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Eikonal Method in Magnetohydrodynamics

STEVEN WEINBERG*
Institute for Defense Analyses, Washington, D. C.
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The eikonal method is extended to waves with several components propagating in inhomogeneous anisotropic media. Formulas are derived for the motion of wave packets, the change in amplitude along a ray path, and the corrections due to diffraction. The method is applied to pure magnetohydrodynamic disturbances, and the problem of computing ray paths in the ionosphere is discussed in some detail. One result is that the fraction of the energy of an isotropic disturbance that eventually gets as low as 200 km drops from 0.99 at 200 km to 0.2 at 450 km, rises again to 0.99 at 700 km, drops to 0.5 at 3000 km, and continues dropping at greater altitudes. Energy trapped between 200 km and 700 km oscillates around 450 km with period 8.5 sec (at the geomagnetic equator). The eikonal method is also applied to a very general problem involving coupled magnetohydrodynamic, electrodynamic, and acoustic modes.

I. INTRODUCTION

FOR a long time mathematical physicists have known how to deal with certain special cases of the propagation of waves in mildly inhomogenous media. The eikonal (or WKB) method has been employed for one-component waves, and has also been extended to multi-component waves[1] in isotropic media.

However, this is not adequate to handle magnetohydrodynamic disturbances, where the "medium" is always anisotropic. For this reason we have developed a full generalization of the eikonal approach, which we describe in Secs. IV through VII.

To exemplify the method, we first treat in Secs. II and III some problems that are complicated enough to motivate the general treatment, yet simple enough to keep the algebra tolerable. In Sec. II we consider pure magnetohydrodynamics, and develop some applications to the behavior of disturbances in the ionosphere. In Sec. III the eikonal approach is applied to the coupling of magnetohydrodynamic, electrodynamic, and acoustic modes in an inhomogeneous plasma of finite conductivity.

Some attempt has been made to keep the discussion in Secs. II and III self contained, and the reader with no interest in magnetohydrodynamics or in generalities may, respectively, skip the first or the second part of this paper.

Many of the results obtained here have perhaps been known for some time. Our purpose is to show that the eikonal method is very general, and that these (and other) results may be derived in a systematic way for any problem involving a medium whose properties change only slowly in space and time.

One explanatory word about earlier work seems called for here. It is well known that shock waves propagate according to the same equations as surfaces of constant phase for ordinary disturbances. The eikonal treatment of shock waves has been discussed previously both for acoustics[2] and magnetohydrodynamics.[3] Our interest here is not in discontinuous shocks, but in continuous modes of propagation. The two problems seem closely related as far as calculating trajectories is concerned, but quite different when it comes to finding the change in amplitude along a ray path. It may be that the analogy is even closer than is realized by the author, but at any rate we shall use the language here of ordinary waves rather than of shock waves.

II. PURE MAGNETOHYDRODYNAMICS

We shall consider here the propagation of disturbances in an inhomogeneous medium of high conductivity

* Permanent address: Physics Department, University of California, Berkeley, California.
[1] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, New York, 1959), p. 109.
[2] Evry Schatzman, Ann. astrophys. **12**, 203 (1949); J. B. Keller, J. Appl. Phys. **25**, 938 (1954).
[3] J. Bazer and O. Fleischman, Phys. Fluids **2**, 366 (1959).

permeated by a varying magnetic field. Effects of pressure, viscosity, displacement current, finite conductivity, etc., are neglected here, but in the next section we show how to take them all into account.

The unperturbed system is stationary, and has mass density $\rho(\mathbf{x})$ and magnetic field $\mathbf{B}^{(0)}(\mathbf{x})$. A disturbance is characterized by a small electric field $\mathbf{E}(\mathbf{x})$, velocity $\mathbf{u}(\mathbf{x})$, current $\mathbf{J}(\mathbf{x})$, and a small increment $\mathbf{B}(\mathbf{x})$ to $\mathbf{B}^{(0)}(\mathbf{x})$; all these small fields have time dependence $e^{-i\omega t}$. The equations describing the system are

$$-ic\nabla\times\mathbf{E}=\omega\mathbf{B}, \tag{1}$$

$$-ic\nabla\times\mathbf{B}=-4\pi i\mathbf{J}, \tag{2}$$

$$\mathbf{E}=-(1/c)\mathbf{u}\times\mathbf{B}^{(0)}, \tag{3}$$

$$-i\omega\rho\mathbf{u}=(1/c)\mathbf{J}\times\mathbf{B}^{(0)}. \tag{4}$$

Equations (2)–(4) immediately give

$$\omega c\mathbf{E}=-i\mathbf{V}\times\mathbf{V}\times\nabla\times\mathbf{B}, \tag{5}$$

where $\mathbf{V}$ is the vector Alfvén velocity,

$$\mathbf{V}=\mathbf{B}^{(0)}/(4\pi\rho)^{\frac{1}{2}}. \tag{6}$$

Here and below when we write a string of cross products like $\mathbf{a}\times\mathbf{b}\times\mathbf{c}\times\cdots$, we shall always mean $\mathbf{a}\times\{\mathbf{b}\times[\mathbf{c}\times\cdots]\}$.

The zeroth order eikonal approximation is obtained by setting

$$\mathbf{E}=\mathbf{E}_0e^{iS},\quad \mathbf{B}=\mathbf{B}_0e^{iS},$$

and neglecting all variations in $\mathbf{E}$ and $\mathbf{B}$ except in the common factor e^{iS}. Equations (1) and (5) then give

$$c\mathbf{k}\times\mathbf{E}_0=\omega\mathbf{B}_0, \tag{7}$$

$$\omega c\mathbf{E}_0=\mathbf{V}\times\mathbf{V}\times\mathbf{k}\times\mathbf{B}_0, \tag{8}$$

where

$$\mathbf{k}\equiv\nabla S. \tag{9}$$

The solutions are of two types:

I. Alfvén Mode:

$$c\mathbf{E}_0=(\mathbf{V}\times\mathbf{V}\times\mathbf{k})(f_0/Vk\sin\theta), \tag{10}$$

$$\omega\mathbf{B}_0=-(\mathbf{k}\cdot\mathbf{V})(\mathbf{V}\times\mathbf{k})(f_0/Vk\sin\theta), \tag{11}$$

$$0=D_0\equiv(\mathbf{k}\cdot\mathbf{V})^2-\omega^2. \tag{12}$$

II. "Fast" Mode:

$$c\mathbf{E}_0=(\mathbf{V}\times\mathbf{k})(g_0/Vk\sin\theta), \tag{13}$$

$$\omega\mathbf{B}_0=-(\mathbf{k}\times\mathbf{k}\times\mathbf{V})(g_0/Vk\sin\theta), \tag{14}$$

$$0=D_0\equiv\mathbf{k}^2\mathbf{V}^2-\omega^2. \tag{15}$$

Here θ is the angle between $\mathbf{V}$ and $\mathbf{k}$. Formulas for f_0 and g_0 will be derived later in this section. (There is also a "slow" mode with $\mathbf{E}_0$ and $\mathbf{B}_0$ parallel to $\mathbf{k}$, but in our approximation it has $\omega=0$.)

Clearly neither (12) nor (15) suffices to determine the vector $\mathbf{k}$; we must use the additional requirement that $\nabla\times\mathbf{k}=0$. The general solution is obtained by constructing ray paths $\mathbf{x}(t)$, $\mathbf{k}(t)$ (see Sec. IV) satisfying

$$\frac{d\mathbf{x}}{dt}=-\frac{\partial D_0}{\partial\mathbf{k}}\bigg/\frac{\partial D_0}{\partial\omega}, \tag{16}$$

$$\frac{d\mathbf{k}}{dt}=-\frac{\partial D_0}{\partial\mathbf{x}}\bigg/\frac{\partial D_0}{\partial\omega}. \tag{17}$$

For the Alfvén mode, the group velocity is the Alfvén velocity,

$$d\mathbf{x}/dt=\mathbf{V} \tag{18}$$

and a disturbance travels only on magnetic field lines; the propagation vector changes at a rate

$$d\mathbf{k}/dt=-k_i\nabla V_i. \tag{19}$$

For the fast mode, the group velocity is

$$d\mathbf{x}/dt=V^2\mathbf{k}/\omega=V(\mathbf{k}/k), \tag{20}$$

so that again its magnitude is V but its direction is now that of $\mathbf{k}$. The propagation vector $\mathbf{k}$ changes at a rate:

$$d\mathbf{k}/dt=-(\omega/V)\nabla V=-k\nabla V. \tag{21}$$

Combining (20) and (21) we obtain for the equation of motion of a disturbance:

$$\left(\frac{1}{V^2}\frac{d}{dt}\right)^2\mathbf{x}=\tfrac{1}{2}\nabla V^{-2}. \tag{22}$$

