Lorentz Invariance and the Kinematic Structure of Vertex Functions*

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The general kinematic properties of vertex functions which follow from the transformation properties of the initial and final single-particle states, and of the vertex (current) operator under proper and improper Lorentz transformations, are studied for the pseudoscalar (pion) and vector (electromagnetic) vertices. The treatment, which relies strongly on the helicity representation for the states of a relativistic particle introduced by Jacob and Wick, is fully relativistic, and applies to particles of arbitrary spin. The number of independent vertex functions or form factors is determined in each case, and the analogs of the nonrelativistic multipole expansions are obtained. One obtains thereby a complete specification of the dependence of the matrix elements on the spins, helicities, and relative parities of the particles, and some information on the limiting behavior of the form factors for small momentum transfers. A theorem first proved by Ernst, Sachs, and Wali in the special case of spin $\frac{1}{2}$, that the matrix elements of the 4-divergence of the electromagnetic current between states of the same (single) particle vanish independently of the assumption of gauge invariance or current conservation, is extended as a by-product of the general results to the case of arbitrary spin and to any vector current having the same transformation properties as j_μ. Finally, the parametrizations obtained for the pseudoscalar and vector vertex functions are used to calculate the cross sections for the general two-particle scattering process $a+b \rightarrow c+d$ in the single-quantum-exchange approximation for relativistic particles of arbitrary spin.

I. INTRODUCTION

THE calculation of transition matrix elements for a number of processes such as the emission and absorption of radiation by nuclei, or the scattering of particles in a single-quantum-exchange approximation, can be reduced to the calculation of matrix elements of the form

$$\langle \mathbf{p}'s'\lambda' | j | \mathbf{p}s\lambda \rangle$$

(vertex functions), where the initial and final states each involve only a single free particle of the indicated spin and momentum, and j is a "current" operator with known properties under Lorentz transformations. The present paper is concerned with a systematic study of the kinematic structure of such vertex functions, that is, with the study of those properties of vertex functions, such as the dependence on the particle spins, which can be deduced from the character of the states and of the operator j under Lorentz transformations. We will obtain thereby a parametrization for the general matrix element in terms of a number of independent quantities, the form factors for the vertex in question.

A familiar example of the parametrization of a vertex function is provided by the multipole decomposition of matrix elements of the electromagnetic current in the nonrelativistic theory of nuclear radiative processes. By invoking the general properties of j_μ and the initial and final states under rotations and reflections, it is possible to treat by the same methods transitions between states of any spin and parity, and in fact, to obtain some information about the limiting behavior of the multipole form factors for small photon momenta. However, the significance of these results in relativistic problems, at least with respect to the definition of the multipole form factors, is perhaps not clear from the usual point of view. It has therefore been customary to introduce instead, appropriate representations for the states of relativistic particles, and to express the matrix elements in a manifestly covariant form. For example, the matrix elements of the electromagnetic current operator for a particle of spin $\frac{1}{2}$, mass m, charge e, and anomalous magnetic moment κ, may be written in the form[1]

$$\langle p' | j_\mu | p \rangle = ie\bar{u}(p')[\gamma_\mu F_1(q^2) + (\kappa/2m)F_2(q^2)\sigma_{\mu\nu}(p'-p)_\nu]u(p),$$

using the Dirac spinor representation for the states. Here q^2 is given by $(p'-p)^2$. The Dirac and Pauli form factors F_1 and F_2 in this expression can be shown to be linear combinations of the charge and magnetic moment form factors for the system. The explicit parametrization of vertex functions in this fashion becomes more difficult for higher spins, and a return to global methods appears to be desirable. However, general studies have apparently been confined simply to the verification that there are the same number of independent matrix elements connecting two states in the relativistic, as in the nonrelativistic theories.[2]

The present work, as noted, consists of a systematic study of the kinematic structure of vertex functions. Results are obtained specifically for the pseudoscalar (pion) and vector (electromagnetic) vertices, but the methods are easily generalized to the other cases of interest. The study is greatly facilitated by the use of the helicity representation for the states of a relativistic particle developed in the elegant paper of Jacob and

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[1] G. Salzman, Phys. Rev. 99, 973 (1955).

[2] D. R. Yennie, M. M. Lévy, and D. G. Ravenhall, Revs. Modern Phys. 29, 144 (1957).

Wick.[3] Because we will need a somewhat more explicit construction of the states than was given by those authors, the definition and properties of the helicity states are considered in some detail in Sec. II. The results are applied in Sec. III to the discussion of the properties of the matrix elements of the pion current between states of arbitrary spin and momentum. A complete parametrization of the general matrix element is obtained in terms of the matrix elements in a special Lorentz frame, the so-called brick wall frame. The restrictions on the spin and helicity dependence of these matrix elements which follow from the properties of j_π and the helicity states under proper and improper Lorentz transformations are determined. These conditions permit all the matrix elements to be expressed in terms of a smaller set of independent matrix elements, for which relativistic multipole-type expansions are then derived. Several examples are considered, including the calculation of the scattering cross section for the process $a+b\rightarrow c+d$ in the one-meson-exchange approximation for arbitrary spins of the initial and final particles. The results are extended in Sec. IV to the vertex functions associated with the electromagnetic current, and multipole expansions of the independent matrix elements are again derived. A theorem first proved by Ernst, Sachs, and Wali[4] in the special case of spin $\frac{1}{2}$, that the matrix elements of the 4-divergence of the electromagnetic current between free states of the same (single) particle vanish independently of the assumption of current conservation or gauge invariance, is extended as a byproduct of the general results to the case of arbitrary spin and to any vector current with the same properties as j_μ under proper and improper Lorentz transformations.[5] Finally, the parametrization of the electromagnetic vertex function is applied to the familiar problem of nuclear radiation, and to the calculation in the single-quantum-exchange approximation of the cross section for the scattering process $a+b\rightarrow c+d$ for relativistic particles of arbitrary spin.

It should be emphasized that some of the present results either are not new or are new only in the context of the relativistic theory. On the other hand, it is felt by the authors that a unified presentation of all the material is worthwhile in itself, and serves furthermore to illustrate more clearly the power of the global approach to the present, essentially kinematic, problems.

II. DEFINITION AND TRANSFORMATION PROPERTIES OF THE HELICITY STATES

a. Definition of the Helicity States

The characterization of the state of a free particle in terms of its helicity and momentum is discussed systematically in the paper of Jacob and Wick.[3,6] We shall follow the conventions of those authors. The construction of the helicity states is based on the assumption that the wave function of a free particle transforms under proper Lorentz transformations according to an irreducible unitary representation of the inhomogeneous Lorentz group. (Extensive accounts of the representation theory are given in the papers of Wigner,[7] Bargmann,[8,9] and, more recently, Shirokov.[10]) Thus, if ψ_l and $\psi_{l'}$ are the wave functions of the particle in the Lorentz frames l and l', and if the transition from l to l' is effected by a Lorentz transformation L, then

$$\psi_{l'}=U(L)\psi_l, \tag{1}$$

where $U(L)$ is a unitary operator. In particular, the wave function of a particle in an arbitrary Lorentz frame can be generated from the wave function in a given frame by an appropriate Lorentz transformation.[7] We shall make use of this fact to construct the helicity representation for the states of particles with a positive, nonzero rest mass. [The case of massless particles is discussed by Jacob and Wick,[3] and by Chou Kuang-Chao.[11]] The natural Lorentz frame in which to start the construction is in this case the rest frame of the particle in question.

The rest states of a particle of mass m may be classified according to the irreducible representations of the total angular momentum $\mathbf{J}^2$ and the z component of the angular momentum J_3.[7] The basic states $|m;s,\lambda\rangle$ for a particle of spin s and z component of the spin λ then satisfy the familiar eigenvalue equations[12–14]:

$$\mathbf{J}^2|m;s,\lambda\rangle=s(s+1)|m;s,\lambda\rangle, \tag{2}$$

$$J_3|m;s,\lambda\rangle=\lambda|m;s,\lambda\rangle. \tag{3}$$

[The total angular momentum operator reduces in the rest frame to the spin operator.] In addition, these states satisfy the relations

$$P_0|m;s,\lambda\rangle=m|m;s,\lambda\rangle,\quad m>0, \tag{4}$$

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[10] Iu. M. Shirokov, J. Exptl. Theoret. Phys. (USSR) **33**, 861, 1196, 1208 (1957); **34**, 717 (1958); **35**, 1005 (1958); [translation: Soviet Phys.—JETP **6**, 664, 919, 929 (1958); **7**, 493 (1958); **8**, 703 (1959)].

[11] Chou Kuang-Chao, J. Exptl. Theoret. Phys. (USSR) **36**, 909 (1959); [translation in Soviet Phys.—JETP **9**, 642 (1959).]

[12] E. P. Wigner, *Group Theory and Its Applications to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959).

[13] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

[14] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

and

$$P^2|m;s,\lambda\rangle=-m^2|m;s,\lambda\rangle, \tag{5}$$

where P_0 is the energy operator, and P^2 is the invariant square of the 4-momentum operator, $P^2=\mathbf{P}^2-P_0^2$. Equations (4) and (5) in essence define the rest system of the particle. In the future, we will suppress the mass parameter, and denote the rest states simply by $|s,\lambda\rangle$. The states defined above can be normalized, and satisfy the spin orthogonality relations

$$\langle s',\lambda'|s,\lambda\rangle=\delta_{ss'}\delta_{\lambda\lambda'}. \tag{6}$$

The relative phases of the states $|s,\lambda\rangle$ for different values of λ may be fixed, following Jacob and Wick,[3] by using the standard conventions for the matrix elements of the raising and lowering operators,

$$(J_1\pm iJ_2)|s,\lambda\rangle=[(s\pm\lambda+1)(s\mp\lambda)]^{\frac{1}{2}}|s,\lambda\pm 1\rangle. \tag{7}$$

The $2s+1$ states $|s,\lambda\rangle$ form a complete set in the rest system of a particle of spin s, and may therefore be used as the basis for the construction of the states in an arbitrary Lorentz frame, Eq. (1). The procedure which we shall use is that of Jacob and Wick.[3] The construction makes use of the fact that a Lorentz transformation along a given axis does not affect the component of the angular momentum along that axis,[15]

$$[J_i,K_j]=i\epsilon_{ijk}K_k, \tag{8}$$

where K_i is the generator of the ordinary Lorentz transformation to a coordinate system moving along the i axis. Thus, if we apply a Lorentz transformation $e^{-i\zeta K_3}$ with velocity v in the positive z direction $\{\zeta=\sinh^{-1}[v/(1-v^2)^{\frac{1}{2}}], \zeta\geq 0\}$ to one of the rest states $|s,\lambda\rangle$, we obtain another state with the same value of λ. However, relative to the fixed coordinate system, the new state has momentum $p=m\sinh\zeta$ along the positive z direction, and energy $p_0=m\cosh\zeta$. This is easily verified using the commutation relations[15] of $\mathbf{K}$ with the 4-momentum operator P_μ,

$$[P_i,K_j]=iP_0\delta_{ij}, \tag{9.1}$$

$$[P_0,K_j]=iP_j, \tag{9.2}$$

and the result given in Eq. (4). We will denote such a state by $|p0s\lambda\rangle$,

$$|p0s\lambda\rangle=e^{-i\zeta K_3}|s,\lambda\rangle, \tag{10.1}$$

$$\zeta=\sinh^{-1}(p/m). \tag{10.2}$$

The second argument in the symbol $|p0s\lambda\rangle$ indicates that the motion of the particle is along the positive z direction. The helicity of a particle is defined as the projection of the total angular momentum of the particle along its direction of motion, that is, the eigenvalue of $\mathbf{J}\cdot\hat{p}$, with $\hat{p}$ a unit vector, and is therefore equal to λ for the state $|p0s\lambda\rangle$.

[15] See, for example, the discussion of the commutation relations given by Shirokov in the first papers of reference 10.

It remains only to construct the helicity states for an arbitrary direction of the particle momentum $\mathbf{p}$. This may be accomplished by applying an appropriate rotation to the states $|p0s\lambda\rangle$. If $\hat{p}$ makes an angle θ with the z axis, a convenient choice is the rotation through the angle θ about the normal to the plane which contains $\hat{p}$ and the z axis, with the sense of the rotation such that the z axis is carried into the direction $\hat{p}$. This rotation may be expressed in terms of the usual Euler angles as $R(\phi,\theta,-\phi)$, where

$$R(\alpha\beta\gamma)=e^{-i\alpha J_3}e^{-i\beta J_2}e^{-i\gamma J_3}, \tag{11}$$

and ϕ is the azimuthal angle of $\hat{p}$ relative to the (arbitrary) x direction. With this convention for the rotation, we define the helicity states for a particle moving with momentum p in the θ,ϕ direction in terms of the helicity states $|p0s\lambda\rangle$ as

$$|p\theta\phi s\lambda\rangle=R(\phi,\theta,-\phi)|p0s\lambda\rangle. \tag{12}$$

For brevity, we will occasionally denote this state by $|\mathbf{p}s\lambda\rangle$. Using the commutation relations of the 3-momentum operator $\mathbf{P}$ with $\mathbf{J}$,[15]

$$[J_i,P_j]=i\epsilon_{ijk}P_k, \tag{13}$$

it is easily verified that the state $|p\theta\phi s\lambda\rangle$ has the indicated momentum. Because the helicity operator $\mathbf{J}\cdot\hat{p}$ transforms as a scalar with respect to rotations, the rotated state clearly has the same helicity λ as the original state $|p0s\lambda\rangle$. As a consequence of our having restricted the discussion to the unitary representations of the inhomogeneous Lorentz group, the states $|p\theta\phi s\lambda\rangle$ furthermore satisfy the same orthogonality conditions in spin space as the rest states, Eq. (6). The corresponding inner product in Hilbert space is

$$\langle\mathbf{p}'s'\lambda'|\mathbf{p}s\lambda\rangle=\delta_{ss'}\delta_{\lambda\lambda'}(2\pi)^3\delta_3(\mathbf{p}'-\mathbf{p})p_0/m; \tag{14}$$

this relation is Lorentz invariant. We note also the corresponding completeness relation

$$|\psi\rangle=\sum_{s,\lambda}\int\frac{d^3p}{(2\pi)^3}\frac{m}{p_0}|\mathbf{p}s\lambda\rangle\langle\mathbf{p}s\lambda|\psi\rangle. \tag{15}$$

The normalization of the states $|\mathbf{p}s\lambda\rangle$ is such that the density of states in momentum space is given by the Lorentz invariant quantity $(m/p_0)d^3p/(2\pi)^3$.

