change). We see that single pion exchange measures $G_{0p(0n)}{}^2$ in proton-proton (neutron-neutron) scattering, and $2G_{\pm p}G_{\pm n}-G_{0p}G_{0n}$ in neutron-proton scattering. We always measure quantities in terms of the neutral pion compton wavelength, and ignore the neutron-proton mass difference, so the quantity f^2M in our equation for proton-proton scattering is to be interpreted as $G_{0p}{}^2(\mu_0{}^2/4M_p)$, and for the ${}^1S_0$ state the discontinuity is simply $\rho(x)=f^2Mx$. For neutron-proton scattering in this state we have two cuts of different length. The cut due to charged-pion exchange will have $\rho_{\pm}(x)=[2G_{\pm p}G_{\pm n}(\mu_0{}^2/4M)\cdot(\mu^2/\mu_0{}^2)x]$ for $0\leqslant x\leqslant(\mu_0{}^2/\mu^2)$, while that due to neutral pion exchange will be of the same magnitude as for proton-proton scattering but of opposite sign, and extend up to $x=1$. For neutron-neutron scattering, we simply replace $G_{0p}{}^2$ by $G_{0n}{}^2$ and set $f=g=h=1$ in the proton-proton equations.

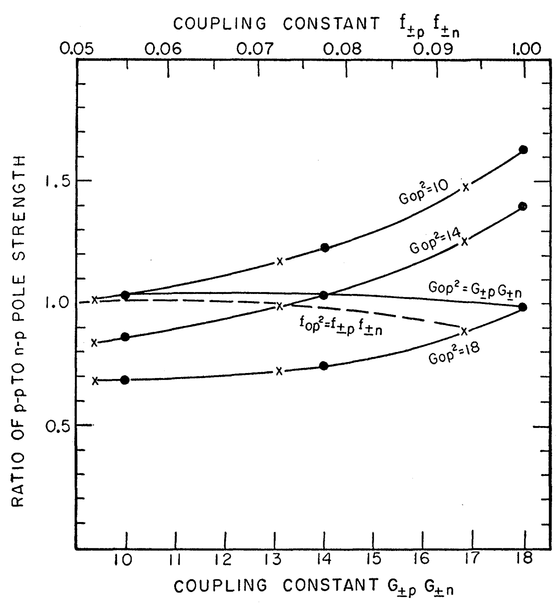

FIG. 1. Ratio of the multiparticle exchange pole strength required to fit low-energy p-p scattering to that needed to fit the n-p scattering length as a function of the coupling constant for charged pions to neutron and proton, for fixed coupling constant for neutral pions to protons.

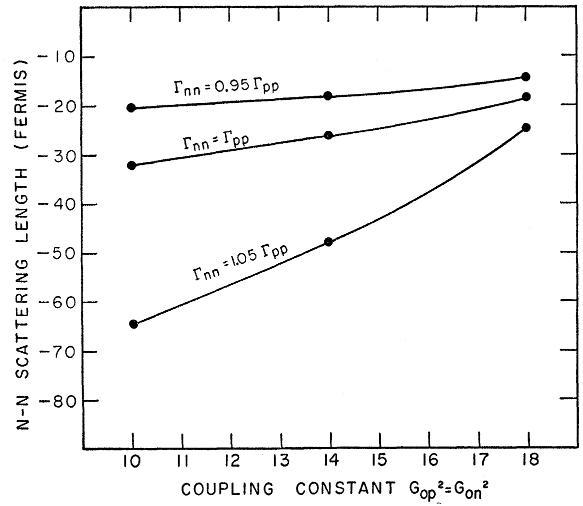

FIG. 2. Prediction of the n-n scattering length, assuming exact charge symmetry or a 5% departure from it for multiparticle exchange.