The geometrical orbit of any wave packet in this mode is identical to that of a mass point moving in a gravitational potential

$$\phi=-\tfrac{1}{2}V^{-2}. \tag{23}$$

To see how these equations for the motion of a fast-mode disturbance may be used in a practical problem, let us first suppose that the Alfvén velocity V in the ionosphere may be regarded as a function of altitude Z alone. Then according to (21) the horizontal component of k is constant; its magnitude is $k\sin\eta$, where η is the angle between $\mathbf{k}$ and the upwards vertical. However, $\omega=kV(Z)$ is also constant, so that

$$\sin(\eta)/V(Z)=\text{const} \tag{24}$$

along any trajectory. If a disturbance starts at altitude Z with angle η, it will be reflected back towards Z at an altitude Z' if

$$V(Z')=V(Z)/\sin(\eta). \tag{25}$$

The condition that a disturbance starting down at Z reach the "bottom" of the ionosphere[4] at $Z_0=200$ km

[4] The altitude 200 km is the "bottom" in the sense that below this height effects of dissipation and finite conductivity become important, and also the properties of the ionosphere change so rapidly that the eikonal method could only be used at these heights for phenomena (e.g., "whistlers") with frequencies very much greater than 1 $\sec^{-1}$. By actually integrating the full wave equations, connection formulas have been obtained to link disturbances as high as 550 km with effects on the ground. See W. E. Francis and Robert Karplus, J. Geophys. Research **65**, 3593 (1960).

before being reflected upwards is that $\sin\eta < V(Z)/V(Z_0)$, where at geomagnetic latitude λ, the Alfvén velocity[5] is

$$V(Z_0) \cong (227 \text{ km/sec}) f(\lambda), \tag{26}$$

$$f(\lambda) = (1+3\sin^2\lambda)^{\frac{1}{2}}. \tag{27}$$

Hence, at altitudes between 700 km and 3000 km, where $V > V(Z_0)$, any disturbance starting downwards will reach 200 km, while any disturbance starting upwards will be reflected back downwards (and reach 200 km) if $\sin\eta > V(Z)/V(Z_1)$; $Z_1 = 3000$ km is the altitude of maximum Alfvén velocity and

$$V(Z_1) \cong (5180 \text{ km/sec}) f(\lambda).$$

Therefore, the proportion of the energy of an isotropic disturbance at 700 km $< Z <$ 3000 km that reaches "bottom" is

$$\begin{aligned}\mathcal{P} &= \frac{1}{2} + \frac{1}{2}\int_{\sin^{-1}[V(Z)/V(Z_1)]}^{\pi/2} \sin\eta d\eta \\ &= \tfrac{1}{2}\{1+(1-[V(Z)/V(Z_1)]^2)^{\frac{1}{2}}\},\end{aligned} \tag{28}$$

which drops from $\mathcal{P} \cong 0.99$ at $Z = 700$ km to $\mathcal{P} = \frac{1}{2}$ at $Z = 3000$ km.

For altitudes above 3000 km, $V(Z) > V(Z_0)$, so any disturbance starting downwards will get to "bottom", unless reflected before reaching 3000 km. Waves starting upwards will not be reflected back, but will leave the earth's environs. The proportion of an isotropic disturbance reaching "bottom" is thus

$$\mathcal{P} = \tfrac{1}{2}\{1-(1-[V(Z)/V(Z_1)]^2)^{\frac{1}{2}}\} \tag{29}$$

which drops from $\mathcal{P} = \frac{1}{2}$ at $Z = 3000$ km to $\mathcal{P} \cong 0.02$ at $Z = 10\,000$ km.

For altitudes between 200 km and 700 km, a disturbance directed downwards reaches 200 km if $\sin\eta < V(Z)/V(Z_0)$, while one directed upwards is reflected downwards if $\sin\eta > V(Z)/V(Z_1)$, and reach 200 km if $\sin\eta < V(Z)/V(Z_0)$. Hence the proportion of an isotropic disturbance reaching bottom from these altitudes is

$$\begin{aligned}\mathcal{P} = \tfrac{1}{2}\{1-(1-[V(Z)/V(Z_0)]^2)^{\frac{1}{2}}\} &+ \tfrac{1}{2}\{(1-[V(Z)/V(Z_1)]^2)^{\frac{1}{2}} \\ &-(1-[V(Z)/V(Z_0)]^2)^{\frac{1}{2}}\}\end{aligned} \tag{30}$$

$$\cong 1-(1-[V(Z)/V(Z_0)]^2)^{\frac{1}{2}}, \tag{31}$$

which has a minimum $\mathcal{P} \cong 0.2$ at about 450 km and rises to $\mathcal{P} \cong 0.99$ at 200 km and 700 km.

What happens after a wave packet reaches 200 km cannot be analyzed here because the effects of finite conductivity become important. These effects are discussed in the next section.

If a disturbance initiated between 200 km and 700 km has $\sin\eta > V(Z)/V(Z_0)$, it will be trapped, oscillating back and forth around the altitude $Z_2 = 450$ km of minimum Alfvén velocity, between two points Z' satisfying (25). As we have seen, about 80% of the energy of an isotropic disturbance at 450 km gets caught in this way. The period of oscillation can be easily estimated, if we approximate

$$\begin{aligned}V(Z) &\cong V_2[1+(Z-Z_2)^2/2h^2], \\ V_2 &\cong (133 \text{ km/sec}) f(\lambda), \\ h &\cong 180 \text{ km}.\end{aligned} \tag{32}$$

Then (22) becomes

$$d^2Z/dt^2 \cong -(V_2{}^2/h^2)(Z-Z_2),$$

and the period of oscillation about Z_2 is

$$T = 2\pi h/V_2 \cong 8.5 \text{ sec}/f(\lambda). \tag{33}$$

Actually, the results here are at best a fair approximation, since the Alfvén velocity does depend on the geomagnetic latitude λ as well as on the altitude Z. In polar coordinates (22) becomes, in the general case,

$$\frac{d^2r}{du^2} - r\left(\frac{d\lambda}{du}\right)^2 - r\cos^2\lambda\left(\frac{d\varphi}{du}\right)^2 = -\frac{1}{V^3}\frac{\partial V}{\partial r}, \tag{34}$$

$$r\frac{d^2\lambda}{du^2} + 2\frac{dr}{du}\frac{d\lambda}{du} - r\sin\lambda\cos\lambda\left(\frac{d\varphi}{du}\right)^2 = -\frac{1}{rV^3}\frac{\partial V}{\partial\lambda}, \tag{35}$$

$$\begin{aligned}r\cos(\lambda)\frac{d^2\varphi}{du^2} &+ 2\frac{dr}{du}\frac{d\varphi}{du}\cos\lambda \\ &+ 2r\frac{d\varphi}{du}\frac{d\lambda}{du}\sin\lambda = -\frac{1}{rV^3\cos\lambda}\frac{\partial V}{\partial\varphi} = 0,\end{aligned} \tag{36}$$

where φ is the geomagnetic longitude, and

$$du = V^2 dt. \tag{37}$$

There are two useful integrals of the motion, an "energy"

$$\begin{aligned}1 &= V^2\left[\left(\frac{dr}{du}\right)^2 + r^2\left(\frac{d\lambda}{du}\right)^2 + r^2\cos^2\lambda\left(\frac{d\varphi}{du}\right)^2\right] \\ &= \frac{1}{V^2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\lambda}{dt}\right)^2 + r^2\cos^2\lambda\left(\frac{d\varphi}{dt}\right)^2\right]\end{aligned} \tag{38}$$

and an "angular momentum"

$$L = r^2\cos^2\lambda\frac{d\varphi}{du} = \frac{r^2\cos^2\lambda}{V^2}\frac{d\varphi}{dt}. \tag{39}$$

Combining them, we obtain

$$1 = \frac{V^2L^2}{r^4\cos^4\lambda}\left[\left(\frac{dr}{d\varphi}\right)^2 + r^2\left(\frac{d\lambda}{d\varphi}\right)^2 + r^2\cos^2\lambda\right]. \tag{40}$$

[5] All Alfvén velocities given here are taken from a table published by A. J. Dessler, W. E. Francis, and E. N. Parker, J. Geophys. Research **65**, 2715 (1960).

A similar formula has been given by Francis, Green, and Dessler[6] for the special case of the geomagnetic equatorial plane, where $\lambda=0$ is fixed. Their derivation made use of Fermat's principle, which is valid here (see Sec. VII), but would not be valid for the more complicated problem treated in Sec. III. Equations (16) and (17) are always valid, even when Fermat's principle is not.