6. Symmetry Properties of the Helicity States

In addition to the translations and rotations in space-time, the full inhomogeneous Lorentz group contains the discrete operations of space and time reflection. The properties of the helicity states under these symmetry operations are easily derived. Under a reflection of the space coordinates $x\rightarrow -x$, the angular momentum operator $\mathbf{J}$ transforms as a pseudovector, while the operators $\mathbf{P}$ and $\mathbf{K}$ transform as vectors. Thus,

denoting the parity operator by $\mathcal{P}$,

$$\mathcal{P}\mathbf{J}\mathcal{P}^{-1}=\mathbf{J}, \tag{16.1}$$

$$\mathcal{P}\mathbf{P}\mathcal{P}^{-1}=-\mathbf{P}, \tag{16.2}$$

$$\mathcal{P}\mathbf{K}\mathcal{P}^{-1}=-\mathbf{K}. \tag{16.3}$$

According to Eq. (16.1) and Eqs. (2), (3), and (6), application of $\mathcal{P}$ to one of the rest states $|s,\lambda\rangle$ can at most result in the multiplication of that state by a factor of unit modulus,

$$\mathcal{P}|s,\lambda\rangle=\eta_p|s,\lambda\rangle, \quad |\eta_p|=1. \tag{17}$$

The phase factor η_p depends on the intrinsic parity of the particle, but does not depend on the spin projection λ [$\mathcal{P}$ commutes with the raising and lowering operators, Eq. (7)]. On the other hand, if $\mathcal{P}$ is applied to a helicity state $|p0s\lambda\rangle$, the original state is not reproduced; the new state is easily seen to have momentum $-p$ and helicity $-\lambda$. It is, therefore, more convenient to consider instead of $\mathcal{P}$ the reflection Y in the x-z plane, $x\rightarrow x$, $y\rightarrow -y$, $z\rightarrow z$. Y can obviously be expressed in terms of $\mathcal{P}$ and a rotation through π about the y axis,

$$Y=e^{-i\pi J_2}\mathcal{P}=\mathcal{P}e^{-i\pi J_2}. \tag{18}$$

From the relation[12–14]

$$e^{-i\pi J_2}|s,\lambda\rangle=(-1)^{s-\lambda}|s,-\lambda\rangle \tag{19}$$

and Eq. (17), it follows that

$$Y|s,\lambda\rangle=\eta_p(-1)^{s-\lambda}|s,-\lambda\rangle, \tag{20}$$

or, since P_3 and K_3 are unchanged by the Y transformation,

$$Y|p0s\lambda\rangle=\eta_p(-1)^{s-\lambda}|p0s\,-\lambda\rangle. \tag{21}$$

The transformed state has momentum p in the positive z direction, but the sign of the helicity has been changed relative to that of the initial state. For a general helicity state $|p\theta\phi s\lambda\rangle$, one obtains

$$Y|p\theta\phi s\lambda\rangle=\eta_p(-1)^{s-\lambda}|p\theta,-\phi,s,-\lambda\rangle, \tag{22}$$

and the transformed momentum differs from the initial momentum only in the sign of its y component.

The operation T of Wigner time inversion[12] is antiunitary, T consisting most generally of a complex conjugation K followed by a unitary transformation U, $T=UK$. An arbitrary state ψ composed of a sum of helicity states,

$$\psi=\sum_\lambda a_\lambda|p\theta\phi s\lambda\rangle, \tag{23.1}$$

therefore transforms according to the relation

$$T\psi=\sum_\lambda a_\lambda{}^*T|p\theta\phi s\lambda\rangle. \tag{23.2}$$

The time-reversed helicity states $T|p\theta\phi s\lambda\rangle$ may easily be expressed in terms of normal helicity states. The operators $\mathbf{P}$ and $\mathbf{J}$ change sign under T, while $\mathbf{K}$ is left unchanged,

$$T\mathbf{P}T^{-1}=-\mathbf{P}, \tag{24.1}$$

$$T\mathbf{J}T^{-1}=-\mathbf{J}, \tag{24.2}$$

$$T\mathbf{K}T^{-1}=\mathbf{K}. \tag{24.3}$$

It is evident from Eq. (24.1) that the time-reversed state $T|p0s\lambda\rangle$ is a state with momentum p in the negative z direction. Since $\mathbf{J}\cdot\hat{p}$ is invariant under T, the transformed state has the initial helicity, λ, and must therefore be proportional to the helicity state $|p\pi0s\lambda\rangle$ with momentum p in the negative z direction and helicity λ,

$$\begin{aligned}T|p0s\lambda\rangle&=|p0s\lambda\rangle^T\\&=\eta_T|p\pi0s\lambda\rangle\\&=\eta_Te^{-i\pi J_2}|p0s\lambda\rangle.\end{aligned} \tag{25}$$

The constant of proportionality η_T is of unit modulus, and is independent of λ and p.[3] In the remaining discussion, we will take advantage of the arbitrariness of the over-all phase in the definition of the basic states $|s,\lambda\rangle$ to choose $\eta_T=1$ for all particles. The rest states then have the familiar properties under time reversal[12,16]:

$$\begin{aligned}T|s,\lambda\rangle&=e^{-i\pi J_2}|s,\lambda\rangle\\&=(-1)^{s-\lambda}|s,-\lambda\rangle,\end{aligned} \tag{26.1}$$

and

$$T^2|s,\lambda\rangle=(-1)^{2s}|s,\lambda\rangle. \tag{26.2}$$

The generalization of Eq. (25) to the arbitrary helicity states is readily shown to be

$$\begin{aligned}T|p\theta\phi s\lambda\rangle&=TR(\phi,\theta,-\phi)|p0s\lambda\rangle\\&=R(\phi,\theta,-\phi)e^{-i\pi J_2}|p0s\lambda\rangle.\end{aligned} \tag{27}$$

The time-reversed state $T|p\theta\phi s\lambda\rangle$ has its momentum directed oppositely to that of the state $|p\theta\phi s\lambda\rangle$, corresponding to the physical description of the time-reversal transformation as a reversal of the direction of all motions. The helicities of the states are of course the same. We note finally a useful identity which follows from the form $T=UK$, namely, for arbitrary states ψ, Φ,

$$(T\psi,T\Phi)=(\psi,\Phi)^*. \tag{28}$$

c. Properties of the Helicity States under Lorentz Transformations

The properties of the helicity states under arbitrary proper Lorentz transformations are also easily

[16] The import of the present definition of the time-reversal states may be illustrated by using the spherical harmonic $\eta_lY_{l,m}$ as the representative of the state $|l,m\rangle$ for a particle of integral spin. Then

$$\begin{aligned}T\eta_lY_{l,m}&=\eta_l\eta_Te^{-i\pi L_2}Y_{l,m}\\&=\eta_l\eta_T(-1)^{l-m}Y_{l,-m}\\&=\eta_l\eta_T(-1)^lY_{l,m}{}^*.\end{aligned}$$

The choice $\eta_T=1$ and the usual requirement that the time-reversed state be the complex conjugate of the original state are satisfied by the choice of the phase factor $\eta_l=i^l$. This is equivalent to incorporating into the spherical harmonics the factors i^l which appear in the expansion of a plane wave.

derived.[17] Let G be such a transformation,

$$x_\mu \xrightarrow{G} x_\mu', \quad \text{with} \quad x_\mu' = a_{\mu\nu} x_\nu.$$

Then the 4-momentum operator transforms under G^{-1} according to

$$G^{-1} P_\mu G = a_{\nu\mu} P_\nu. \tag{29}$$

Thus, if p_μ is the 4-momentum of the initial helicity state $|\mathbf{p}s\lambda\rangle$, the transformed state clearly has 4-momentum $p_\mu' = a_{\nu\mu} p_\nu$ referred to the fixed coordinate system,

$$\begin{aligned} p_\mu' G|\mathbf{p}s\lambda\rangle &= P_\mu G|\mathbf{p}s\lambda\rangle \\ &= a_{\nu\mu} p_\nu G|\mathbf{p}s\lambda\rangle, \end{aligned} \tag{30}$$

and can therefore be expanded as a linear combination of the helicity states with this momentum,

$$G|\mathbf{p}s\lambda\rangle = \sum_{\lambda'} |g\mathbf{p}s\lambda'\rangle\langle g\mathbf{p}s\lambda'|G|\mathbf{p}s\lambda\rangle. \tag{31}$$

We have used the symbol $g\mathbf{p}$ for the 3-momentum of the transformed state, $gp_i = a_{\nu i} p_\nu$. Denoting by $H(\mathbf{p})$ the Lorentz transformation used to define the helicity state $|\mathbf{p}s\lambda\rangle = |p\theta\phi s\lambda\rangle$,

$$|\mathbf{p}s\lambda\rangle = H(\mathbf{p})|s,\lambda\rangle = R(\phi,\theta,-\phi) e^{-i\zeta_p K_3}|s,\lambda\rangle, \tag{32}$$

and by $H(g\mathbf{p})$, the defining transformation for the state $|g\mathbf{p}s\lambda\rangle$, we can rewrite Eq. (31) in the form

$$G|\mathbf{p}s\lambda\rangle = \sum_{\lambda'} |g\mathbf{p}s\lambda'\rangle\langle s\lambda'|H^{-1}(g\mathbf{p}) G H(\mathbf{p})|s\lambda\rangle. \tag{33}$$

Since the Lorentz transformation $H^{-1}(g\mathbf{p})GH(\mathbf{p})$ connects two states in the rest frame of the particle, it can at most represent a rotation R_g,

$$R_g = H^{-1}(g\mathbf{p}) G H(\mathbf{p}). \tag{34}$$

The transformed helicity state may therefore be expressed in the form

$$G|\mathbf{p}s\lambda\rangle = \sum_{\lambda'} |g\mathbf{p}s\lambda'\rangle D_{\lambda'\lambda}^{(s)}(R_g), \tag{35}$$

where the D's are the familiar representation coefficients of the rotation group in three dimensions,[12–14,18]

$$\begin{aligned} D_{\lambda'\lambda}^{(s)}(R) &= D_{\lambda'\lambda}^{(s)}(\alpha\beta\gamma) \\ &= \langle s\lambda'|R(\alpha\beta\gamma)|s\lambda\rangle. \end{aligned} \tag{36}$$

The result given in Eq. (35) may be extended at once to include improper Lorentz transformations. It is most convenient in this case to factor the reflections out of G, so that these may be applied directly to the initial or final state using the results of Sec. II(b).

The transformation formula for the helicity states given in Eq. (35) is remarkably simple. Its physical significance is furthermore clear. The $2s+1$ initial states $|\mathbf{p}s\lambda\rangle$ represent plane waves, with the $2s+1$ possible values for the projection of the total angular momentum of the particle along its direction of motion. The transformed states $|g\mathbf{p}s\lambda'\rangle$ again represent plane waves. If the quantization axis for these states is taken as the new direction of motion, as in the helicity representation, there are again only $2s+1$ possible values for the spin projection λ'. The helicity in the transformed state may nevertheless be different from the helicity in the initial state. The resulting unitary transformation on the helicity indices is described by the matrices $D^{(s)}$, and is of an essentially geometrical character.

Simple illustrations of Eq. (35) are provided by pure rotations and by ordinary Lorentz transformations along the direction of motion of a particle. Since the helicity operator $\mathbf{J}\cdot\hat{p}$ transforms under rotations as a scalar, a rotation does not change the helicity of a state. It may, nevertheless, change the phase of the wave function. One easily verifies that if θ,ϕ and θ',ϕ' specify, respectively, the initial direction of the momentum $\mathbf{p}$ and the direction resulting from a rotation $R(\alpha\beta\gamma)$, the rotation R_g of Eq. (34) is a rotation about the z axis, and is given by

$$R_g = e^{i\Phi J_3} = R^{-1}(\phi',\theta',-\phi') R(\alpha\beta\gamma) R(\phi,\theta,-\phi), \tag{37.1}$$

whence

$$R(\alpha\beta\gamma)|p\theta\phi s\lambda\rangle = e^{i\lambda\Phi}|p\theta'\phi' s\lambda\rangle. \tag{37.2}$$

In the case of a Lorentz transformation $G = e^{i\zeta K_3}$ which reverses the direction of the momentum of the helicity state $|p0s\lambda\rangle$, the rotation R_g is again easily evaluated, and is found to be $e^{i\pi J_2}$,

$$D_{\lambda'\lambda}^{(s)}(R_g) = (-1)^{s+\lambda}\delta_{\lambda',-\lambda}. \tag{38}$$

Thus the sign of the helicity is reversed, as would be expected from the observation that the Lorentz transformation does not affect the z component of J_3, but reverses the direction of the momentum.