We see that in the framework there are two possible unknown sources of charge dependence, the possibility that $G_{\pm p}G_{\pm n}\neq G_{0p}{}^2$ or that the multimeson exchange singularity differs for n-p and p-p scattering. We represent the multipion exchange by a single pole[7] and for assumed value of $G_{0p}{}^2$ adjust the position and strength of this pole until the phase shifts predicted by the solution of the integral equation agree with the p-p experimental values at 1.397 and 2.425 Mev.[19] For n-p scattering, since the effective range is not well known experimentally, we keep the multimeson pole at the same position, and for assumed values of $G_{\pm p}G_{\pm n}$ and $G_{0p}G_{0n}$ adjust the pole strength to fit the n-p scattering length (we assume charge symmetry to the extent that $G_{0p}=G_{0n}$). The results of the calculation are presented in Fig. 1. We see that if $G_{0p}{}^2=G_{\pm p}G_{\pm n}$, then the multimeson exchange pole strengths are also required to be equal within the expected limits of $3\frac{1}{2}\%$. Since Breit *et al.* have shown[20] that the coupling constants in question are equal, we conclude that the multimeson exchange part of the interaction is charge independent within the expected limits. (It is not clear to us whether we should not have assumed $f_{0p}{}^2=f_{\pm p}f_{\pm n}$ instead; fortunately, this assumption is also consistent with the

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charge independence of the multimeson exchange as is shown by the dotted curve in Fig. 1.)

If we knew G_{0n}^2, we could use these results to predict the *n-n* scattering length, using charge symmetry to evaluate the multipion exchange pole parameters. In principle G_{0n} could be obtained from an analysis of photoproduction of π^0's from neutrons, but such an analysis is currently unavailable. If we assume charge symmetry ($G_{0p}^2=G_{0n}^2$ and $\Gamma_{pp}=\Gamma_{nn}$), we obtain the predictions given in Fig. 2. At present G_{0p}^2 can be obtained more accurately from *p-p* scattering than from photoproduction of π^0's from protons or pion-nucleon scattering. Unfortunately the value of 12 ± 2 obtained by MacGregor, Moravcsik, and Stapp[21] at 310 Mev is suspect, since Signell[22] has shown that the same data do not give the correct value of the neutral pion mass. He shows that the mass is correctly given by the 95-Mev data if only S, P, and D waves are treated phenomenologically, but again the coupling-constant value is suspect because it is not stable when the coupling parameter $\bar{\epsilon}_2$ is included in the analysis.[23] Consequently the best available value is that of 13.5 ± 0.9 obtained by the energy-dependent analysis of Breit *et al.*[20]; It is difficult to estimate how much this uncertainty should be increased for the arbitrariness arising from the initial values of phase shifts used in their search, which is difficult to treat statistically, and from the presumed systematic errors which exist in some of the data used. Making the optimistic assumption that these uncertainties are minor, and assuming exact charge symmetry for multipion exchange, we predict an *n-n* scattering length of -27 ± 1.4 f. We note that if the multipion pole departs by as little as 5% from charge symmetry, the scattering length could lie anywhere between -53 and -18 f, so that a measurement of this quantity gives a very sensitive test of charge symmetry. Recently Ilakovac, Kuo, Petravić, and Šlaus[24] have measured D$(n,p)2n$ at 14.4 Mev and shown that, using a simplified analysis, the *n-n* scattering length is -22 f with a statistical uncertainty of only 2 f, which would indicate a slight departure from charge symmetry. It would therefore be of interest to see if the uncertainties due to three-body effects in the final state are sufficiently manageable to put this result on a firm theoretical basis. It might turn out to be easier to perform the experiment proposed by McVoy,[25] which is somewhat more tractable from the point of view of analysis.

ACKNOWLEDGMENTS

We are indebted to R. Carpenter for developing the IBM 650 code for solving the integral equations and carrying out the calculations, and to M. Gonzales for assistance with the numerical work. We wish to thank Dr. K. Edwards for pointing out an algebraic error.

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