There is one conclusion that we can draw without solving any of these equations. If we define χ to be the angle between the propagation vector and due east, then according to (40),

$$L=r\cos(\lambda)\cos(\chi)/V.$$

Hence, a disturbance beginning at a point r, λ, φ with angle χ from due east, cannot reach a point r_0, λ_0, φ_0 where the Alfvén velocity is V_0, unless

$$\cos\chi< Vr_0\cos(\lambda_0)/V_0 r\cos(\lambda). \tag{41}$$

Of course, not all disturbances satisfying this condition do reach r_0.

After this excursion into calculating orbits, our next task is to say something about the amplitudes f_0, g_0 [defined by (10) and (13)] of the two types of mode. Let us write

$$\begin{aligned}\mathbf{E}&=(\mathbf{E}_0+\mathbf{E}_1)e^{iS},\\ \mathbf{B}&=(\mathbf{B}_0+\mathbf{B}_1)e^{iS},\end{aligned} \tag{42}$$

where S, $\mathbf{E}_0$, $\mathbf{B}_0$ are given by (9), (10), (11) or (9), (13), (14). Inserting these formulas in the fundamental Eqs. (1) and (5) we obtain

$$c\mathbf{k}\times\mathbf{E}_1-\omega\mathbf{B}_1=ic\nabla\times[\mathbf{E}_0+\mathbf{E}_1], \tag{43}$$

$$-\omega c\mathbf{E}_1+\mathbf{V}\times\mathbf{V}\times\mathbf{k}\times\mathbf{B}_1=i\mathbf{V}\times\mathbf{V}\times\nabla\times[\mathbf{B}_0+\mathbf{B}_1]. \tag{44}$$

The first conclusion to be drawn is that $\mathbf{E}_0$ and $\mathbf{B}_0$ are larger than $\mathbf{E}_1$ and $\mathbf{B}_1$, respectively, by a factor kh, where h is roughly the scale length within which $\mathbf{V}$ varies appreciably. Hence, if $kh\gg1$ we will be justified in dropping $\mathbf{E}_1$ and $\mathbf{B}_1$ on the right sides of (43) and (44). This is just the point where the characteristic WKB approximation enters in our approach. We now have

$$\begin{aligned}c\mathbf{k}\times\mathbf{E}_1-\omega\mathbf{B}_1&=ic\nabla\times\mathbf{E}_0,\\ -\omega c\mathbf{E}_1+\mathbf{V}\times\mathbf{V}\times\mathbf{k}\times\mathbf{B}_1&=i\mathbf{V}\times\mathbf{V}\times\nabla\times\mathbf{B}_0.\end{aligned} \tag{45}$$

Although these equations involve the unknown $\mathbf{E}_1$ and $\mathbf{B}_1$, they have an important consequence for $\mathbf{E}_0$ and $\mathbf{B}_0$. If we multiply (43) and (44), by $\mathbf{B}_0$ and $(c/V^2)\mathbf{E}_0$, respectively, and add, then the right side gives

$$c\mathbf{B}_0\cdot\mathbf{k}\times\mathbf{E}_1-\omega\mathbf{B}_0\cdot\mathbf{B}_1-(\omega c^2/V^2)\mathbf{E}_0\cdot\mathbf{E}_1-V^2\mathbf{E}_0\cdot\mathbf{k}\times\mathbf{B}_1, \tag{47}$$

where we have used the fact that $\mathbf{E}_0$ is perpendicular to $\mathbf{V}$. Now this expression is unchanged by the substitution

$$\mathbf{E}_0\leftrightarrow\mathbf{E}_1,\quad \mathbf{B}_0\leftrightarrow\mathbf{B}_1, \tag{48}$$

and hence vanishes because the right sides of (43) and (44) vanish when we replace $\mathbf{E}_1$ and $\mathbf{B}_1$ by $\mathbf{E}_0$ and $\mathbf{B}_0$. We have thus proven that

$$\mathbf{B}_0\cdot\nabla\times\mathbf{E}_0-\mathbf{E}_0\cdot\nabla\times\mathbf{B}_0=0, \tag{49}$$

or in other words

$$\nabla\cdot\mathbf{E}_0\times\mathbf{B}_0=0. \tag{50}$$

In the next section this equation is seen to hold even in the presence of dissipative effects.

We will now show how to use (50) to obtain f_1 and g_0. For the Alfvén mode

$$c\mathbf{E}_0\times\mathbf{B}_0=\pm\mathbf{V}f_0^2, \tag{51}$$

so that

$$\nabla\cdot(\mathbf{V}f_0^2)=0, \tag{52}$$

and hence

$$(\mathbf{V}\cdot\nabla)\ln f_0=-\tfrac{1}{2}\nabla\cdot\mathbf{V}.$$

Using (18), this becomes

$$(d/dt)\ln f_0=-\tfrac{1}{2}\nabla\cdot\mathbf{V}. \tag{53}$$

But noting the solenoidal character of $\mathbf{B}^{(0)}$, we have

$$\nabla\cdot\mathbf{V}=-\tfrac{1}{2}\mathbf{V}\cdot\nabla\ln\rho=-\tfrac{1}{2}(d/dt)\ln\rho \tag{54}$$

and hence along any ray path

$$f_0\rho^{-\frac{1}{4}}=\text{const.} \tag{55}$$

The magnitudes of $\mathbf{E}_0$ and $\mathbf{B}_0$ vary along a ray path as $V\rho^{+\frac{1}{4}}$ and $\rho^{\frac{1}{4}}$, respectively. This result has been known for a long time.[7]

For the fast mode,

$$\omega c(\mathbf{E}_0\times\mathbf{B}_0)=\mathbf{k}g_0^2, \tag{56}$$

so that

$$\nabla\cdot(\mathbf{k}g_0^2)=0, \tag{57}$$

and hence

$$(\mathbf{k}\cdot\nabla)\ln g_0=-\tfrac{1}{2}\nabla\cdot\mathbf{k}.$$

Using (20), this becomes

$$(d/dt)\ln g_0=-(V/2k)\nabla\cdot\mathbf{k}. \tag{58}$$

Here there is no trick available, and we must resign ourselves to actually doing an integral to get g_0.

We see that (45) and (46) contain enough information about $\mathbf{E}_0$ and $\mathbf{B}_0$ to allow us to obtain f_0 and g_0. We now use these equations to solve for the first-order fields, obtaining for the Alfvén mode

$$\mathbf{E}_1=\frac{-iV^2\omega}{c(k^2V^2-\omega^2)^2}(\mathbf{V}\times\mathbf{k})\times\left[(\mathbf{V}\times\mathbf{k})\cdot\left\{\nabla\times\mathbf{B}_0+\frac{c}{\omega}\mathbf{k}\times\nabla\times\mathbf{E}_0\right\}\right], \tag{59}$$

[6] W. E. Francis, M. I. Green, and A. J. Dessler, J. Geophys. Research **64**, 1643 (1959).

[7] C. Walén, Arkiv mat. astron. Fysik **30A**, No. 15 (1944).

and for the fast mode

$$\mathbf{E}_1=\frac{iV^2}{c[\omega^2-(\mathbf{k}\cdot\mathbf{V})^2]^2}(\mathbf{V}\times\mathbf{V}\times\mathbf{k})$$
$$\times\left[(\mathbf{V}\times\mathbf{V}\times\mathbf{k})\cdot\left\{\nabla\times\mathbf{B}_0+\frac{c}{\omega}\mathbf{k}\times\nabla\times\mathbf{E}_0\right\}\right]. \quad (60)$$

The magnetic field $\mathbf{B}_1$ is, in both cases,

$$\omega\mathbf{B}_1=c\mathbf{k}\times\mathbf{E}_1-ic\nabla\times\mathbf{E}_0. \quad (61)$$

An equally good solution could be obtained by a term parallel to $\mathbf{V}\times\mathbf{V}\times\mathbf{k}$ to (59), or a term parallel to $\mathbf{V}\times\mathbf{k}$ to (60). Such terms, which are in the direction of $\mathbf{E}_0$, can clearly be ignored compared to $\mathbf{E}_0$.

From (45) and (46) we may also derive an exact formula

$$0=\mathbf{B}_0\cdot\nabla\times[\mathbf{E}_0+\mathbf{E}_1]-\mathbf{E}_0\cdot\nabla\times[\mathbf{B}_0+\mathbf{B}_1], \quad (62)$$

so that

$$\nabla\cdot(\mathbf{E}_0\times\mathbf{B}_0)=\mathbf{E}_0\cdot\nabla\times\mathbf{B}_1-\mathbf{B}_0\cdot\nabla\times\mathbf{E}_1. \quad (63)$$

Inserting (59) or (60) and (61) on the right side yields a formula for the diffraction loss along a ray path. We will not venture on an adequate treatment of diffraction in this paper.