The transformation law for matrix elements in the helicity representation follows at once from Eq. (35). Let A be an operator with known properties under Lorentz transformations,

$$GAG^{-1} = A_G. \tag{39}$$

Then the matrix elements of A and the transformed operator A_G are related by the equation

$$\begin{aligned} \langle \mathbf{p}'s'\mu'|A|\mathbf{p}s\mu\rangle &= \langle \mathbf{p}'s'\mu'|G^{-1}A_G G|\mathbf{p}s\mu\rangle = \sum_{\lambda\lambda'} D_{\lambda'\mu'}^{(s')*}(R_g') \\ &\quad \times \langle g\mathbf{p}'s'\lambda'|A_G|g\mathbf{p}s\lambda\rangle D_{\lambda\mu}^{(s)}(R_g). \end{aligned} \tag{40}$$

The rotations R_g and R_g' are defined separately for the initial and final states in accordance with Eq. (34). The foregoing result is of great utility; it is often much simpler to evaluate the matrix elements of A in some particular Lorentz frame than it is in a general frame. We shall in fact make use of this observation in the following sections.

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[17] The authors are indebted to Professor G. C. Wick for an illuminating discussion of the properties of the helicity states under arbitrary Lorentz transformations.

[18] The representation coefficients $D_{m'm}^{(j)}(\alpha\beta\gamma)$ defined in Eq. (36) agree with the convention of Rose.[14] The definition of Edmonds[13] and that given in the translation of Wigner's book[12] correspond to $D_{m'm}^{(j)}(-\alpha,-\beta,-\gamma)$ in the present convention. Jacob and Wick,[3] also used the D's as defined by Rose.

III. PROPERTIES OF VERTEX FUNCTIONS: THE PION CURRENT

In the remainder of this paper, we will be concerned primarily with the properties of matrix elements of the form

$$\langle \mathbf{p}'s'\mu'|A|\mathbf{p}s\mu\rangle$$

("vertex functions") which follow from the properties of the operator A and of the helicity states under inhomogeneous Lorentz transformations. As a first, and relatively simple, example, we will consider the matrix elements between arbitrary single-particle states of the current operator for the neutral pion, defined by

$$j_{\pi,0}=(\Box+m_\pi^2)\phi_0, \qquad (41)$$

where ϕ_0 is the π^0 field operator. It will be convenient to use Eq. (40) and the scalar character of $j_{\pi,0}$ under proper Lorentz transformations to express the general matrix element in terms of matrix elements in a special Lorentz frame, the so-called "brick wall" or Breit frame. In this frame, the initial particle has momentum p in the positive z direction, while the outgoing particle, which may or may not be the same, has momentum p in the negative z direction. The only rotation which appears in the definition of the helicity states is then a trivial rotation $e^{-i\pi J_2}$ in the final state. We therefore write for the momentum space matrix element in a general coordinate system

$$\langle \mathbf{p}'s'\mu'|j_{\pi,0}|\mathbf{p}s\mu\rangle =\sum_{\lambda\lambda'} D_{\lambda'\mu'}{}^{(s')*}(R_g')\Pi_{s'\lambda';s\lambda}D_{\lambda\mu}{}^{(s)}(R_g), \qquad (42.1)$$

where the functions $\Pi_{s'\lambda';s\lambda}$ are the matrix elements of $j_{\pi,0}$ in the brick wall coordinate system,

$$\Pi_{s'\lambda';s\lambda}=\langle p0s'\lambda'|e^{i\pi J_2}j_{\pi,0}|p0s\lambda\rangle. \qquad (42.2)$$

We turn now to the study of these functions.

a. Symmetry Properties of the Vertex Functions

A number of properties of the matrix elements $\Pi_{s'\lambda';s\lambda}$ can be deduced at once using conservation laws and invariance under the symmetry operations of the inhomogeneous Lorentz group. Conservation of J_3 implies that $\lambda+\lambda'=0$, and that $|\lambda|\leq s, s'$. Since J_3 commutes with the operator $j_{\pi,0}$ and changes sign under a rotation through π about the y axis, we have

$$\begin{aligned}\lambda\Pi_{s'\lambda';s\lambda}&=\langle p0s'\lambda'|e^{i\pi J_2}j_{\pi,0}J_3|p0s\lambda\rangle\\&=-\langle p0s'\lambda'|J_3e^{i\pi J_2}j_{\pi,0}|p0s\lambda\rangle\\&=-\lambda'\Pi_{s'\lambda';s\lambda}.\end{aligned} \qquad (43)$$

Thus either $\lambda+\lambda'=0$, or $\Pi_{s'\lambda';s\lambda}=0$, as stated. Since $|\lambda|\leq s$ and $|\lambda'|\leq s'$, we obtain the desired result,

$$\Pi_{s'\lambda';s\lambda}=\delta_{\lambda',-\lambda}\Pi_{s',-\lambda;s\lambda}, \quad |\lambda|\leq s, s'. \qquad (44)$$

The change in the sign of the helicity between the initial and final states corresponds to the opposite directions of motion of the two particles; the scalar operator $j_{\pi,0}$ does not change the projection of $\mathbf{J}$ along the fixed z axis, but the final state is quantized along $-z$.

The pion current transforms as a pseudoscalar under reflections,

$$Yj_{\pi,0}Y^{-1}=-j_{\pi,0}, \qquad (45)$$

where Y is the reflection in the x-z plane, Eq. (18). Utilizing the transformation properties of the helicity states given in Eq. (21), and the fact that $\mathbf{J}$ is a pseudovector $YJ_2Y^{-1}=J_2$, we obtain

$$\begin{aligned}\Pi_{s'\lambda';s\lambda}&=\langle p0s'\lambda'|e^{i\pi J_2}Y^{-1}(Yj_{\pi,0}Y^{-1})Y|p0s\lambda\rangle\\&=-\eta_P'^*\eta_P(-1)^{s+s'-\lambda-\lambda'}\\&\qquad\times\langle p0s'-\lambda'|e^{i\pi J_2}j_{\pi,0}|p0s-\lambda\rangle.\end{aligned} \qquad (46)$$

Using the relation $\lambda+\lambda'=0$ and the definition of the functions Π, this may be written as

$$\Pi_{s'\lambda';s\lambda}=(-1)^{s+s'+\pi+1}\Pi_{s',-\lambda';s,-\lambda}, \qquad (47)$$

where the factor $\eta_P'^*\eta_P$ which specifies the relative parity of the initial and final states has been replaced by $(-1)^\pi$, $\pi=0$ (1) for even (odd) relative parity.

It remains only to consider the restrictions on the matrix elements which follow from time reversal. The neutral pion current changes sign under this operation

$$Tj_{\pi,0}T^{-1}=-j_{\pi,0}. \qquad (48)$$

Applying the identity given in Eq. (28), and using Eq. (25) to relate the time reversed helicity states to the ordinary helicity states, we obtain the result

$$\begin{aligned}\Pi_{s'\lambda';s\lambda}{}^*&={}^T\langle p0s'\lambda'|Te^{i\pi J_2}j_{\pi,0}T^{-1}|p0s\lambda\rangle^T\\&=-\langle p0s'\lambda'|e^{i\pi J_2}j_{\pi,0}|p0s\lambda\rangle\\&=-\Pi_{s'\lambda';s\lambda}.\end{aligned} \qquad (49)$$

We have again used the fact that $j_{\pi,0}$ transforms as a scalar with respect to rotations, and have chosen $\eta_T=1$ for all particles in Eq. (25). Summarizing the foregoing results, we find that the imposition of the symmetry operations of the full inhomogeneous Lorentz group leads to the following set of relations among the matrix elements of $j_{\pi,0}$ in the brick wall coordinate system:

$$\Pi_{s'\lambda';s\lambda}=\delta_{\lambda',-\lambda}\Pi_{s',-\lambda;s,\lambda}, \quad |\lambda|\leq s, s', \quad (J_3) \qquad (44')$$

$$\Pi_{s',-\lambda';s,-\lambda}=(-1)^{s+s'+\pi+1}\Pi_{s'\lambda';s\lambda}, \quad (Y) \qquad (47')$$

$$\Pi_{s'\lambda';s\lambda}{}^*=-\Pi_{s'\lambda';s\lambda}, \quad (T). \qquad (49')$$

These conditions severely restrict the number of independent vertex functions. If we label the rows of the matrix $\Pi_{s'\lambda';s\lambda}$ by λ', and the columns by λ, we obtain a matrix with $2s'+1$ rows and $2s+1$ columns, which could in general have $(2s+1)(2s'+1)$ complex entries. However, according to Eq. (49′), the matrix elements are pure imaginary numbers, and the nonzero entries are restricted to the skew diagonal, $\lambda'+\lambda=0$, Eq. (44′). Finally, the parity condition of Eq. (47′) relates the matrix elements with opposite signs of the helicities. It is a simple matter to determine the remaining number of independent matrix elements; the results are sum-

TABLE I. The number of independent vertex functions for the π^0 current is given for the different types of incoming and outgoing particles and the two values of the symmetry parameter $(-1)^\gamma$. Here γ is equal to $(s+s'+\pi+1)$, with $(-1)^\pi$ the relative parity of the initial and final states, and j is the lesser of s and s'.

Type of particles	$(-1)^\gamma=+1$	$(-1)^\gamma=-1$
Fermions	$j+\frac{1}{2}$	$j+\frac{1}{2}$
Bosons	$j+1$	j

marized in Table I. The matrix elements of $j_{\pi,0}$ in a general coordinate frame are completely specified in terms of those in the brick wall frame by Eq. (42.1); the number of independent vertex functions is therefore precisely the number of the independent functions Π.

Simple examples of the foregoing results are provided by the $N-N-\pi^0$ and the $N-N^*-\pi^0$ vertices [the first nucleon isobar N^* is regarded as a stable particle for illustrative purposes]. Applying the conditions imposed by Eqs. (44′), (47′), and (49′), one obtains as the matrix representations for the vertex functions:

$N\to N+\pi^0$: $s=s'=\frac{1}{2}$, $\pi=0$,

$$\Pi_{\frac{1}{2}\lambda';\frac{1}{2}\lambda}\to\begin{pmatrix}0 & ia\\ ia & 0\end{pmatrix},\quad a\text{ real};\tag{50}$$

$N\to N^*+\pi^0$: $s=\frac{1}{2}$, $s'=\frac{3}{2}$, $\pi=0$,

$$\Pi_{\frac{3}{2}\lambda';\frac{1}{2}\lambda}\to\begin{bmatrix}0 & 0\\ 0 & ib\\ -ib & 0\\ 0 & 0\end{bmatrix},\quad b\text{ real}.\tag{51}$$

The matrix elements a and b in these examples are functions of the momentum p of the particles in the brick wall frame.

b. Multipole Decomposition of the Pion Vertex Function

While the results of the preceding section are quite general, the parametrization of the vertex functions in terms of specific matrix elements for different values of λ is somewhat inconvenient. We shall therefore construct the analog of the usual nonrelativistic multipole expansion of the matrix element in Eq. (42.2). We begin by expressing the helicity state $|p0s\lambda\rangle$ in the brick wall system in terms of the rest states $|s,\lambda\rangle$ using Eqs. (10.1) and (10.2). The matrix elements may then be written in the form

$$\begin{aligned}\Pi_{s'\lambda';s\lambda}&=\langle p0s'\lambda'|e^{i\pi J_2}j_{\pi,0}|p0s\lambda\rangle\\&=\langle s'\lambda'|e^{i\zeta'K_3}e^{i\pi J_2}j_{\pi,0}e^{-i\zeta K_3}|s,\lambda\rangle\\&=(-1)^{s'-\lambda'}\langle s',-\lambda'|j_{\pi,0}e^{-i(\zeta+\zeta')K_3}|s,\lambda\rangle.\end{aligned}\tag{52}$$

We have used successively the fact that K_3 changes sign under a rotation through π about the y axis, the scalar character of $j_{\pi,0}$ under proper Lorentz transformations, and the relation given in Eq. (19). The Lorentz transformation parameters ζ and ζ' are equal, respectively, to $\sinh^{-1}(p/m)$ and $\sinh^{-1}(p/m')$, where p is the 3-momentum of the incident particle in the brick wall frame. The sum $\zeta+\zeta'$, for which we shall use the symbol ν, can be expressed conveniently in terms of the 4-momentum p and p' of the initial and final particles in the Lorentz invariant form

$$\nu=\zeta+\zeta'=\sinh^{-1}[(p\cdot p'/mm')^2-1]^{\frac{1}{2}}.\tag{53}$$

The form of the matrix element given in Eq. (52) is similar to that which would occur in a nonrelativistic theory of particles with structure. This suggests that we follow the usual procedure and expand the exponential in a series of tensor operators. We therefore write

$$j_{\pi,0}e^{-i\nu K_3}=\sum_{n=0}^{\infty}\frac{(-i\nu)^n}{n!}j_{\pi,0}K_3{}^n,\tag{54}$$

and decompose the products $j_{\pi,0}K_3{}^n$ into sets of spherical tensors. The irreducible tensor operator T_J of rank J is a set of $2J+1$ operators $T_{J,M}$ which transform under rotations in the same fashion as the spherical harmonics $Y_{J,M}$,[13]

$$RT_{J,M}R^{-1}=\textstyle\sum_{M'}T_{J,M'}D_{M'M}{}^{(J)}(R),\tag{55}$$

and which consequently satisfy the commutation relations

$$[J_3,T_{J,M}]=MT_{J,M},\tag{56.1}$$

$$[J_1\pm iJ_2,T_{J,M}]=[(J\pm M+1)(J\mp M)]^{\frac{1}{2}}T_{J,M\pm1}.\tag{56.2}$$

The product $j_{\pi,0}K_3{}^n$, composed of the scalar $j_{\pi,0}$ and the n-fold product of the third component of the vector operator $\mathbf{K}$ is readily seen to be a sum of tensor operators of rank $J\le n$ having $M=0$,

$$j_{\pi,0}K_3{}^n=\sum_{J=0}^{n}T_{J,0}{}^{(n)}.\tag{57}$$

Furthermore, only those terms contribute to the sum in Eq. (57) for which J is even (or odd) accordingly as n is even (or odd). This may be shown at once by applying a rotation through π about the y axis to both sides of Eq. (57), and using Eq. (55).