III. MAGNETO-ACOUSTI-ELECTRO-HYDRODYNAMICS

We shall now extend the development of Sec. I to cover the interplay between magnetohydrodynamic, electrodynamic, and acoustic modes in an inhomogeneous neutral plasma at rest in a nonconstant magnetic field.

The plasma has mass density $\rho(\mathbf{x})$, and complex characteristic frequency $\omega_P(\mathbf{k})$, where

$$\omega_P^2=(4\pi ne^2/m_e)[\omega/(\omega+i\omega_c)]. \quad (64)$$

Here $n(\mathbf{x})$ is the electron number density and $\omega_c(\mathbf{x})$ is the collision frequency; both may depend on position. The unperturbed magnetic field is $\mathbf{B}^{(0)}(\mathbf{x})$. A disturbance is characterized by a small electric field $\mathbf{E}(\mathbf{x})$, current $\mathbf{J}(\mathbf{x})$, ion velocity $\mathbf{u}(\mathbf{x})$, and an increment $\mathbf{B}(\mathbf{x})$ to $\mathbf{B}^{(0)}(\mathbf{x})$; all these quantities have time dependence $e^{-i\omega t}$. The system is described by the equations

$$-ic\nabla\times\mathbf{E}=\omega\mathbf{B}, \quad (65)$$

$$-ic\nabla\times\mathbf{B}=-\omega\mathbf{E}-4\pi i\mathbf{J}, \quad (66)$$

$$-4\pi i\omega\mathbf{J}=\omega_P^2[\mathbf{E}+(1/c)\mathbf{u}\times\mathbf{B}^{(0)}], \quad (67)$$

$$-i\omega\rho\mathbf{u}=(1/c)\mathbf{J}\times\mathbf{B}^{(0)}+\mathbf{F}(\mathbf{u}), \quad (68)$$

where $\mathbf{F}$ is the sum of pressure, gravitational, and viscous forces. We are neglecting a term in (67) proportional to the cyclotron frequency, though it could easily be taken into account.

In the zeroth-order eikonal approximation these equations become

$$c\mathbf{k}\times\mathbf{E}_0=\omega\mathbf{B}_0, \quad (69)$$

$$c\mathbf{k}\times\mathbf{B}_0=-\omega\mathbf{E}_0-4\pi i\mathbf{J}_0, \quad (70)$$

$$-4\pi i\omega\mathbf{J}_0=\omega_P^2[\mathbf{E}_0+(1/c)\mathbf{u}_0+\mathbf{B}^{(0)}], \quad (71)$$

$$-i\omega\rho\mathbf{u}_0=(1/c)\mathbf{J}_0\times\mathbf{B}^{(0)}+\mathbf{F}(\mathbf{u}_0). \quad (72)$$

Also, to this order

$$\mathbf{F}=(\rho/\omega)[-ia^2\mathbf{k}(\mathbf{k}\cdot\mathbf{u}_0)-\omega\nu k^2\mathbf{u}_0], \quad (73)$$

where $a(\mathbf{x})$ is the local speed of sound and $\nu(\mathbf{x})$ the local kinematic viscosity. The exact expression for $\mathbf{F}$ in an inhomogeneous medium is much more complicated.

Equations (69), (70), and (71) may be solved to give

$$\mathbf{E}_0=\frac{\omega_P^2}{c(\omega_P^2-\omega^2)}$$
$$\times\left[\mathbf{B}^{(0)}\times\mathbf{u}_0+\left(\frac{c^2}{\omega_P^2-\omega^2+c^2k^2}\right)\mathbf{k}\times\mathbf{k}\times\mathbf{B}^{(0)}\times\mathbf{u}_0\right], \quad (74)$$

$$\mathbf{B}_0=\frac{\omega_P^2}{\omega(\omega_P^2-\omega^2+c^2k^2)}\mathbf{k}\times\mathbf{B}^{(0)}\times\mathbf{u}_0, \quad (75)$$

$$-4\pi i\mathbf{J}_0=\frac{\omega_P^2\omega}{c(\omega_P^2-\omega^2)}$$
$$\times\left[\mathbf{B}^{(0)}\times\mathbf{u}_0+\left(\frac{c^2}{\omega^2(\omega_P^2-\omega^2+c^2k^2)}\right)\mathbf{k}\times\mathbf{k}\times\mathbf{B}^{(0)}\times\mathbf{u}_0\right], \quad (76)$$

and inserting these and (73) in (72) yields

$$\omega(\omega+i\nu k^2)\mathbf{u}_0=a^2\mathbf{k}(\mathbf{k}\cdot\mathbf{u}_0)+\frac{\omega_P^2}{\omega_P^2-\omega^2}$$
$$\times\left[\frac{\omega^2}{c^2}\mathbf{V}\times\mathbf{V}\times\mathbf{u}+\frac{\omega_P^2}{\omega_P^2-\omega^2+c^2k^2}\mathbf{V}\times\mathbf{k}\times\mathbf{k}\times\mathbf{V}\times\mathbf{u}_0\right], \quad (77)$$

where

$$\mathbf{V}\equiv\mathbf{B}^{(0)}/(4\pi\rho)^{\frac{1}{2}}. \quad (78)$$

There are two types of solution

I. $$\mathbf{u}_0\sim(\mathbf{V}\times\mathbf{k}) \quad (79)$$

$$0=D_0\equiv\omega(\omega+i\nu k^2)(\omega_P^2-\omega^2)(\omega_P^2-\omega^2+c^2k^2)$$
$$-(\mathbf{V}\cdot\mathbf{k})^2\omega_P^4+\omega^2\omega_P^2(V^2/c^2)(\omega_P^2-\omega^2+c^2k^2). \quad (80)$$

II. $$\mathbf{u}_0\sim\{a^2\mathbf{k}[V^2k^2-(\mathbf{V}\cdot\mathbf{k})^2]$$
$$+\mathbf{V}\times\mathbf{V}\times\mathbf{k}[a^2k^2-\omega(\omega+i\nu k^2)]\} \quad (81)$$

$$0=D_0\equiv(\omega_P^2-\omega^2)\{c^2\omega(\omega+i\nu k^2)$$
$$\times[k^2a^2-\omega(\omega+i\nu k^2)](\omega_P^2-\omega^2+c^2k^2)$$
$$+\omega_P^2(\omega^2-c^2k^2)[a^2(\mathbf{k}\cdot\mathbf{V})^2-V^2\omega(\omega+i\nu k^2)]\}. \quad (82)$$

These formulas for D_0 suffice to determine the path of any disturbance, according to the equations (see Sec. IV)

$$\frac{d\mathbf{x}}{dt}=-\frac{\partial D_0/\partial\mathbf{k}}{\partial D_0/\partial\omega}, \tag{83}$$

$$\frac{d\mathbf{k}}{dt}=\frac{\partial D_0/\partial\mathbf{x}}{\partial D_0/\partial\omega}. \tag{84}$$

Clearly the group velocity $d\mathbf{x}/dt$ will lie in the plane spanned by $\mathbf{V}$ and $\mathbf{k}$, though in general it will not lie along either direction.

For simplicity, from now on we shall neglect pressure and viscosity effects, setting $a=0$ and $\nu=0$. It should be emphasized that this is not an essential omission; such effects could readily be handled. Also, we are not assuming here that dissipation is absent, since ω_P may have a complex part due to collisions. The two modes are now defined by

I. $$\mathbf{u}_0=(\mathbf{V}\times\mathbf{k})[f_0/Vk(\sin\theta)(4\pi\rho)^{\frac{1}{2}}] \tag{85}$$

$$0=D_0\equiv\omega^2(\omega_P{}^2\omega^2+c^2k^2)\times(\omega_P{}^2[1+V^2/c^2]-\omega^2)-\omega_P{}^4(\mathbf{V}\cdot\mathbf{k})^2; \tag{86}$$

II. $$\mathbf{u}_0=(\mathbf{V}\times\mathbf{V}\times\mathbf{k})(g_0/4\pi\rho Vk\sin\theta) \tag{87}$$

$$0=D_0\equiv\omega_P{}^2V^2(\omega^2-c^2k^2)+\omega^2c^2(\omega_P{}^2-\omega^2+c^2k^2). \tag{88}$$

(In II we are assuming $\omega\not\equiv\omega_P$ and $\omega\not\equiv 0$. Here θ is the angle between $\mathbf{V}$ and $\mathbf{k}$.) The equations of motion (83) and (84) are quite messy here, partly because D_0 has such a complicated ω dependence. (Even ω_P depends on ω.) We will not give them explicitly.

In order to obtain further information about the behavior of the fields, we write as in Sec. I

$$\mathbf{u}(\mathbf{x})=[\mathbf{u}_0(\mathbf{x})+\mathbf{u}_1(\mathbf{x})]e^{iS}(\mathbf{x}) \tag{89}$$

and likewise for $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{J}$. Here $\mathbf{u}_0$ given by (85) or (87), and $u_1\ll u_0$. The fields $\mathbf{E}_0$, $\mathbf{B}_0$, $\mathbf{J}_0$ are related to $\mathbf{u}_0$ by (74), (75), (76). The exact Eqs. (65)–(68) may now be rewritten (with $\mathbf{F}$ assumed zero)

$$c\mathbf{k}\times\mathbf{E}_1-\omega\mathbf{B}_1=ic\boldsymbol{\nabla}\times(\mathbf{E}_0+\mathbf{E}_1), \tag{90}$$

$$c\mathbf{k}\times\mathbf{B}_1+\omega\mathbf{E}_1+4\pi i\mathbf{J}_1=ic\boldsymbol{\nabla}\times(\mathbf{B}_0+\mathbf{B}_1), \tag{91}$$

$$-4\pi i\omega\mathbf{J}_1-\omega_P{}^2[\mathbf{E}_1+(1/c)\mathbf{u}_1\times\mathbf{B}^{(0)}]=0, \tag{92}$$

$$-i\omega\rho\mathbf{u}_1-(1/c)\mathbf{J}_1\times\mathbf{B}^{(0)}=0. \tag{93}$$

By multiplying, respectively, by $\mathbf{B}_0$, $-\mathbf{E}_0$, $(4\pi i/\omega_P{}^2)\mathbf{J}_0$, and $-4\pi i\mathbf{u}_0$, and adding, we can show that again,

$$\mathbf{B}_0\cdot\nabla\times(\mathbf{E}_0+\mathbf{E}_1)-\mathbf{E}_0\cdot\nabla\times(\mathbf{B}_0+\mathbf{B}_1)=0$$

or in other words

$$\boldsymbol{\nabla}\cdot(\mathbf{E}_0\times\mathbf{B}_0)=\mathbf{E}_0\cdot\boldsymbol{\nabla}\times\mathbf{B}_1-\mathbf{B}_0\cdot\boldsymbol{\nabla}\times\mathbf{E}_1. \tag{94}$$

Neglecting $\mathbf{E}_1$ and $\mathbf{B}_1$ on the right gives us

$$\boldsymbol{\nabla}\cdot(\mathbf{E}_0\times\mathbf{B}_0)=0, \tag{95}$$

which determines the variation of f_0 or g_0 along a ray path, as we now shall show.

In mode I,

$$\mathbf{E}_0\times\mathbf{B}_0=f_0{}^2\mathbf{P}, \tag{96}$$

$$\mathbf{P}\equiv\frac{\omega_P{}^4(\mathbf{V}\cdot\mathbf{k})}{\omega c(\omega_P{}^2-\omega^2)(\omega_P{}^2-\omega^2+c^2k^2)}\times\left[\mathbf{V}-\frac{c^2(\mathbf{V}\cdot\mathbf{k})\mathbf{k}}{(\omega_P{}^2-\omega^2+c^2k^2)}\right]; \tag{97}$$

and in mode II,

$$\mathbf{E}_0\times\mathbf{B}_0=g_0{}^2\mathbf{Q}, \tag{98}$$

$$Q=k\frac{\omega_P{}^4V^4}{\omega c(\omega_P{}^2-\omega^2+c^2k^2)}. \tag{99}$$

It is important to note now that in both cases, $\mathbf{E}_0\times\mathbf{B}_0$ is in the direction of the group velocity. If for convenience we define a parameter u by

$$du=-2dt/(\partial D_0/\partial\omega), \tag{100}$$

then in mode I

$$\frac{d\mathbf{x}}{du}=\omega^2c^2\left(\omega_P{}^2\left[1+\frac{V^2}{c^2}\right]-\omega^2\right)\mathbf{k}-\omega_P{}^4(\mathbf{V}\cdot\mathbf{k})\mathbf{V}=\alpha\mathbf{P}, \tag{101}$$

$$\alpha=\frac{-\omega^3c(\omega_P{}^2-\omega^2)(\omega_P{}^2-\omega^2+c^2k^2)^2(\omega_P{}^2[1+V^2/c^2]-\omega^2)}{\omega_P{}^4(\mathbf{V}\cdot\mathbf{k})^2}; \tag{102}$$

and in mode II

$$d\mathbf{x}/du=c^2(\omega^2c^2-\omega_P{}^2V^2)\mathbf{k}=\beta\mathbf{Q}, \tag{103}$$

$$\beta=\frac{c^3\omega(\omega^2c^2-\omega_P{}^2V^2)(\omega_P{}^2-\omega^2+c^2k^2)}{\omega_P{}^4V^4}. \tag{104}$$

For mode I, $\boldsymbol{\nabla}\cdot(f_0{}^2\mathbf{P})=0$, so

$$(d/du)\ln f_0=-\tfrac{1}{2}\alpha\boldsymbol{\nabla}\cdot\mathbf{P}. \tag{105}$$

For mode II, $\boldsymbol{\nabla}\cdot(g_0{}^2\mathbf{Q})=0$, so

$$(d/du)\ln g_0=-\tfrac{1}{2}\beta\boldsymbol{\nabla}\cdot\mathbf{Q}. \tag{106}$$

Hence the evaluation of f_0 or g_0 along a particular ray path is accomplished just by performing an integration. The integrands are complicated, and there is no simple trick available like the one used in the case of a pure Alfvén mode.

In order to find the small corrections $\mathbf{u}_1$, $\mathbf{E}_1$, $\mathbf{B}_1$, $\mathbf{J}_1$, in terms of the (presumably) known $\mathbf{E}_0$ and $\mathbf{B}_0$, it is now necessary to solve the algebraic Eqs. (90)–(93), neglecting $\mathbf{E}_1$ and $\mathbf{B}_1$ on the right sides of (90) and (91). In order to evaluate the diffraction corrections to (105) and (106), we must insert these solutions for $\mathbf{E}_1$ and $\mathbf{B}_1$ into the right side of (94). We will not actually do any of this here, though in principle it is straightforward.

IV. THE GENERAL EIKONAL METHOD

We will consider here a set of n coupled linear homogeneous partial-differential equations, which may be written

$$M(\mathbf{x}, -i\nabla)\psi(\mathbf{x})=0. \tag{107}$$

The field $\psi(\mathbf{x})$ has n components (e.g., ψ might consist of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{J}$, and $\mathbf{u}$) characterizing a small disturbance, and M is an $n\times n$ matrix characterizing the undisturbed medium.

For problems in which M is time independent, we may assume that ψ depends on time through a factor $e^{-i\omega t}$; then M depends on ω as well as on the position $\mathbf{x}$. On the other hand, if the properties of the medium (and hence M) depend upon time, then $\mathbf{x}$ must be regarded as a four-vector, and ∇ as a gradient in space-time.

By suitably introducing field derivatives as new components of ψ, it is always possible to rewrite (1) as a system of first-order equations,

$$[-i\mathbf{A}(\mathbf{x})\cdot\nabla+B(\mathbf{x})]\psi(\mathbf{x})=0. \tag{108}$$

For example, the single second-order equations,

$$[\nabla^2+k^2(\mathbf{x})]\phi=0$$

may be replaced by the four first-order equations,

$$-i\nabla\cdot\boldsymbol{\phi}+k^2\phi=0,$$
$$i\nabla\phi-\boldsymbol{\phi}=0.$$

From time to time we will refer to (108) for illustrative purposes.

The zeroth order eikonal approximation consists of assuming that M depends so weakly on $\mathbf{x}$ that all the spatial dependence of the components of $\psi(\mathbf{x})$ is essentially given by a common factor $\exp[iS(\mathbf{x})]$. With this assumption, (107) becomes

$$M(\mathbf{x},\mathbf{k})\psi=0, \tag{109}$$

where

$$\mathbf{k}(\mathbf{x})\equiv\nabla S(\mathbf{x}). \tag{110}$$

Equation (109) has, in general, a number of solutions. We will pick out a particular one which we call ψ_0, and will assume that the vanishing of $M\psi_0$ entails a condition on $\mathbf{k}$ of the form

$$D_0(\mathbf{x},\mathbf{k})=0. \tag{111}$$

A convenient way to define ψ_0 and D_0 is to define $\psi_0(\mathbf{x},\mathbf{q})$ and $D_0(\mathbf{x},\mathbf{q})$ for general $\mathbf{q}$ as particular solutions of the eigenvalue problem

$$M(\mathbf{x},\mathbf{q})\psi_0(\mathbf{x},\mathbf{q})=D_0(\mathbf{x},\mathbf{q})\psi_0(\mathbf{x},\mathbf{q}), \tag{112}$$

and then define $\mathbf{k}(\mathbf{x})$ by using the restriction (111). With (112), the determinant of M is just the product of the D_0 for the various solutions.