The irreducible components of the mixed tensor $j_{\pi,0}K_3{}^n$ can be constructed using the transformation relation given in Eq. (55) and the orthogonality conditions for the representation coefficients $D^{(J)}$,[12–14]

$$\int D_{m_1'm_2'}{}^{(j')*}(R)D_{m_1m_2}{}^{(j)}(R)dR=\frac{8\pi^2}{2J+1}\delta_{jj'}\delta_{m_1m_1'}\delta_{m_2m_2'}.\tag{58}$$

The integration in Eq. (58) is extended over the entire group space of the rotation group, $0\le\alpha\le2\pi$, $0\le\beta\le\pi$, $0\le\gamma\le2\pi$, where $R=R(\alpha\beta\gamma)$, and dR represents the volume element $d\alpha d(\cos\beta)d\gamma$. Thus, applying a rotation

R to the operator $j_{\pi,0}K_3^n$, and integrating over the group space with the weight $D_{M,0}^{(J)*}(R)$, we obtain from Eqs. (57) and (58) the representation for the tensor operator $T_{J,M}^{(n)}$

$$T_{J,M}^{(n)}=\frac{2J+1}{8\pi^2}\int D_{M,0}^{(J)*}(R)Rj_{\pi,0}K_3^nR^{-1}dR. \quad (59)$$

Substitution of this result in Eq. (54) and summation over n, leads to the desired decomposition of $j_{\pi,0}e^{-i\nu K_3}$ into a series of irreducible tensor operators,

$$j_{\pi,0}e^{-i\nu K_3}=\sum_{J=0}^{\infty} T_{J,0}, \quad (60.1)$$

where

$$T_{J,0}=\sum_{n=J}^{\infty}\frac{2J+1}{8\pi^2}\frac{(-i\nu)^n}{n!}\int D_{0,0}^{(J)*}(R)$$
$$\times Rj_{\pi,0}K_3^nR^{-1}dR, \quad (-1)^n=(-1)^J. \quad (60.2)$$

Thus, from Eq. (52),

$$\Pi_{s'\lambda';s\lambda}=(-1)^{s'-\lambda'}\sum_{J=0}^{\infty}\langle s',-\lambda'|T_{J,0}|s,\lambda\rangle$$
$$=(-1)^{2s'}\sum_{J=0}^{\infty}\begin{pmatrix} s' & J & s\\ \lambda' & 0 & \lambda\end{pmatrix}\langle s'\|T_J\|s\rangle. \quad (61)$$

In the last line, we have used the Wigner-Eckart theorem[13] to express the matrix elements of the tensor operators $T_{J,0}$, *taken between rest states*, in terms of the Wigner $3j$ symbols[12,13] (symmetrized vector coupling coefficients) and the reduced matrix elements $\langle s'\|T_J\|s\rangle$. Equation (61) gives the desired multipole-type decomposition of the matrix elements $\Pi_{s'\lambda';s\lambda}$. The $3j$ symbols vanish unless the sum of the lower indices is zero, that is, $\lambda+\lambda'=0$, and also if s, s', and J fail to satisfy the triangular inequalities, $s+J\geq s'$, $s'+J\geq s$, $s+s'\geq J$. Equation (61) may be inverted through the use of the orthogonality relations for the $3j$ symbols[12,13]; the resulting equation expresses the reduced matrix elements as linear combinations of the Π's,

$$\langle s'\|T_J\|s\rangle$$
$$=(-1)^{2s'}(2J+1)\sum_{\lambda}\begin{pmatrix} s' & J & s\\ -\lambda & 0 & \lambda\end{pmatrix}\Pi_{s',-\lambda;s,\lambda}. \quad (62)$$

Those properties of the reduced matrix elements which follow from symmetry considerations are easily deduced. From time-reversal invariance, it follows that the matrix elements $\Pi_{s'\lambda';s\lambda}$, hence also the $\langle s'\|T_J\|s\rangle$, are pure imaginary numbers, Eq. (49′). The parity condition of Eq. (47′) and the symmetry relations for the $3j$ symbols,[12,13]

$$\begin{pmatrix} s' & J & s\\ -\lambda' & -\mu & -\lambda\end{pmatrix}=(-1)^{s'+s+J}\begin{pmatrix} s' & J & s\\ \lambda' & \mu & \lambda\end{pmatrix}, \quad (63)$$

imply that the only nonzero reduced matrix elements are those for which J is even or odd accordingly as the relative parity of the initial and final particles is odd or even, $(-1)^J=(-1)^{\pi+1}$. Thus Eq. (61) becomes

$$\Pi_{s'\lambda';s\lambda}=(-1)^{2s'}\sum_{J=0}\begin{pmatrix} s' & J & s\\ \lambda' & 0 & \lambda\end{pmatrix}\langle s'\|T_J\|s\rangle, \quad (61)$$
$$(-1)^J=(-1)^{\pi+1}\text{ only.}$$

These restrictions, together with those which follow from the properties of the $3j$ symbols mentioned above, are sufficient to determine the number of nonvanishing terms in Eq. (61), that is, the number of independent matrix elements, as given in Table I. We note finally that the condition $(-1)^n=(-1)^J$ which restricts the powers of ν which appear in Eq. (60.2) may be restated for the nonvanishing matrix elements in terms of the relative parity of the initial and final states as $(-1)^n=(-1)^{\pi+1}$. The inverse hyperbolic sines which enter the definition of ν, Eq. (53), are odd functions of the 3-momentum p of the particles in the brick wall coordinate system. Furthermore, ν vanishes linearly with p for $p\to 0$, $\nu\to p(m+m')/(mm')$ for $p\ll m,m'$. The nonvanishing reduced matrix elements $\langle s'\|T_J\|s\rangle$ are therefore even or odd functions of p for odd or even relative parities of the initial and final particles, and vanish as p^J for $p\to 0$. These properties of the reduced matrix elements with respect to the possible values of J and the powers of p are precisely the same as would be obtained in a nonrelativistic theory in which matrix elements of the pion current multiplied by the usual retardation factor $e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}}$ were calculated between nuclear states with angular momenta s and s' and relative parity $(-1)^\pi$.

It should be emphasized that the reduced matrix elements $\langle s'\|T_J\|s\rangle$ are dynamical quantities, which depend in value on the details of the mutual interactions of the particles s, s', and the pion field. Dynamical considerations are beyond the scope of this paper. The number and symmetry properties of the matrix elements, and some general requirements on their dependence on p, are nevertheless determined by purely kinematical requirements relating to Lorentz invariance. These properties of the matrix elements and the parametrization of the general vertex function given in Eqs. (42) and (61) are quite useful in the analysis of certain types of experiments.

c. Generalizations and Examples

The discussion of the preceding sections was confined to the properties of matrix elements of the neutral pion current operator $j_{\pi,0}$. The results may easily be extended to include the charged pion current. Let T, T_3 and T', T_3' denote the total isotopic spin and its third component for the initial and final particles, and let

$j_{\pi,q}$ denote the q component of the pion current,

$$j_{\pi,q}=(\Box+m_\pi^2)\phi_q,\quad q=0,\pm1. \tag{64}$$

The indices $q=+1,0,-1$ refer, respectively, to the positive, neutral, and negative pion fields. The matrix elements of $j_{\pi,q}$ between single particle states may be reduced through the use of the Wigner-Eckart theorem applied to the matrix elements in isotopic spin space,

$$\langle T'T_3'|j_{\pi,q}|T,T_3\rangle =(-1)^{T'-T_3'}\begin{pmatrix} T' & 1 & T\\ -T_3' & q & T_3\end{pmatrix}\langle T'\|j_\pi\|T\rangle, \tag{65}$$

where we have suppressed the remaining quantum numbers. This reduction depends first on the fact that the isotopic spin operators satisfy the commutation relations for angular momentum, and second, on the independence of the internal isotopic spin symmetry, and the properties of the fields under Lorentz transformations. The reduced matrix elements $\langle T'\|j_\pi\|T\rangle$ may be expressed at once in terms of the matrix elements of the neutral pion current for some particular choice of the initial and final charge states, say $T_3=t$ equal to the lesser of T and T'. In the notation of the preceding sections, we then write

$$\langle p0s'\lambda'T't|e^{i\pi J_2}j_{\pi,0}|p0s\lambda Tt\rangle=\Pi_{s'\lambda';s\lambda}(T',T). \tag{66}$$

Comparison of this definition with Eq. (65) indicates that the reduced matrix elements $\langle T'\|j_\pi\|T\rangle$ are related to the Π's by

$$\langle T'\|j_\pi\|T\rangle =(-1)^{T'-t}\begin{pmatrix} T' & 1 & T\\ -t & 0 & t\end{pmatrix}^{-1}\Pi_{s'\lambda;s\lambda}(T',T). \tag{67}$$

Consequently, the number of independent matrix elements $\langle T'\|j_\pi\|T\rangle$ is the same as the number of independent functions $\Pi_{s'\lambda';s\lambda}$, Table I, and no new vertex functions are added by the generalization from the neutral to the charged pion fields. It is clear that these results can be extended without difficulty to incorporate other internal symmetries relative to the initial and final states, for example, the conservation of strangeness and baryon number, or, in symmetry models for the strong interactions, global symmetry.

The reduction techniques which led to the multipole decomposition of the matrix elements $\Pi_{s'\lambda';s\lambda}$, Eqs. (60) and (61), are quite general, and permit the simultaneous discussion of matrix elements between states of arbitrary spin. In the simple cases of spins 0, $\frac{1}{2}$, and 1, the same reduction may be carried through fairly easily by constructing explicit representations for the states. The general methods are perhaps best illustrated in these familiar cases. We shall therefore apply the general techniques to the reduction of the pion-nucleon vertex function using the standard Dirac representation for the nucleon spinors. While this representation for spin $\frac{1}{2}$ is nonunitary, contrary to our assumptions, a consistent scheme is obtained by replacing the adjoint states $\langle \mathbf{p}\frac{1}{2}\lambda|$ by the Dirac adjoint spinor $\bar{u}(\mathbf{p},\lambda)=u^+(\mathbf{p},\lambda)\gamma_4$, and the states $|\mathbf{p}\frac{1}{2}\lambda\rangle$ by the spinors $u(\mathbf{p},\lambda)$. The requisite normalization of the spinors according to Eq. (14) is the invariant normalization $\bar{u}(\mathbf{p},\lambda)u(\mathbf{p},\lambda')=\delta_{\lambda\lambda'}$. From Eq. (61), we obtain for the matrix element of $j_{\pi,0}$ between nucleon states

$$\Pi_{\frac{1}{2}\lambda';\frac{1}{2}\lambda}=(-1)^{\frac{1}{2}-\lambda'}\sum_{J=0}^{\infty}\frac{2J+1}{8\pi^2}\int D_{00}{}^{(J)}(R) \times\langle\tfrac{1}{2},-\lambda'|Rj_{\pi,0}e^{-2i\zeta K_3}R^{-1}|\tfrac{1}{2},\lambda\rangle dR, \tag{68}$$

where $R=R(\alpha\beta\gamma)$, and $dR=d\alpha d(\cos\beta)d\gamma$. We remark first the $D_{00}{}^{(J)}(R)=P_J(\cos\beta)$. Furthermore, the rotation $e^{-i\gamma J_3}$ commutes with $e^{-2i\zeta K_3}$ and thus drops out of Eq. (68). The rotation $e^{-i\alpha J_3}$ may be applied directly to the states, and the integrals over α and γ may then be performed. There results the simpler expression

$$\Pi_{\frac{1}{2}\lambda';\frac{1}{2}\lambda}=(-1)^{\frac{1}{2}+\lambda}\delta_{\lambda',-\lambda}\sum_J(J+\tfrac{1}{2})\int P_J(\cos\beta) \times\langle\tfrac{1}{2},\lambda|e^{-i\beta J_2}j_{\pi,0}e^{-2i\zeta K_3}e^{i\beta J_2}|\tfrac{1}{2},\lambda\rangle d(\cos\beta). \tag{69}$$