Of course, Eq. (111) alone does not determine $\mathbf{k}$; we must use the additional fact that $\mathbf{k}$ is a gradient. The solution to the problem of constructing an irrotational $\mathbf{k}$ satisfying a condition of form (111) has been known since the work of Hamilton a century ago. [In one dimension, where the single Eq. (111) does determine k uniquely, the condition that $\mathbf{k}$ is a gradient carries no information. The rest of this section is needed only for problems in more than one dimension.]

To construct $\mathbf{k}$, we begin by introducing a family of ray paths through "phase space":

$$\mathbf{x}=\mathbf{x}(\tau),\quad \mathbf{k}=\mathbf{k}(\tau),$$

defined [for any initial conditions $\mathbf{x}(\tau_1)$, $\mathbf{k}(\tau_1)$] by the equations

$$dx_i/d\tau=\partial D_0/\partial k_i, \tag{113}$$

$$dk_i/d\tau=-\partial D_0/\partial x_i. \tag{114}$$

The parameter τ has no direct significance.

Clearly, along any ray path D_0 will be constant, since

$$dD_0/d\tau=(\partial D_0/\partial x_i)(dx_i/d\tau)+(\partial D_0/\partial k_i)(dk_i/d\tau)=0. \tag{115}$$

If we choose at $\tau=\tau_1$ an $\mathbf{x}_1$ and $\mathbf{k}_1$ satisfying the condition

$$D_0(\mathbf{x}_1,\mathbf{k}_1)=0,$$

then $\mathbf{x}$ and $\mathbf{k}$ are determined for all τ, and (111) stays satisfied.

Now we must ask for any given $\mathbf{x}$ what initial conditions we impose to determine the path through $\mathbf{x}$ and hence $\mathbf{k}(\mathbf{x})$. Let us consider an "initial" surface Γ in coordinate space with unit normal $\mathbf{n}(\mathbf{x})$, and suppose that for each $\mathbf{x}_1$ on Γ the conditions

$$D_0(\mathbf{x}_1,\mathbf{k}_1)=0, \tag{116}$$

$$\mathbf{k}_1/|\mathbf{k}_1|=\mathbf{n}(\mathbf{x}_1), \tag{117}$$

determine a unique $\mathbf{k}_1(\mathbf{x}_1)$. Equation (117) ensures that ∇S is normal to Γ on Γ, so that S is constant over Γ. For any given $\mathbf{x}$ there will be a unique ray path leading from Γ to $\mathbf{x}$, defined by the conditions

$$\mathbf{x}(\tau_1)\text{ on }\Gamma,\quad \mathbf{k}(\tau_1)=\mathbf{k}_1(\mathbf{x}(\tau_1)). \tag{118}$$

Then $\mathbf{k}(\mathbf{x})$ is given as

$$\mathbf{k}(\mathbf{x})=\mathbf{k}(\tau_2), \tag{119}$$

where τ_2 is the value of τ for which

$$\mathbf{x}(\tau_2)=\mathbf{x}. \tag{120}$$

In order to show that this solves the problem we will construct $S(\mathbf{x})$ and show directly that its gradient is $\mathbf{k}(\mathbf{x})$. The formula for $S(\mathbf{x})$ is

$$S(\mathbf{x})=\int_{\tau_1}^{\tau_2}\mathbf{k}(\tau)\cdot\dot{\mathbf{x}}(\tau)d\tau. \tag{121}$$

Here a dot means $d/d\tau$.

If $\mathbf{x}$ is varied by an amount $\delta\mathbf{x}$, we get a new ray path

$$\mathbf{x}'(\tau)=\mathbf{x}(\tau)+\Delta\mathbf{x}(\tau),$$
$$\mathbf{k}'(\tau)=\mathbf{k}(\tau)+\Delta\mathbf{k}(\tau).$$

We may hold τ_1 fixed but then τ_2 will be changed by $\Delta\tau_2$; from (119) we see that

$$\Delta\mathbf{x}(\tau_2)+\dot{\mathbf{x}}(\tau_2)\Delta\tau_2=\delta\mathbf{x}.$$

The variation of $S(\mathbf{x})$ is

$$\begin{aligned}\delta S(\mathbf{x})&=\mathbf{k}(\tau_2)\cdot\dot{\mathbf{x}}(\tau_2)\Delta\tau_2\\&\quad+\int_{\tau_1}^{\tau_2}\Delta\mathbf{k}(\tau)\cdot\dot{\mathbf{x}}(\tau)d\tau+\int_{\tau_1}^{\tau_2}\mathbf{k}(\tau)\Delta\dot{\mathbf{x}}(\tau)d\tau\\&=\mathbf{k}(\tau_2)\cdot[\dot{\mathbf{x}}(\tau_2)\Delta\tau_2+\Delta\mathbf{x}(\tau_2)]-\mathbf{k}(\tau_1)\cdot\Delta\mathbf{x}(\tau_1)\\&\quad+\int_{\tau_1}^{\tau_2}[\Delta\mathbf{k}(\tau)\cdot\dot{\mathbf{x}}(\tau)-\dot{\mathbf{k}}(\tau)\cdot\Delta\mathbf{x}(\tau)]d\tau\\&=\mathbf{k}(\tau_2)\cdot\delta\mathbf{x}-\mathbf{k}(\tau_1)\cdot\Delta x(\tau_1)\\&\quad+\int_{\tau_1}^{\tau_2}\left(\frac{\partial D_0}{\partial k_i}\Delta k_i+\frac{\partial D_0}{\partial x_i}\Delta x_i\right)d\tau.\end{aligned}\tag{122}$$

The last term is the integral of ΔD_0, and vanishes since $D_0=0$ for all paths. The penultimate term vanishes because $\Delta x(\tau_1)$ is the difference between two vectors on Γ and hence is tangent to Γ, while $\mathbf{k}(\tau_1)$ is normal to Γ. Hence,

$$\delta S(\mathbf{x})=\mathbf{k}(\mathbf{x})\cdot\delta\mathbf{x},\tag{123}$$

and so $\mathbf{k}$ is indeed the gradient of S.

By definition $\mathbf{k}(\mathbf{x})$ is always normal to the surface of constant S passing through $\mathbf{x}$. [Γ is one such surface, defined by (121) to have $S=0$.] It should be noted that the direction of a ray path, $\dot{\mathbf{x}}$, is not in general the same as that of $\mathbf{k}$. For instance for an Alfvén wave (see Sec. II)

$$D_0=(\mathbf{V}\cdot\mathbf{k})^2-\omega^2,$$

where $\mathbf{V}$ is the Alfvén velocity (in the direction of the magnetic field), and so here

$$d\mathbf{x}/d\tau=2\omega\mathbf{V}.$$

On the other hand, in geometric optics,

$$D_0=c^2\mathbf{k}^2-n^2\omega^2,$$

so that the tangent to the ray path,

$$d\mathbf{x}/d\tau=2c^2\mathbf{k},$$

is in fact in the direction of $\mathbf{k}$.

V. SOLUTION FOR THE AMPLITUDES

Having constructed the rapidly varying phase $S(\mathbf{x})$, we must now say something about the spatial dependence of the direction and normalization of $\psi(\mathbf{x})$. Let us write

$$\psi(\mathbf{x})=[\psi_0(\mathbf{x})+\psi_1(\mathbf{x})]\exp[iS(\mathbf{x})],\tag{124}$$

where ψ_0 satisfies

$$M(\mathbf{x},\mathbf{k})\psi_0=0.\tag{125}$$

We expect that $\psi_0\gg\psi_1$, and that both ψ_0 and ψ_1 will vary much more slowly than S.

Inserting (124) in (107) gives the exact equations

$$0=M(\mathbf{x},\mathbf{k}-i\boldsymbol{\nabla})[\psi_0(\mathbf{x})+\psi_1(\mathbf{x})],\tag{126}$$

or, using (125),

$$M(\mathbf{x},\mathbf{k})\psi_1(\mathbf{x})=i[\mathbf{A}(\mathbf{x},\mathbf{k})\cdot\boldsymbol{\nabla}+C(\mathbf{x},\mathbf{k})+\cdots]\times[\psi_0(\mathbf{x})+\psi_1(\mathbf{k})],\tag{127}$$

where

$$A^i=(\partial M/\partial k_i),\tag{128}$$

$$C=\frac{1}{2}\frac{\partial^2 M}{\partial k_i\partial k_j}\frac{\partial k_j}{\partial x_i}.\tag{129}$$

For example, the first-order system (108) gives exactly

$$(\mathbf{A}\cdot\mathbf{k}+B)\psi_1=i(\mathbf{A}\cdot\boldsymbol{\nabla})(\psi_0+\psi_1),\tag{130}$$

the matrix C and higher derivatives vanishing in this case.

Clearly (127) shows that ψ_1 is less than ψ_0 by a factor kh, where h is the typical scale length within which M varies appreciably. If $kh\gg1$, then we may neglect ψ_1 on the right-hand side; also neglecting terms of the same order in M we have the "WKB" approximation:

$$M\psi_1=i[\mathbf{A}\cdot\boldsymbol{\nabla}+C]\psi_0.\tag{131}$$

We will now make a new assumption, that M is a symmetric matrix. This assumption is not very restrictive, since even if M is not symmetric, we can usually rewrite the system of equations (1) as $M'\psi=0$, where $M'=NM$ and N is a matrix chosen to make M' symmetric. In the complicated example treated in Sec. III M could be juggled into a symmetric form, though it was *not* Hermitian because of the presence of dissipation.

Multiplying (131) on the left by the row vector ψ_0^T now gives

$$\psi_0^T(\mathbf{A}\cdot\boldsymbol{\nabla}+C)\psi_0=0,\tag{132}$$

where we make use of the symmetry of M and the vanishing of $M\psi_0$. It is Eq. (132) that determines the spatial dependence of ψ_0. Actually the solution is much easier than it looks.

The vector ψ_0 is partially determined by the condition $M\psi_0=0$. We will assume here that D_0 is a nondegenerate eigenvalue, so that the direction of ψ_0 is fixed by this condition, and its magnitude is to be found by integrating (132). (Actually, in the geometric optics of isotropic media, M has twofold degenerate eigenvalues.[1] In the examples treated in Secs. II and III there is no degeneracy.)

To integrate (132) it is convenient to capitalize on the fact that we already know the direction of ψ_0; we define

$$\psi_0=f_0\varphi_0,\tag{133}$$

where φ_0 is some solution of $M\varphi_0=D_0\varphi_0$ with arbitrarily chosen normalization (e.g., $\varphi_0^T\varphi_0\equiv1$) and f_0

is an amplitude to be found. Then (132) gives

$$(\varphi_0^T\mathbf{A}\varphi_0)\cdot\nabla \ln f_0=-\varphi_0^T(\mathbf{A}\cdot\nabla+C)\varphi_0. \quad (134)$$

To simplify this further, let us go back for a moment to Eq. (112). Differentiating with respect to $\mathbf{q}$ and then setting $\mathbf{q}=\mathbf{k}(\mathbf{x})$ we obtain

$$\mathbf{A}\psi_0+M\partial\psi_0/\partial\mathbf{k}=(\partial D_0/\partial\mathbf{k})\psi_0.$$

Multiplying by ψ_0^T gives then

$$(\psi_0^T\mathbf{A}\psi_0)=(\partial D_0/\partial\mathbf{k})(\psi_0^T\psi_0)$$

and hence

$$\frac{(\varphi_0^T\mathbf{A}\varphi_0)}{(\varphi_0^T\varphi_0)}=\frac{\partial D_0}{\partial\mathbf{k}}=\frac{d\mathbf{x}}{d\tau}. \quad (135)$$

The circumstance that $\varphi_0^T\mathbf{A}\varphi_0$ points in the direction of the tangent to the ray path allows us to write now instead of (134),

$$(d/d\tau)\ln f_0=-\varphi_0^T(\mathbf{A}\cdot\nabla+C)\varphi_0/\varphi_0^T\varphi_0 \quad (136)$$

and $f_0(\mathbf{x})$ can be found by a single quadrature if it is given at any point on each ray path.

It is the particular form of (136) that allows us to speak of disturbances being propagated along ray paths. For if we pick an initial surface Γ (on which the phase S is zero) we are free to give $f_0(\mathbf{x})$ any arbitrary values for $\mathbf{x}$, on Γ. The value of $f_0(\mathbf{x})$ is then given by (136) for all points; its value at $\mathbf{x}$ depends only on the value it has at the point $\mathbf{x}_1$, where the ray path through $\mathbf{x}$ intersects Γ.

Frequently our Eq. (136) for f_0 may be regarded as the expression of a simple conservation law. Let us suppose that

$$\partial^2M(\mathbf{x},\mathbf{k})/\partial x_i\partial k_i=0, \quad (137)$$

so that

$$\nabla\cdot\mathbf{A}=2C. \quad (138)$$

[For the system of first-order equations (108) for which $C=0$, this condition becomes

$$\nabla\cdot\mathbf{A}=0 \quad (139)$$

which is satisfied in geometrical optics and the problem of Sec. III because $\mathbf{A}$ is constant, though of course B is not.] Using (138), Eq. (132) becomes a conservation condition

$$\nabla\cdot\mathbf{\Pi}_0=0, \quad (140)$$

where the conserved current $\mathbf{\Pi}_0$ is

$$\mathbf{\Pi}_0=\psi_0^T\mathbf{A}\psi_0. \quad (141)$$

If we now choose φ_0 so that

$$\varphi_0^T\varphi_0=1,$$

then using (136) and (138) [or directly using (139)] we have in place of (136)

$$(d/d\tau)\ln F_0=-\nabla\cdot(\varphi_0^T\mathbf{A}\varphi_0)=-\nabla\cdot(d\mathbf{x}/d\tau). \quad (142)$$

The obvious significance here is that f_1 changes along a ray path only because the paths converge or diverge.

In some special cases the integral for f_0 may be done trivially. A famous example is the one-dimensional Schrödinger equation

$$[d^2/dx^2+U^2(x)]\psi(x)=0,$$

where $M=-k^2+U^2$ and hence

$$A=-2k,\quad C=-(dk/dx).$$

Here (136) (choosing $\varphi_0\equiv 1$) becomes

$$(d/d\tau)\ln f_0=dk/dx.$$

Since in this case $D_0=-k^2+U^2$,

$$dx/d\tau=-2k,$$

and we may write

$$(d/d\tau)\ln f_0=(-1/2k)dk/d\tau,$$

with solution

$$f_0\sqrt{k}=\text{const.}$$

The conserved flux here is

$$\mathbf{\Pi}_0=A\psi_0^2=-2kf_0^2.$$

The Schrödinger equation may also be written as a first-order system

$$(-iAd/dx+B)\chi=0,$$

where

$$\chi=\begin{bmatrix}\psi(x)\\ -i\psi'(x)\end{bmatrix},$$

$$A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad B=-\begin{bmatrix}U^2 & 0\\ 0 & 1\end{bmatrix}.$$

The conserved flux here is again

$$\mathbf{\Pi}_0=\chi_0^TA\chi_0=-i\psi_0\psi_0'\cong k\psi_0^2.$$

It may be noted that $\mathbf{\Pi}_0$ is conserved only in the lowest eikonal approximation. There is a different current $\mathrm{Im}\psi_0{}^*\psi'$, which is conserved exactly, but only if U is real.

Another important case where f_0 may be evaluated immediately is that of the Alfvén mode in magnetohydrodynamics, discussed in Sec. II.

It should be emphasized that no assumption of reality has been made anywhere. In fact the phase $S(\mathbf{x})$, the propagation vector $\mathbf{k}(\mathbf{x})$, and the amplitude $\psi_0(\mathbf{x})$ may all be complex. Equation (136) tells us how both real and imaginary parts of f_0 vary along a ray path.

Having presumably been able to calculate f_0, it is now just a matter of algebra to solve the equation

$$M\psi_1=i(\mathbf{A}\cdot\nabla+C)\psi_0$$

to get ψ_1. Its solution, which we will symbolically write

$$\psi_1=iM^{-1}(\mathbf{A}\cdot\nabla+C)\psi_0, \quad (143)$$

is clearly unique up to a term in ψ, parallel to ψ_0, which can be ignored since it makes a small contribution to ψ.

We can go on and derive higher-order equations. For example if we write the first-order system (108) in the form (130), and insert (143) on the right side, we obtain

$$\psi_0^T(\mathbf{A}\cdot\nabla)[1+i(\mathbf{A}\cdot\mathbf{k}+B)^{-1}\mathbf{A}\cdot\nabla]\psi_0=0 \tag{144}$$

and, hence, setting $\psi_0=f_0\varphi_0$,

$$\frac{d}{d\tau}\ln f_0=-\frac{1}{(\varphi_0^T\varphi_0)}\Big[\varphi_0^T\mathbf{A}\cdot\nabla\varphi_0+\left(\frac{i}{f_0}\right)\varphi_0^T(\mathbf{A}\cdot\nabla)(\mathbf{A}\cdot\mathbf{k}+B)^{-1}(\mathbf{A}\cdot\nabla)\varphi_0f_0\Big]. \tag{145}$$

The second term on the right represents the change in f_0 due to diffraction of the disturbance from one ray path into another. Its evaluation is again straightforward, in spite of its unpleasant appearance.