At this point, we shall introduce the Dirac representation for the states and use the standard form for the pion current to lowest order in perturbation theory,

$$j_{\pi,0}=ig\bar{\psi}\gamma_5\tau_0\psi. \tag{70}$$

Using the additional relations $K_3=\frac{1}{2}\gamma_3\gamma_4$ and $J_2=-\frac{1}{2}\gamma_3\gamma_1$, the operator in Eq. (69) may be brought into the matrix form

$$ig\tau_0\begin{pmatrix} -(p/m)\sigma_3\cos\beta & (p_0/m)+(p/m)\sigma_1\sin\beta\\ (p_0/m)+(p/m)\sigma_1\sin\beta & -(p/m)\sigma_3\cos\beta\end{pmatrix}, \tag{71}$$

where the corresponding Dirac spinors are expressed in terms of two-component Pauli spinors χ_λ quantized along the positive z axis by

$$u(\mathbf{p},\lambda)=\frac{1}{(2m)^{\frac{1}{2}}}\begin{pmatrix}(p_0+m)^{\frac{1}{2}}\chi_\lambda\\ -(p_0-m)^{\frac{1}{2}}\sigma_3\chi_\lambda\end{pmatrix}. \tag{72}$$

Multiplying out the matrices and performing the integration over β, one finds that only the $J=1$ term survives (P-wave pion-nucleon interaction), and $\Pi_{\frac{1}{2}\lambda';\frac{1}{2}\lambda}$ may be represented as a matrix in helicity space by

$$\Pi_{\frac{1}{2}\lambda';\frac{1}{2}\lambda}\to ig\frac{p}{m}\langle\tau_0\rangle\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{73}$$

This result, which is of the form given in Eq. (50), provides the necessary connection between the lowest order pion-nucleon interaction expressed in the helicity representation, and the standard Dirac representation.

The same result can be obtained with a trivial amount of effort by computing directly the matrix element of $j_{\pi,0}$ in the brick wall coordinate frame, using the Dirac spinors of Eq. (72). [Note, however, that the quantization axes are different in the initial and final states.] The use of the general method in the present case is obviously unnecessary; it may nevertheless be useful in more complicated problems.

As a second example, we shall apply the results of the preceding sections to the calculation of the differential cross section for the scattering of particles of arbitrary spin in the one pion exchange approximation, corresponding to the diagram of Fig. 1. For simplicity, we will suppress any isotopic spin dependence, and consider only the exchange of a neutral pion. The contribution of this diagram to the scattering amplitude M may be expressed in terms of the pion vertex functions for the pairings a, c and b, d of the incident and final particles, and the pion propagator, which we shall approximate by its leading term, $[(p_a-p_c)^2+m_\pi^2]^{-1}$,[19]

$$M_{\lambda_c\lambda_d;\lambda_a\lambda_b}=(2\pi)^4\delta^4(p_a+p_b-p_c-p_d)\times[(p_a-p_c)^2+m_\pi^2]^{-1}F_{\lambda_c\lambda_d;\lambda_a\lambda_b} \quad (74)$$

where

$$\begin{aligned}F_{\lambda_c\lambda_d;\lambda_a\lambda_b}&=\langle\mathbf{p}_cs_c\lambda_c|j_{\pi,0}|\mathbf{p}_as_a\lambda_a\rangle\langle\mathbf{p}_ds_d\lambda_d|j_{\pi,0}|\mathbf{p}_bs_b\lambda_b\rangle\\&=\sum_{\lambda_a'\lambda_b'\lambda_c'\lambda_d'}D_{\lambda_c'\lambda_c}{}^{(s_c)*}(R_c)D_{\lambda_a'\lambda_a}{}^{(s_a)}(R_a)D_{\lambda_b'\lambda_b}{}^{(s_b)}(R_b)\\&\qquad\times D_{\lambda_d'\lambda_d}{}^{(s_d)*}(R_d)\Pi_{s_c\lambda_c';s_a\lambda_a'}\,\Pi_{s_d\lambda_d';s_b\lambda_b'}.\end{aligned} \quad (75)$$

The second form for F follows from the transformation properties of the vertex functions given in Eq. (42.1). The differential cross section for the scattering of unpolarized particles is obtained from M by a familiar calculation,

$$\begin{aligned}d\sigma=&(2\pi)^4\delta^4(p_a+p_b-p_c-p_d)\\&\times\frac{m_c}{p_{0c}}\frac{m_d}{p_{0d}}\frac{d^3p_c}{(2\pi)^3}\frac{d^3p_d}{(2\pi)^3}\frac{m_am_b}{f}[(p_a-p_c)^2+m_\pi^2]^{-2}\\&\times(2s_a+1)^{-1}(2s_b+1)^{-1}\sum_{\lambda_a\lambda_b\lambda_c\lambda_d}|F_{\lambda_c\lambda_d;\lambda_a\lambda_b}|^2,\end{aligned} \quad (76)$$

where f is the Møller factor,

$$f=[(p_a\cdot p_b)^2-(m_am_b)^2]^{\frac{1}{2}} \quad (77)$$

and we have assumed that the final helicities are not observed. The seemingly complicated sums over the helicities simplify considerably when use is made of the unitary character of the representation coefficients [cf. Eq. (36)],

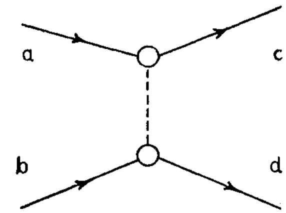

FIG. 1. The diagram considered in the discussion of scattering processes in the single-quantum-exchange approximation.

$$\textstyle\sum_{\lambda'}D_{\lambda\lambda'}{}^{(s)*}(R)D_{\mu\lambda'}{}^{(s)}(R)=\delta_{\mu\lambda}. \quad (78)$$

The resulting expression involves no D functions, and it is consequently unnecessary to determine the rotations R_i. Substituting finally for the functions the multipole decomposition given in Eq. (61), and using orthogonality relations for the $3j$ symbols,[12,13] one obtains for the sum

$$\begin{aligned}&\sum_{\lambda_a\lambda_b\lambda_c\lambda_d}|F_{\lambda_c\lambda_d;\lambda_a\lambda_b}|^2\\&=\sum_{\lambda_a\lambda_b\lambda_c\lambda_d}|\Pi_{s_c\lambda_c;s_a\lambda_a}|^2|\Pi_{s_d\lambda_d;s_b\lambda_b}|^2\\&=\sum_{JJ'}[(2J+1)(2J'+1)]^{-1}|\langle s_c\|T_J(p_{ac})\|s_a\rangle|^2\\&\qquad\times|\langle s_d\|T_{J'}(p_{bd})\|s_b\rangle|^2,\end{aligned} \quad (79.1)$$

where the summation indices J and J' are restricted to the values

$$|s_a-s_c|\le=J\le s_a+s_c,\quad(-1)^J=(-1)^{\pi_{ac}}, \quad (79.2)$$

$$|s_b-s_d|\le J'\le s_b+s_d,\quad(-1)^{J'}=(-1)^{\pi_{bd}}. \quad (79.3)$$

Substitution of Eq. (79.1) in Eq. (76) yields the desired expression for the cross section. The (trivial) generalization necessary to include the exchange of charged pions is left to the reader. Specialization to the case of pion-nucleon scattering using Eq. (73) leads to familiar result for that problem in lowest order perturbation theory,

$$d\sigma=\delta^4(p_a+p_b-p_c-p_d)d^3p_cd^3p_d(m^2/p_{0c}p_{0d})m^2/F\times4(q^2/4m^2)^2G^4[q^2+m_\pi^2]^{-2}, \quad (80)$$

where $G^2=g^2/4\pi\simeq14$, and q^2 is the invariant square of the 4-momentum transfer, $q^2=(p_a-p_c)^2$. Note that p^2, the square of the 3-momentum of any of the particles in the appropriate brick wall coordinate system, is equal to $\frac{1}{4}q^2$.

It is clear that the methods which we have used to study the general kinematic structure of matrix elements of the pion current are applicable also in the case of other pseudoscalar or scalar currents. The symmetry properties of the matrix elements will, of course, be different if the properties of the current in question under parity and time reversal differ from those of the

[19] The transition matrix element should actually be expressed in terms of the truncated vertex functions of H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **2**, 425 (1955), and the complete pion propagator $D_F'(p_a-p_c)$. However, the truncated matrix elements are related to the Π's of the present paper by $\Pi^t=D_F(D_F')^{-1}\Pi$ for the initial and final particles on the mass shell. In the approximation $D_F'\approx D_F$, which should be valid for $|(p_a-p_c)^2|\approx m_\pi^2$, the two sets of functions are the same, and one obtains Eq. (74). The approximation used neglects all vacuum polarization-type diagrams on the exchanged pion line.

pion current. Cases of interest include the K-meson vertex (pseudoscalar), and the vertices associated with the (hypothetical) scalar K and π mesons. The parity condition analogous to that in Eq. (47′) must, however, be treated with some care for the strange mesons, because for these unstable particles, the parity is defined only relative to the baryon systems. For example, one can speak of the parity of the K meson relative to the ΣN system, but not of an absolute K-meson parity. The factor (-1) in Eq. (47′) which arose from the π-meson parity would consequently be absent in such a case, and the factor $(-1)^{\pi}$ would represent the parity of the combined system.

IV. PROPERTIES OF VERTEX FUNCTIONS: THE ELECTROMAGNETIC CURRENT

We will complete our discussion of the properties of vertex functions in the helicity representation by studying the matrix elements of the electromagnetic current between single-particle states. This is in some respects a more interesting case than that of the pion current, if only because of the greater utility of the results which follows from the relative weakness of the electromagnetic interactions. The electromagnetic current transforms as a 4-vector under proper Lorentz transformation; thus, if a transformation of the coordinate system is specified by $x_\mu \to x_\mu'$, with $x_\mu'=a_{\mu\nu}x_\nu$, j_μ transforms as

$$Gj_\mu G^{-1}=a_{\mu\nu}j_\nu. \tag{81}$$

The matrix elements of j_μ in a general Lorentz frame may be related through Eq. (40) to the matrix elements in the standard brick wall frame, but in contrast to the case of the scalar field, the transformation now depends explicitly on the matrix $a_{\mu\nu}$,

$$\langle \mathbf{p}'s'\mu'|j_\mu|\mathbf{p}s\mu\rangle = \sum_{\lambda\lambda'\nu} a_{\mu\nu}D_{\lambda'\mu'}{}^{(s')*}(R_g')\Gamma_{s'\lambda';s\lambda}{}^{(\nu)}D_{\lambda\mu}{}^{(s)}(R_g), \tag{82.1}$$

where $\Gamma_{s'\lambda';s\lambda}{}^{(\nu)}$ is the matrix element of j_ν in the brick wall system,

$$\Gamma_{s'\lambda';s\lambda}{}^{(\nu)}=\langle p0s'\lambda'|e^{i\pi J_2}j_\nu|p0s\lambda\rangle. \tag{82.2}$$

a. Symmetry Properties of the Vertex Functions

It will be convenient in the ensuing discussion to use spherical components for the spacelike part of j_μ; these are related to the Cartesian components of $\mathbf{j}=(j_1,j_2,j_3)$ as

$$j_{\pm1}=\mp\frac{1}{\sqrt{2}}(j_1\pm ij_2),\quad j_3=j_3. \tag{83}$$

We will use the symbol j_0 for the Hermitian fourth component of j_μ, that is, for the charge density operator. The spacelike part of j_μ transforms under rotations as a polar vector, while j_0 is a scalar. Under the reflection Y in the x-y plane, j_2 changes sign,

$$Yj_\mu Y^{-1}=(j_1,-j_2,j_3,j_0), \tag{84}$$

while under time reversal, $\mathbf{j}$ changes sign,

$$Tj_\mu T^{-1}=(-j_1,-j_2,-j_3,j_0). \tag{85}$$

We note finally the usual assumption of current conservation,

$$\partial_\mu j_\mu=0. \tag{86}$$

Of the matrix elements of j_μ in the brick wall frame, the simplest is that which involves j_0. The results obtained in Sec. III(a) for the pion vertex function may be taken over at once, with simple changes in sign in the relations which arise from the reflection and time inversion symmetries. The results corresponding to those of Eqs. (44′), (47′), and (49′) are then

$$\Gamma_{s'\lambda';s\lambda}{}^{(0)}=\delta_{\lambda',-\lambda}\Gamma_{s',-\lambda;s\lambda}{}^{(0)},\quad (J_3) \tag{87}$$

$$\Gamma_{s',-\lambda';s,-\lambda}{}^{(0)}=(-1)^{s'+s+\pi}\Gamma_{s'\lambda';s\lambda}{}^{(0)},\quad (Y) \tag{88}$$

$$\Gamma_{s'\lambda';s\lambda}{}^{(0)*}=\Gamma_{s'\lambda';s\lambda}{}^{(0)}.\quad (T) \tag{89}$$

From Eqs. (87) and (89), the matrix elements of j_0 between single-particle states are real, and vanish unless $\lambda+\lambda'=0$. The parity relation, Eqs. (88), serves as in the case of the pion current to reduce the number of independent matrix elements to roughly one-half of the nominal number.