VI. TEMPORAL BEHAVIOR

One way to approach the problem of the time dependence of ψ is to regard all the manipulations of the previous sections as referring to space-time vectors rather than just positions. Then to the equations determining the spatial orbit of a disturbance,

$$d\mathbf{x}/d\tau=\partial D_0/\partial\mathbf{k},\quad d\mathbf{k}/d\tau=-\partial D_0/\partial\mathbf{x}, \tag{146}$$

we must add a fourth pair:

$$dt/d\tau=-\partial D_0/\partial\omega,\quad d\omega/d\tau=\partial D_0/\partial t, \tag{147}$$

which relates the time t to the parameter τ. The sign changes in going from (146) to (147) are due to the fact that in following the conventional definition of ω we must set

$$\omega=-\partial S/\partial t. \tag{148}$$

If we use t instead of τ to parametrize the path of a disturbance, then (146) and (147) give

$$\frac{d\mathbf{x}}{dt}=-\frac{\partial D_0/\partial\mathbf{k}}{\partial D_0/\partial\omega}, \tag{149}$$

$$\frac{d\mathbf{k}}{dt}=\frac{\partial D_0/\partial\mathbf{x}}{\partial D_0/\partial\omega}, \tag{150}$$

$$\frac{d\omega}{dt}=-\frac{\partial D_0/\partial t}{\partial D_0/\partial\omega}. \tag{151}$$

Equation (149) is the general formula for a group velocity; in the one-dimensional case it reduces to the familiar expression $d\omega/dk$. Clearly (149), (150), and (151) are unchanged if we replace D_0 by any decent function of D_0, although this will alter the definition of τ.

It may be of some interest to see how the formula for $dt/d\tau$ actually expresses the time taken in the transit of a wave packet along a ray path. We shall consider here a time-independent medium, for which $\partial D_0/\partial t=0$ and ω is constant. The wave function ψ is then a linear combination of form

$$\psi(\mathbf{x},t)\cong\int d\omega\,\psi_0(\mathbf{x},\omega)\exp i[S(\mathbf{x},\omega)-\omega t]. \tag{152}$$

We will assume that $\psi_0(\mathbf{x},\omega)$ varies slowly over a range of ω within which $S(\mathbf{x},\omega)$ varies rapidly. Then the only values of $\mathbf{x}$ and t for which $\psi(\mathbf{x},t)$ is appreciable are those for which $S-\omega t$ is stationary, i.e.,

$$t=(\partial/\partial\omega)S(\mathbf{x},\omega). \tag{153}$$

Hence, near a point $\mathbf{x}_1$ on the initial surface Γ, where

$$S\cong\mathbf{k}(\omega,\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_1),$$

we have

$$t\cong[\partial\mathbf{k}(\omega,\mathbf{x}_1)/\partial\omega]\cdot(\mathbf{x}-\mathbf{x}_1).$$

This shows that the wave packet we have constructed is one that is essentially confined to Γ at $t=0$; Eq. (153) tells how long it takes to reach $\mathbf{x}$.

The evaluation of $\partial S/\partial\omega$ follows along the same lines as the evaluation of ∇S in Sec. IV. If ω is varied by $\delta\omega$, the paths $\mathbf{x}(\tau)$, $\mathbf{k}(\tau)$ will be varied by $\Delta\mathbf{x}(\tau)$, $\Delta\mathbf{k}(\tau)$, giving again Eq. (122). This time $\delta\mathbf{x}=0$, and again $\mathbf{k}(\tau_1)\cdot\Delta\mathbf{x}(\tau_1)=0$ for the same reason as before, so here the first two terms of (122) vanish. However, now the third term is not zero, but instead (since $\Delta D_0=0$)

$$\delta S(\mathbf{x},\omega)=-\int_{\tau_1}^{\tau_2}\frac{\partial D_0}{\partial\omega}\delta\omega d\tau.$$

Thus

$$t=\frac{\partial S(\mathbf{x},\omega)}{\partial\omega}=-\int_{\tau_1}^{\tau_2}\frac{\partial D_0}{\partial\omega}d\tau \tag{154}$$

and the relation between t and τ is

$$dt=-(\partial D_0/\partial\omega)d\tau. \tag{155}$$

VII. FERMAT'S PRINCIPLE

The derivation of Eq. (123) shows that if we vary the ray paths holding the end point $\mathbf{x}$ and the value $D_0=0$ (but not τ_2) fixed, then $\delta S=0$. It is natural to ask whether the principle of stationary S implies a principle of least time. In fact it does, for sufficiently simple time-independent problems in which D_0 is a homogeneous function of $\mathbf{k}$ and ω, i.e., in which

$$D_0(\alpha\mathbf{k},\alpha\omega)\equiv\alpha^mD_0(\mathbf{k},\omega) \tag{156}$$

for arbitrary values of $\mathbf{k}$.

For example, in geometric optics $D_0=c^2\mathbf{k}^2-n^2\omega^2$, so $m=2$, while for the Alfvén mode we could take $D_0=\mathbf{V}\cdot\mathbf{k}-\omega$, with $m=1$.

To derive Fermat's principle we differentiate (156) with respect to α and then set $\alpha=1$, obtaining Euler's relation:

$$(\partial D_0/\partial\mathbf{k})\cdot\mathbf{k}+(\partial D_0/\partial\omega)\omega=mD_0=0.$$

Hence

$$\begin{aligned}\mathbf{k}\cdot d\mathbf{x}&=\mathbf{k}\cdot(\partial D_0/\partial\mathbf{k})d\tau\\&=-\omega(\partial D_0/\partial\omega)d\tau\\&=\omega dt\end{aligned}\tag{157}$$

and therefore

$$S=\omega t.\tag{158}$$

If the medium is time independent ω is just a constant, so $\delta t=0$ follows from $\delta S=0$.

In more complicated problems there appear parameters, such as a plasma frequency ω_P, a collision frequency ω_c, a viscosity ν, etc., which destroy the homogeneity of D_0 and hence invalidate Fermat's principle. We can only use Fermat's principle if the medium is entirely characterized by a set of velocities, like the local speed of light, or of Alfvén waves, or of sound. (In particular, Fermat's principle could be applied to Sec. II but not in Sec. III of this paper; its use by Francis, Green, and Dessler[6] was actually justified.) At any rate, the only use usually made of Fermat's principle is in deriving Eqs. (146) and (147), and these equations are always valid.

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Lifetime Effects in Condensed Helium-3

PIERRE MOREL
French Embassy, New York, New York

AND

PHILIPPE NOZIÈRES
*Laboratoire de Physique, Ecole Normale Supérieure, Paris, France**

(Received February 13, 1962)

The condensation of a Fermion system by forming d-type bound pairs is discussed with the help of time-dependent correlation functions both at absolute zero and finite temperatures, for the purpose of applying this study to the case of liquid helium-3. We use essentially Gor'kov's method, suitably generalized to take into account the anisotropy of the bound pairs and also the finite lifetime of the quasi-particles which make up the pairs. The treatment proposed here goes one step further than the Hartree approximation in the sense that the finite decay rate of the quasi-particles is introduced by means of a model spectral density for the renormalized propagator (Green's function). This model features a single broad peak instead of the infinitely sharp peak which characterizes the Hartree approximation. Considerable care is taken to relate this microscopic model to the available experimental data on the scattering probability in liquid helium-3. It is concluded that the effect of scattering on the condensation can be adequately described by a cutoff Λ of the order of 1°K, limiting the domain in momentum space of the quasi-particles which participate effectively in the condensation process. This entails a reduction of the transition temperature estimated previously on the basis of the Hartree approximation, down to a value of the order of 0.02–0.03°K.

I. INTRODUCTION

THE idea of particle pairing, which is the basis of Bardeen, Cooper, and Schrieffer's (BCS) theory of superconductivity,[1] naturally leads us to ask whether this condensation could also obtain by forming pairs in a finite angular momentum state instead of the s state considered by BCS. Many authors[2–5] have agreed, on theoretical grounds, that this is indeed possible provided that the interaction potential is favorable, i.e., produces a larger binding energy in a finite l state than in the s state. The interaction potential between two bare helium-3 atoms comprises a strong repulsive core and a weak long-range attraction; consequently, the effective interaction in the liquid is thought to be attractive for two quasi-particles in a large relative angular momentum state.[6] Actual computations[3,7] showed that this interaction is indeed the most attractive in the d state ($l=2$), although it is repulsive for $l=0$ and 1. The existence of a condensed state of liquid helium-3, stable

* Part of this work was performed while this author was at the Bell Telephone Laboratories, Murray Hill, New Jersey.

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