The matrix elements of j_3 may be obtained from those of j_0 using the equation of current conservation, Eq. (86). This may be rewritten in momentum space in the form

$$(p'-p)_\mu\langle p'|j_\mu|p\rangle=0. \tag{90}$$

In the brick wall frame the initial and final particles move along the positive and negative z axes with equal momenta p. Equation (90) therefore yields the relation

$$2p\Gamma_{s'\lambda',s\lambda}{}^{(3)}=(p_0-p_0')\Gamma_{s'\lambda',s\lambda}{}^{(0)}, \tag{91}$$

and the matrix elements $\Gamma^{(3)}$ are seen to be multiples of the $\Gamma^{(0)}$. If the masses of the initial and final particles are different, then $p_0'\neq p_0$, and the matrix elements of j_3 are nonzero. On the other hand, if the initial and final particles are identical or if they have the same mass, the matrix elements of j_3 vanish identically. *The vanishing of $\Gamma^{(3)}$ for identical particles is in fact independent of the assumption of current conservation.*[5,20] This remarkable result is easily proved. Because j_3 is unchanged by rotations about the z axes, $[j_3,J_3]=0$, and

$$\Gamma_{s\lambda';s\lambda}{}^{(3)}=\delta_{\lambda',-\lambda}\Gamma_{s,-\lambda;s\lambda}{}^{(3)}. \tag{92}$$

From the properties of j_3 under space and time inversions, Eqs. (84) and (85), one deduces in addition that

$$\Gamma_{s\lambda;s,-\lambda}{}^{(3)}=(-1)^{2s}\Gamma_{s,-\lambda;s\lambda}{}^{(3)},\quad (Y) \tag{93}$$

[20] The special case of spin $\frac{1}{2}$ particles was considered by Ernst, Sachs, and Wali.[4] The generalization to arbitrary spin was first given in reference 5.

and

$$\Gamma_{s,-\lambda;s\lambda}{}^{(3)*}=\Gamma_{s,-\lambda;s\lambda}{}^{(3)}. \quad (T) \tag{94}$$

But because we are dealing with matrix elements of a Hermitian operator between states which describe the same particle, the last relation can be re-expressed as follows:

$$\begin{aligned}\Gamma_{s,-\lambda;s\lambda}{}^{(3)*}&=\langle p0s\,-\lambda|e^{i\pi J_2}j_3|p0s\lambda\rangle^* \\ &=\langle p0s\lambda|j_3e^{-i\pi J_2}|p0s\,-\lambda\rangle \\ &=(-1)^{2s+1}\langle p0s\lambda|e^{i\pi J_2}j_3|p0s\,-\lambda\rangle \\ &=(-1)^{2s+1}\Gamma_{s,\lambda;s,-\lambda}{}^{(3)}.\end{aligned} \tag{95}$$

We have used the fact that

$$e^{2\pi iJ_2}|p0s\lambda\rangle=(-1)^{2s}|p0s\lambda\rangle, \tag{96}$$

and have noted that j_3 changes sign under a rotation through π about the y axis. Comparing Eq. (95) with the combined results of Eqs. (93) and (94), we see at once that $\Gamma_{s\lambda';s\lambda}{}^{(3)}\equiv 0$. Because the matrix elements of the scalar quantity $\partial_\mu j_\mu$, the 4-divergence of the current, are equal in the brick wall frame to $2p\Gamma_{s'\lambda';s\lambda}{}^{(3)}$, it follows that the matrix elements of this quantity vanish in any Lorentz frame, *independently of whether the current j_μ is conserved or not*, $\langle\partial\cdot j\rangle\equiv 0$.[20]

It is clear that this result will be valid also for general vector currents which carry additional internal quantum numbers, provided the internal symmetries of the current and of the initial and final states do not alter their assumed space-time transformation properties. The theorem would apply, for example, to the very interesting case of a charge independent interaction of the [unstable] $T=1$ vector meson ρ[21] with the nucleon field, as well to the interactions with nucleons of the $T=0$ vector mesons ω[22] and η.[23] To the extent to which one can neglect differences in the masses of particles which are presumably of electromagnetic origin, it is applicable to the vector current of beta decay theory. It would not apply to the axial vector beta decay current because of a changed sign in Eq. (93).

The discussion of the symmetry properties of the remaining matrix elements is quite straightforward, and the details will for the most part be omitted. From the commutation relations of a vector operator with J_3 [this is a special case of Eq. (56.1)],

$$[J_3,j_q]=qj_q, \tag{97}$$

it follows that

$$\Gamma_{s'\lambda';s\lambda}{}^{(\pm1)}=\delta_{\lambda',-\lambda\mp1}\Gamma_{s',-\lambda\mp1;s,\lambda}{}^{(\pm1)}, \quad |\lambda|<s,\ |\lambda'|<s', \quad (J_3) \tag{98}$$

while, from the properties of $\mathbf{j}$ under the Y transformation, Eq. (84), we obtain

$$\Gamma_{s',-\lambda';s,-\lambda}{}^{(\pm1)}=(-1)^{s+s'+\pi}\Gamma_{s'\lambda';s\lambda}{}^{(\mp1)}. \quad (Y) \tag{99}$$

The factor $(-1)^\pi$ again represents the relative parity of the initial and final states. Note that the upper indices are interchanged in this relation corresponding to the interchange of j_1 and j_{-1} under Y. Finally, the time-reversal transformation, Eq. (85), implies that the matrix elements are real,

$$\Gamma_{s'\lambda';s\lambda}{}^{(\pm1)*}=\Gamma_{s'\lambda';s\lambda}{}^{(\pm1)}. \quad (T) \tag{100}$$

In the special case in which the initial and final particles are the same, we may derive an additional relation, analogous to that obtained above for $\Gamma^{(3)}$, by using the Hermitian character of the operators j_1 and j_2 to re-express the left-hand member of Eq. (100) in the form

$$\Gamma_{s\lambda';s\lambda}{}^{(\pm1)*}=(-1)^{2s+1}\Gamma_{s\lambda;s\lambda'}{}^{(\pm1)}, \quad s'\equiv s. \tag{101.1}$$

Combining this result with that of Eq. (100), we obtain the extra condition in the convenient form

$$\Gamma_{s\lambda';s\lambda}{}^{(\pm1)}=(-1)^{2s+1}\Gamma_{s\lambda;s\lambda'}{}^{(\pm1)}, \quad s'\equiv s. \tag{101.2}$$

The symmetry relations given in Eqs. (87)–(89) and (98)–(101) limit the number of independent vertex functions for the vector current. The matrix elements of j_3, if nonzero, can be expressed in terms of the matrix elements of j_0 using Eq. (91); in addition, Eq. (99) relates the matrix elements of j_{-1} to those of j_{+1}. To obtain a complete parametrization of the vector vertex, it is therefore necessary only to specify the matrix elements of j_0 and j_{+1}. The possible numbers of independent nonzero matrix elements of these operators as determined from the symmetry conditions are summarized in Table II. The number of matrix elements of each kind in the present, relativistic theory is, not un-

TABLE II. The number of independent vertex functions for the components j_0 and j_{+1} of the electromagnetic current. Here j is the lesser of s and s', and γ is equal to $s'+s+\pi$, where $(-1)^\pi$ is the relative parity factor for the initial and final states. The total number of independent vertex functions is equal to the sum of the numbers for j_0 and j_{+1}. The results are applicable to general conserved vector currents which transform like j_μ. The definition of identical particles can in this case be broadened to include particles which differ only in an internal quantum number.

		$(-1)^\gamma=+1$		$(-1)^\gamma=-1$	
Type of particle	Spins	j_0	j_{+1}	j_0	j_{+1}
Nonidentical fermions	$s'=s$	$j+\frac{1}{2}$	$2j$	$j+\frac{1}{2}$	$2j$
	$s'\neq s$	$j+\frac{1}{2}$	$2j+1$	$j+\frac{1}{2}$	$2j+1$
Nonidentical bosons	$s'=s$	$j+1$	$2j$	j	$2j$
	$s'\neq s$	$j+1$	$2j+1$	j	$2j+1$
Identical fermions	$s'=s$	$\cdots$	$\cdots$	$j+\frac{1}{2}$	$j+\frac{1}{2}$
Identical bosons	$s'=s$	$j+1$	j	$\cdots$	$\cdots$

[21] See, for example, E. Pickup, D. K. Robinson, and E. O. Salant, Phys. Rev. Letters **7**, 192 (1961), and the numerous references contained therein.

[22] B. C. Maglić, L. W. Alvarez, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. Letters **7**, 178 (1961); N. H. Xuoung and G. R. Lynch, *ibid.* **7**, 327 (1961); and the references to theoretical papers given therein.

[23] A. Pevsner, R. Kraemer, M. Nussbaum, C. Richardson, P. Schlein, R. Strand, T. Toohig, M. Block, A. Engler, R. Gessaroli, and C. Meltzer, Phys. Rev. Letters **7**, 421 (1961). Also A. Pevsner *et al.*, Bull. Am. Phys. Soc. **6**, 433 (1961).

expectedly, the same as occurs in the nonrelativistic theory of multipole radiation.[24]

b. Multipole Decomposition of the Electromagnetic Vertex Function

It is natural at this point to seek a multipole decomposition of the electromagnetic vertex function analogous to that which is familiar in the nonrelativistic theory.[25] The procedure to be followed is similar to that which was used in the discussion of the pion vertex function, but slight complications appear because the operator j_μ, in contrast to $j_{\pi,0}$, does not commute with the proper Lorentz transformations. Thus, expressing the matrix element $\Gamma_{s'\lambda';s\lambda}{}^{(0)}$, Eq. (82.2), in terms of the rest states $|s,\lambda\rangle$ for the ingoing and outgoing particles by using the definitions of Eqs. (10), we obtain

$$\begin{aligned}\Gamma_{s'\lambda';s\lambda}{}^{(0)}&=\langle s',\lambda'|e^{i\zeta'K_3}e^{i\pi J_2}j_0e^{-i\zeta K_3}|s,\lambda\rangle\\&=(-1)^{s'-\lambda'}\langle s',-\lambda'|e^{-i(\nu/2)K_3}j_0'\\&\qquad\times e^{-i(\nu/2)K_3}|s,\lambda\rangle,\end{aligned}\quad(102)$$

where ν is defined in Eq. (53), and we have written the result in a form symmetric with respect to the initial and final momenta and energies. The operator j_0' is defined by a Lorentz transformation,

$$\begin{aligned}j_0'&=e^{i(\sigma/2)K_3}j_0e^{-i(\sigma/2)K_3}\\&=j_0\cosh(\sigma/2)+j_3\sinh(\sigma/2)\end{aligned}\quad(103)$$

where

$$\sigma=\zeta-\zeta'=\sinh^{-1}\left\{\frac{(m'^2-m^2)}{mm'}\frac{[(p\cdot p')^2-(mm')^2]^{\frac{1}{2}}}{-(p+p')^2}\right\}.\quad(104)$$

The matrix element of j_3 can be written in a similar form,

$$\begin{aligned}\Gamma_{s'\lambda';s\lambda}{}^{(3)}=(-1)^{s'-\lambda'}\langle s',-\lambda'|e^{-i(\nu/2)K_3}j_3'\\\times e^{-i(\nu/2)K_3}|s,\lambda\rangle,\end{aligned}\quad(105)$$

with

$$j_3'=j_0\sinh(\sigma/2)+j_3\cosh(\sigma/2).\quad(106)$$

One may now use the current conservation relation, Eq. (91), or in the case of identical ingoing and outgoing particles, the fact that $\Gamma^{(3)}$ is equal to zero whether the current is conserved or not, to eliminate the operator j_3 between Eqs. (102) and (105), obtaining thereby

$$\begin{aligned}\Gamma_{s'\lambda';s\lambda}{}^{(0)}=W(p,p')(-1)^{s'-\lambda'}\langle s',-\lambda'|e^{-i(\nu/2)K_3}j_0\\\times e^{-i(\nu/2)K_3}|s,\lambda\rangle.\end{aligned}\quad(107)$$

$W(p,p')$ is a kinematic factor which is symmetric in the 4-momenta p and p',

$$W(p,p')=\left[\cosh(\sigma/2)+\frac{p_0'-p_0}{2p}\sinh(\sigma/2)\right]^{-1}.\quad(108)$$

For initial and final particles of equal mass, $W=1$. A more compact, but less symmetric form for Eq. (107) can be obtained by commuting one of the remaining Lorentz transformations with j_0, and again eliminating j_3 by using the connections between $\Gamma^{(3)}$ and $\Gamma^{(0)}$.

The multipole decomposition of $\Gamma^{(0)}$ may now be constructed by expanding the exponentials in Eq. (107), and expressing the result in terms of irreducible tensor operators. The construction is similar to that given in Sec. III(b), and the details will be omitted. The result is

$$\Gamma_{s'\lambda';s\lambda}{}^{(0)}=(-1)^{2s'}\sum_{J=0}\begin{pmatrix}s'&J&s\\\lambda'&0&\lambda\end{pmatrix}Q_J(s',s),$$
$$(-1)^J=(-1)^\pi.\quad(109)$$

The sum is restricted to even or odd values of J accordingly as the relative parity of the initial and final particles is even or odd, the restriction following from the Y symmetry. The form factors $Q_J(s's)$ are real functions of p, the 3-momentum in the brick wall frame,

$$Q_J(s',s)=\langle s'\|T_J{}^{(0)}\|s\rangle,\quad(110.1)$$

with

$$T_{J,M}{}^{(0)}=W(p,p')\sum_{n=J}^{\infty}\frac{2J+1}{8\pi^2}\frac{(-i\nu/2)^n}{n!}\int D_{M,0}{}^{(J)*}(R)$$
$$\times\sum_{m=0}^{n}\binom{n}{m}RK_3{}^{n-m}j_0K_3{}^mR^{-1}dR,$$
$$(-1)^n=(-1)^J.\quad(110.2)$$

For nonidentical particles in the initial and final states, the Q_J have the significance of charge transition form factors. On the other hand, for identical particles, $J=0$, $2,\cdots 2s$ only, and the form factors Q_J describe the electric charge (monopole), quadrupole, and higher charge moments of the system. It is interesting to note in this case that for a vanishing 4-momentum transfer,

$$T_{J,M}{}^{(0)}\rightarrow j_0\delta_{J,0}\delta_{M,0},\quad p\rightarrow 0.\quad(111)$$

Therefore, the form factors Q_J vanish for $J>0$, and Q_0 is simply related to the total charge of the particle. From Eqs. (107) and (109),

$$Q_0\rightarrow(2s+1)^{\frac{1}{2}}\langle s,\lambda|j_0|s,\lambda\rangle=(2s+1)^{\frac{1}{2}}e,\quad p\rightarrow 0.\quad(112)$$

One may easily deduce the connection between the remaining Q's and the static multipole moments of the particle in the usual nonrelativistic theory by identifying the irreducible tensor operators in the two cases.[25] Similar identifications can, of course, be made for the transition moments. We remark finally that the form

[24] A proof of this result was apparently first given by Yennie, Lévy, and Ravenhall, reference 2, in the special case of identical initial and final particles. The generalization to different initial and final states follows easily from their results.

[25] See, for example, the treatment of multipole fields given in M. E. Rose, *Multipole Fields* (John Wiley & Sons, Inc., New York, 1955).

factors $Q_J(s',s)$ vanish in general as p^J for $p\to 0$, $p\ll m,m'$.

The multipole decomposition of the matrix elements of the transverse components of the current, $j_{\pm 1}$, is also easily accomplished. In this case, there is no difficulty with the Lorentz transformations, $j_{\pm 1}$, commuting with K_3, and the matrix elements may be brought into the form

$$\Gamma_{s'\lambda';s\lambda}{}^{(\pm 1)}=(-1)^{s'-\lambda'}\langle s',-\lambda'|j_{\pm 1}e^{-i\nu K_3}|s,\lambda\rangle, \quad (113)$$

where ν is again defined by Eq. (53). However, it is clear from the commutation relations of $j_{\pm 1}$ with J_3, Eq. (97), and Eq. (56.1) that the operator in Eq. (113) is the ± 1 component of a reducible tensor operator. The multipole decomposition of the matrix elements consequently assumes the form

$$\Gamma_{s'\lambda';s\lambda}{}^{(\pm 1)}=(-1)^{2s'}\sum_{J=1}\begin{pmatrix} s' & J & s \\ \lambda' & \pm 1 & \lambda\end{pmatrix} \times\langle s'\|T_J{}^{(\pm 1)}\|s\rangle, \quad (114)$$

where the irreducible tensor operators $T_{J,M}{}^{(\pm 1)}$ are defined by

$$T_{J,M}{}^{(\pm 1)}=\sum_{n=J-1}^{\infty}\frac{2J+1}{8\pi^2}\frac{(-i\nu)^n}{n!}\int D_{M,\pm 1}{}^{(J)*}(R) \times Rj_{\pm 1}K_3{}^nR^{-1}dR. \quad (115)$$

The summation index n is restricted in those operators which lead to nonvanishing reduced matrix elements to odd or even n values accordingly as the relative parity of the initial and final states is even or odd, $(-1)^n=(-1)^{\pi+1}$. This restriction on n may be derived by substituting in Eq. (113) the identity

$$j_{\pm 1}e^{-i\nu K_3}=e^{i\pi J_2}j_{\mp 1}e^{+i\nu K_3}e^{-i\pi J_2}, \quad (116)$$

and applying the rotations to the rest states using the result given in Eq. (19). One obtains in this fashion the alternative expressions for $\Gamma^{(\pm 1)}$

$$\Gamma_{s'\lambda';s\lambda}{}^{(\pm 1)}=(-1)^{s+s'+1}(-1)^{s'+\lambda'} \times\langle s',\lambda'|j_{\mp 1}e^{+i\nu K_3}|s,-\lambda\rangle. \quad (117)$$

But from Eqs. (99) and (113), we obtain also

$$\Gamma_{s'\lambda';s\lambda}{}^{(\pm 1)}=(-1)^{s+s'+\pi}(-1)^{s'+\lambda'} \times\langle s',\lambda'|j_{\mp 1}e^{-i\nu K_3}|s,-\lambda\rangle. \quad (118)$$

Expansion of the exponentials in Eqs. (117) and (118) and comparison of the results indicates that only those terms can contribute to the matrix elements for which the powers of ν are odd (or even), if the relative parity of the initial and final states is even (or odd). This is the stated restriction.

It is convenient to separate the tensor operators of Eq. (115) into two classes accordingly as $(-1)^J=(-1)^n$ or $(-1)^J=(-1)^{n+1}$; $(-1)^n$ is the orbital parity absorbed at the vertex by the transverse, (odd parity) part of j_μ. Noting that $(-1)^n=(-1)^{\pi+1}$, we therefore write

$$\Gamma_{s'\lambda';s\lambda}{}^{(\pm 1)}=(-1)^{2s'}\sum_{J=1}\begin{pmatrix} s' & J & s \\ \lambda' & 1 & \lambda\end{pmatrix} \times\{\tfrac{1}{2}[1+(-1)^{J+\pi}]E_J(s',s) +\tfrac{1}{2}[1-(-1)^{J+\pi}]M_J(s',s)\}. \quad (119)$$

The form factors M_J and E_J correspond to the magnetic and electric transition multipole moments of radiation theory.[25] The matrix elements $\Gamma^{(-1)}$ may be obtained from the $\Gamma^{(+1)}$ using Eqs. (99) and (63),

$$\Gamma_{s'\lambda';s\lambda}{}^{(-1)}=(-1)^{2s'}\sum_{J=1}\begin{pmatrix} s' & J & s \\ \lambda' & -1 & \lambda\end{pmatrix} \times\{\tfrac{1}{2}[1+(-1)^{J+\pi}]E_J(s',s) -\tfrac{1}{2}[1-(-1)^{J+\pi}]M_J(s',s)\}. \quad (120)$$

In the special case in which the initial and final particles are the same, the matrix elements $\Gamma^{(\pm)}$ satisfy the extra symmetry relation given in Eq. (101.2). It may be shown through the use of the symmetry relations for the $3j$ symbols[13] that this restricts J to odd values in Eqs. (119) and (120). The form factors E_J are consequently zero in this case, and we obtain

$$\Gamma_{s'\lambda';s\lambda}{}^{(\pm 1)}=\pm(-1)^{2s'}\sum_{J}\begin{pmatrix} s' & J & s \\ \lambda' & \pm 1 & \lambda\end{pmatrix}M_J, \quad s'\equiv s,\quad J \text{ odd}. \quad (121)$$

An elementary particle for which time reversal and parity are symmetry operations cannot, therefore have any electric multipole moments E_J.

From Eq. (115) and the restrictions on J and n, it follows that the form factors $E_J(s',s)$ and $M_J(s',s)$ approach zero, respectively, as p^{J-1} and p^J for $p\to 0$, $p\ll m,m'$. For example, the form factor E_1 can approach a constant value for $p\to 0$, M_1 and E_2 vanish as p, M_2 and E_3 vanish as p^2, and so forth. This behavior is necessary. It is, of course, possible for the form factors to vanish more rapidly. This is generally the case in nonrelativistic problems involving the emission or absorption of radiation by complex nuclei. The current operator connecting states $\psi_{s\lambda}$ and $\psi_{s'\lambda'}$ is then of the (approximate) form[26]

$$\mathbf{j}=\sum_j(e_j/2m_j)[\psi_{s'\lambda'}{}^*\mathbf{p}_j\psi_{s\lambda}+(\mathbf{p}_j\psi_{s'\lambda'})^*\psi_{s\lambda} +i\mathbf{p}_j\times(\psi_{s'\lambda'}{}^*\boldsymbol{\sigma}_j\psi_{s\lambda})], \quad (121.1)$$

$$j_0=\sum_j e_j\psi_{s'\lambda'}{}^*\psi_{s\lambda}. \quad (121.2)$$

The single-particle momentum operators p_j effectively introduce an additional power of p into the limiting powers given above for the multipole form factors E_J

[26] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

and M_J, leading to the familiar nonrelativistic estimates for the momentum dependence.[25,26] The limiting forms for the Q_J are not changed. On the other hand, the limiting powers of p are actually those given above in the case of elementary particles with the intrinsic charge and magnetic multipole moments defined by the Q_J and M_J.

The multipole decompositions of the matrix elements $\Gamma_{s'\lambda';s\lambda}{}^{(\nu)}$ which we have given in this section can be generalized to the matrix elements of any vector current with the same transformation properties as j_μ. However, if that current is not conserved $\partial_\mu j_\mu \neq 0$ the vertex functions $\Gamma^{(0)}$ and $\Gamma^{(3)}$ are independent, and a new set of multipole form factors must be introduced for the latter. Internal symmetries of the system may be incorporated in the manner discussed in connection with the pion current. Cases of interest include the vertex functions for the currents associated with the unstable ρ,[21] ω,[22] η,[23] and K_V[27] mesons, if such currents exist, and with the vector current of beta decay theory. With simple modifications, the results may be extended to the case of axial vector currents as well.

c. Applications and Examples

The foregoing results are of immediate utility in the description of many processes which involve the coupling of particles to the electromagnetic field. If the coupling is sufficiently weak, the interaction may be regarded as a perturbation, and, to lowest order in e, one is led to calculate matrix elements of $-j\cdot A$. Since $j\cdot A$ is a Lorentz scalar, the analog of Eq. (82.1) may be written

$$\langle p's'\mu'|-j\cdot A|ps\mu\rangle = -\sum_{\lambda\lambda'} D_{\lambda'\mu'}{}^{(s')*}(R_g')\Gamma_{s'\lambda';s\lambda}{}^{(\nu)}A_\nu D_{\lambda\mu}{}^{(s)}(R_g), \quad (123.1)$$

where, suppressing the spin indices for simplicity,

$$-\Gamma^{(\nu)}A_\nu = \Gamma^{(0)}A_0 - \Gamma^{(3)}A_3 + \Gamma^{(-1)}A_{+1} + \Gamma^{(+1)}A_{-1}. \quad (123.2)$$

A_μ is here the electromagnetic field in the brick wall coordinate system. The photon [real or virtual] which is emitted at the vertex travels along the positive z axis with 4-momentum $q=(2p,\, p_0-p_0')$ and helicity specified by the index on Γ. The current conservation relation, Eq. (91), may be used to write the matrix element of $j\cdot A$ between "brick wall" states in the obviously gauge-invariant form

$$\langle p0s'\lambda'|e^{i\pi J_2}(-j\cdot A)|p0s\lambda\rangle = \Gamma_{s'\lambda';s\lambda}{}^{(0)}[A_0-(p_0-p_0')A_3/2p] + \Gamma_{s'\lambda';s\lambda}{}^{(1)}A_{-1} + \Gamma_{s'\lambda';s\lambda}{}^{(-1)}A_{+1}. \quad (124)$$

[27] M. Alston, L. W. Alvarez, P. Eberhard, M. L. Good, W. Graziano, H. K. Ticho, and S. G. Wojcicki, Phys. Rev. Letters **5**, 11, 520 (1960). M. A. B. Bég and P. C. DeCelles, *ibid.* **6**, 145, 428(E) (1961).

The gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$, or, in momentum space, $A_\mu \to A_\mu + q_\mu \chi$, does not affect the transverse components of the field $A_{\pm1}$, while the terms added to A_0 and A_3 drop out when combined as in Eq. (124). We remark also that the term in Eq. (124) which involves A_0 and A_3 does not contribute to the emission of real photons. The field in that case satisfies the transversality condition $q\cdot A=0$ with $q^2=0$, and only $A_{\pm1}$ contribute to the radiation. The emission of real radiation therefore involves in lowest order, only the form factors E_J and M_J for the transverse current, and then only for a fixed value of p for each transition. The charge form factors Q_J can of course contribute to processes such as the Auger effect, the internal conversion of radiation, or electromagnetic scattering, which involve virtual photons. The matrix elements for these processes are usually labeled by the multipolarity as EJ, for example, the $E0$ processes in internal conversion. We have chosen in the present paper to use a different nomenclature to emphasize the fact that the charge matrix elements Q_J are fundamentally different from, and independent of, the matrix elements E_J and M_J which are associated with the transverse components of the current.

A simple example is provided by the radiative decay of a polarized particle. Let the initial particle have spin s and helicity λ along the positive z axis in its rest system, and let the final particle, with spin s' and helicity λ', emerge in the direction specified by the angles θ, ϕ. The transition matrix element for the decay is then given to lowest order in the electromagnetic coupling by

$$M_{s'\lambda';s\lambda} = (2\pi)^4\delta^4(p'+q-p)F_{s'\lambda';s\lambda}, \quad (125.1)$$

where

$$F_{s'\lambda';s\lambda} = \langle p'\theta\phi s'\lambda'|-j\cdot A|s,\lambda\rangle. \quad (125.2)$$

One easily sees that the appropriate transformation to the brick wall frame is given by

$$G = e^{-i\zeta K_3}e^{-i\pi J_2}R^{-1}(\phi,\theta,-\phi), \quad (126)$$

with $\zeta=\sinh^{-1}(p/m)$, p the 3-momentum of either particle in the brick frame. The rotations R_g and R_g' of Eq. (123) are then found to be

$$R_g = e^{-i\pi J_2}R^{-1}(\phi,\theta,-\phi) \quad (127.1)$$

and

$$R_g' = 1. \quad (127.2)$$

We may therefore express $F_{s'\lambda';s\lambda}$ in terms of the Γ's as

$$F_{s'\lambda';s\lambda} = \sum_\mu \langle p0s'\lambda'|e^{i\pi J_2}(-j\cdot A)|p0s\mu\rangle \times D_{\mu\lambda}{}^{(s)}(-\phi,\pi-\theta,-\phi) = \sum_\mu [\Gamma_{s'\lambda';s\mu}{}^{(+1)}A_{-1} + \Gamma_{s'\lambda';s\mu}{}^{(-1)}A_{+1}] \times(-1)^{s+\mu}D_{\lambda\mu}{}^{(s)}(\phi,\theta,-\phi). \quad (128)$$

We have omitted the term in Eq. (124) which cannot

contribute to the real radiative process. The transition probability for the decay is proportional to the absolute square of $F_{s'\lambda';s\lambda}$, summed over the helicity indices of the photon and the final particle if those helicities are not observed. It is a simple, if tedious, matter to verify that the usual angular distributions for multipole radiation are obtained when appropriate terms in the multipole expansions of the Γ's are substituted in Eq. (128). The details of the calculation are left to the reader. We shall only note, first, that the natural Lorentz frame in which to consider the angular distributions is the center of mass frame of the outgoing particles, that is, the rest frame of the initial particle, and second, that multipole form factors and multipole radiation appear naturally in the relativistic theory. However, the multipole form factors are defined by matrix elements of certain irreducible tensor operators calculated in an unphysical system, the common rest system of the initial and final particles.

An interesting application of Eq. (128) which involves in addition to the results given so far, the notation of the polarization of a relativistic particle, occurs in the discussion of the radiative decay $\Sigma^0 \to \Lambda^0+\gamma$ for polarized Σ's. The correlation between the polarization of the Λ^0 and the linear polarization of the γ provides a test for the $\Sigma^0-\Lambda^0$ relative parity.[28] The necessary discussion of polarization phenomena is given by Jacob and Wick.[3] The calculation, which will be left to the reader, leads quickly to the well known result of Feldman and Fulton.[28].

As a final example, we shall calculate the cross section for the electromagnetic scattering of particles of arbitrary spin to lowest order in the fine structure constant. The interaction may be represented by the diagram of Fig. 1. The calculation is similar to that which was carried out in Sec. III(c) for the scattering of particles of arbitrary spin in the one pion exchange approximation. The transition matrix element for the electromagnetic scattering can be written in the form

$$M_{\lambda_c\lambda_d;\lambda_a\lambda_b}=(2\pi)^4\delta^4(p_a+p_b-p_c-p_d)\times[(p_a-p_c)^{-2}]F_{\lambda_c\lambda_d;\lambda_a\lambda_b}, \quad (129)$$

where

$$F_{\lambda_c\lambda_d;\lambda_a\lambda_b}=-\langle \mathbf{p}_c s_c\lambda_c|j_\mu|\mathbf{p}_a s_a\lambda_a\rangle\times\langle \mathbf{p}_d s_d\lambda_d|j_\mu|\mathbf{p}_b s_b\lambda_b\rangle. \quad (130)$$

The individual matrix elements may be expressed in terms of the matrix elements in the brick wall system using Eq. (82.1). The (unitary) representation coefficients $D^{(s)}$ disappear in the expression for the cross section for the scattering of unpolarized particles, if the final helicities are not observed, and one is left with expressions of the form (a, c vertex)

$$\begin{aligned}S_{\mu\nu}(s_cs_a)&=\sum_{\lambda_a\lambda_c}\Gamma_{s_c\lambda_c;s_a\lambda_a}{}^{(\mu)}\Gamma_{s_c\lambda_c;s_a\lambda_a}{}^{(\nu)*}\\&=\sum_{\lambda_a\lambda_c}\{(\Gamma_{s_c\lambda_c;s_a\lambda_a}{}^{(0)})^2[\delta_{\mu0}\delta_{\nu0}-(p_{0a}-p_{0c})\\&\quad\times(\delta_{\mu0}\delta_{\nu3}+\delta_{\mu3}\delta_{\nu0})/2p_{ac}+(p_{0a}-p_{0c})^2\\&\quad\times\delta_{\mu3}\delta_{\nu3}/(2p_{ac})^2]+(\Gamma_{s_c\lambda_c;s_a\lambda_a}{}^{(+1)})^2\\&\qquad\times[\delta_{\mu\nu}-\delta_{\mu3}\delta_{\nu3}+\delta_{\mu0}\delta_{\nu0}]\}. \quad (131)\end{aligned}$$

Equation (131) holds in the brick wall system of particles a and c. It may be generalized to an arbitrary frame rather easily. Let us introduce two orthogonal 4-vectors,

$$P_\mu=(p_a+p_c)_\mu \quad (132.1)$$

and

$$K_\mu=(p_a-p_c)_\mu-P_\mu(m_c^2-m_a^2)/P^2, \quad (132.2)$$

$$K\cdot P=0. \quad (132.3)$$

In the brick wall frame, P_μ has only a time like component; the only nonvanishing component of K_μ is the z component, with the magnitude $+2p_{ac}$. We may consequently replace the delta functions in Eq. (131) by appropriate combinations of P and K. The resulting expression gives the desired generalization of $S_{\mu\nu}(s_cs_a)$ to an arbitrary Lorentz frame,[29]

$$\begin{aligned}S_{\mu\nu}(s_cs_a)&=\sum_{\lambda_a\lambda_c}\{(\Gamma_{s_c\lambda_c;s_a\lambda_a}{}^{(0)})^2(P^2)^{-1}[-P_\mu P_\nu+(P_\mu K_\nu+P_\nu K_\mu)\\&\quad\times(m_c^2-m_a^2)/K^2-K_\mu K_\nu(m_c^2-m_a^2)^2/K^4]\\&\quad+(\Gamma_{s_c\lambda_c;s_a\lambda_a}{}^{(+1)})^2[\delta_{\mu\nu}-K_\mu K_\nu/K^2-P_\mu P_\nu/P^2]\}. \quad (133)\end{aligned}$$

It is easily verified that $S_{\mu\nu}$ satisfies the conditions of current conservation or gauge invariance,

$$(p_a-p_c)_\mu S_{\mu\nu}=S_{\mu\nu}(p_a-p_c)_\nu=0. \quad (134)$$

The remaining sums over the helicities λ_a and λ_c may be performed using the multipole decompositions of the vertex functions $\Gamma^{(0)}$ and $\Gamma^{(+1)}$ given in Eqs. (109) and (119),

$$\sum_{\lambda_a\lambda_c}(\Gamma_{s_c\lambda_c;s_a\lambda_a}{}^{(0)})^2=\sum_J(2J+1)^{-1}\tfrac{1}{2}[1+(-1)^{J+\pi_{ac}}]Q_J{}^2(s_cs_a), \quad (135.1)$$

$$\begin{aligned}\sum_{\lambda_a\lambda_c}(\Gamma_{s_c\lambda_c;s_a\lambda_a}{}^{(+1)})^2&=\sum_J(2J+1)^{-1}\{\tfrac{1}{2}[1+(-1)^{J+\pi_{ac}}]E_J{}^2(s_cs_a)\\&\qquad+\tfrac{1}{2}[1-(-1)^{J+\pi_{ac}}]M_J{}^2(s_cs_a)\}, \quad (135.2)\end{aligned}$$

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[28] G. Feldman and T. Fulton, Nuclear Phys. **8**, 106 (1958). A different parametrization for this problem which uses explicitly the properties of photons and spin ½ particles under Lorentz transformations, is given by L. Michel and H. Rouhaninejad, Phys. Rev. **122**, 242 (1961).

[29] This result was given for the special case of elastic scattering by Yennie, Lévy, and Ravenhall, reference 2.

where in each case $|s_c-s_a|\leq J\leq s_c+s_a$, and $J\geq 1$ in Eq. (135.2).

Combining the result given in Eq. (133) with the analogous result for the b, d vertex, and supplying the appropriate kinematic factors, we obtain for the electromagnetic scattering cross section to lowest order in the electromagnetic coupling

$$d\sigma=(2\pi)^4\delta^4(p_a+p_b-p_c-p_d)\frac{m_c}{p_{0c}}\frac{m_d}{p_{0d}}\frac{d^3p_c}{(2\pi)^3}\frac{d^3p_d}{(2\pi)^3}\frac{m_am_b}{f}$$
$$\times[(p_a-p_c)^2]^{-2}(2s_a+1)^{-1}(2s_b+1)^{-1}$$
$$\times S_{\mu\nu}(s_c,s_a)S_{\mu\nu}(s_d,s_b). \quad (136)$$

This expression encompasses both elastic and inelastic scattering, and is completely covariant. The result may obviously be generalized to the case of scattering by the exchange of a single vector meson associated with a conserved vector current j_μ by making the appropriate change in the propagator[30,31] in the transition matrix element in Eq. (129),

$$\delta_{\mu\nu}/q^2\rightarrow[\delta_{\mu\nu}+q_\mu q_\nu/m^2][q^2+m^2]^{-1}, \quad (137)$$

where $q_\mu=(p_a-p_c)_\mu=(p_d-p_b)_\mu$, and m is the mass of the meson. The calculation of the vertex functions and the summations over the helicities of the initial and final particles are carried out as before. It is necessary only to note that, because of the conservation conditions given in Eq. (134), the q-dependent terms in the numerator in Eq. (137) do not contribute to the final result. The generalized cross section may therefore be obtained from Eq. (136) simply by replacing the factor $[(p_a-p_c)^2]^{-2}$ by $[(p_a-p_c)^2+m^2]^{-2}$. The internal symmetries which may be involved in the transitions if the vector meson carries additional quantum numbers, such as the isotopic spin in the case of the $T=1$ vector meson ρ,[21] on the isotopic spin and strangeness in the case of the strange $T=\frac{1}{2}$ vector meson K_V,[27] may be treated as in the discussion of the pion vertex function, Sec. III(c).[32] The matrix elements $\Gamma^{(\nu)}$ are then to be regarded as reduced matrix elements with respect to the internal symmetries, and Eq. (136) must be generalized to include the specification of the internal quantum numbers. We recall also that the parity conditions must be treated with care in the case of the strange mesons.

[30] For a discussion of the theory of vector mesons, see, for example, N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959).

[31] See, however, the remarks in reference 19 on the use of the complete vs the free propagator, and on the complete rather than the truncated vertex functions.

[32] To the extent to which it is reasonable to consider a sharp scattering resonance as representing an unstable particle, and to treat that particle as stable in some approximation.

Considerable simplifications result in the factors $S_{\mu\nu}$ in Eq. (136) if attention is restricted to elastic processes. It is a simple matter to check, for example, that Eq. (136) reproduces the well-known Rosenbluth cross section[33] for the scattering of electrons by protons. The necessary relations between the charge and magnetic moment form factors Q_0 and M_1 for the proton, and the customary Dirac and Pauli form factors F_1 and F_2 in the expression for the current[1]

$$\langle p'|j_\mu|p\rangle=ie\bar{u}(p')[\gamma_\mu F_1(q^2)$$
$$+(\kappa/2m)F_2(q^2)\sigma_{\mu\nu}(p'-p)_\nu]u(p),$$
$$q^2=(p'-p)^2, \quad (138)$$

are as follows[2,34]:

$$Q_0(p)=\sqrt{2}e[F_1-\kappa(p/m)^2F_2], \quad (139.1)$$

$$M_1(p)=2-6^{\frac{1}{2}}e(p/m)[F_1+\kappa F_2]. \quad (139.2)$$

We have noted that $q^2=4p^2$, where p is the 3-momentum of the incident particle in the brick wall system. In the case of the electron, κ, the anomalous part of the magnetic moment, is equal to zero, but M_1 is not zero because of the Dirac magnetic moment. These relations may be derived either by reducing the expressions in Eq. (138) to two-component form, or by making a formal reduction using the definitions given in Eqs. (110), (114), and (119), and the techniques which were illustrated in Sec. III(c).

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[33] M. N. Rosenbluth, Phys. Rev. **79** 615 (1950).

[34] J. P. Walecka, Nuovo cimento **11**, 821 (1959).