

Theory of Transient Space-Charge-Limited Currents in Solids in the Presence of Trapping

A. MANY*

Department of Physics, The Hebrew University, Jerusalem, Israel

AND

G. RAKAVY

Department of Theoretical Physics, The Hebrew University, Jerusalem, Israel

(Received January 17, 1962)

The problem of transient space-charge-limited currents in insulating and conducting crystals is treated mathematically. With a number of simplifying assumptions, solutions are derived for the time-dependent current and space-charge distribution following the onset of injection, the latter taking place via an ohmic contact under an applied voltage-pulse. Exact analytical solutions are given for the two limiting cases of no trapping and fast trapping. For flow in an insulator under slow trapping, approximate expressions are derived which are valid for trapping-times larger than twice the transit time. For shorter trapping times the equations of flow are solved numerically and the solutions presented in graphical form.

I. INTRODUCTION

THE mechanism of space-charge-limited currents (SCLC) in solids has been first proposed by Mott and Gurney¹ who have given a simplified analysis of space-charge injection into an ideal insulator devoid of traps. These studies have been extended by Shockley and Prim² and Dacey³ for the case of injection into a semiconductor by means of a p - n junction, taking into account the diffusive contribution to the current but again omitting trapping effects. Although a rigorous analytic solution to this problem could be obtained, its range of applicability in practice is rather limited since in most solids, trapping levels play a dominant role. The pioneering work of Rose,^{4,5} Smith⁶ and Lampert,⁷ whose primary concern was the effect of trapping on SCLC, has turned the latter into a powerful tool for studying defect levels in solids. A number of workers have since investigated SCLC problems, both theoretically and experimentally.⁸

Aside from a few experimental observations on the transient response,^{6,9,10} most of the work reported on SCLC has been confined to steady-state conditions. Recent work in this laboratory, on the other hand, has shown that a detailed study of transient SCLC can yield valuable information on transport, trapping, recombination, and photogeneration processes in solids.

In this paper the theory of transient SCLC in solids, in the presence of trapping, is presented. The analysis

covers both an insulating and a conducting crystal. In the latter case, however, space-charge effects are significant only for low-conductivity materials for which the dielectric relaxation time is not too small compared to the transit time of an injected carrier. A single-carrier flow through a plane-parallel crystal provided with an ohmic (injecting) contact is considered. Solutions are sought for the time-dependent SCLC and charge distribution along the sample following the onset of injection, the latter arising, e.g., from an applied voltage pulse. A number of simplifying assumptions are introduced. First, the diffusive contribution to the conduction current is omitted. If this term is not neglected, the analysis is most difficult; even for the much simpler case of steady-state flow in a trap-free insulating crystal the solution is quite complicated.² The error incurred by this procedure, however, is much the same as that for steady-state conditions where, as has been discussed by Lampert⁷ and Wright,⁸ it is usually small under normal operating conditions. Regarding the traps, we shall assume that they are all situated at one discrete energy level, and that the density of trapped carriers is always small compared to that of the unoccupied levels at thermal equilibrium. As will be discussed below, this is not too serious a limitation as the treatment of the general case may often be reduced to that of the simpler one assumed here, over a considerable range of the current decay. With these assumptions, exact analytical solutions are obtained for the two limiting cases of no trapping and fast trapping. Approximate solutions are given for an insulating crystal characterized by slow trapping. These solutions are shown to be valid for trapping times larger than about twice the transit time. For shorter trapping times, the equations of flow must be solved numerically. Results obtained in this manner are presented graphically for various trapping times.

A partial analysis of transient SCLC for the ideal case of an insulating solid devoid of traps, together

* Present address: RCA Laboratories, Princeton, New Jersey.

¹ N. F. Mott and R. W. Gurney, *Electronic Processes in Ionic Crystals* (Oxford University Press, New York, 1940), 1st ed., p. 172.

² W. Shockley and R. C. Prim, *Phys. Rev.* **90**, 753 (1953).

³ G. C. Dacey, *Phys. Rev.* **90**, 759 (1953).

⁴ A. Rose, *RCA Rev.* **12**, 362 (1951).

⁵ A. Rose, *Phys. Rev.* **97**, 1538 (1955).

⁶ R. W. Smith and A. Rose, *Phys. Rev.* **97**, 1531 (1955).

⁷ M. A. Lampert, *Phys. Rev.* **103**, 1648 (1956).

⁸ G. T. Wright, *Solid State Electronics* **2**, 165 (1961); references therein.

⁹ K. W. Böer, J. Dziesiaty, and U. Kümmel, *Z. Naturforsch.* **13a**, 560 (1958).

¹⁰ G. T. Wright, *Proc. Inst. Elect. Engrs.* **B106**, 915 (1959).

with some preliminary experimental results obtained on iodine have already been presented previously.¹¹ Similar work on anthracene has also been reported.¹² In both cases a very good agreement has been obtained between those features of the theory and experiment which are not markedly affected by trapping. In the following paper,¹³ more detailed results on iodine, with particular emphasis on trapping, are presented and compared with the theory discussed here.

II. GENERAL FORMULATION OF THE PROBLEM

In this section the problem of the time-dependent flow of mobile charge carriers injected at one electrode into the bulk of a solid, in the presence of trapping, will be mathematically formulated. A single-carrier system will be assumed throughout. The neutral sample under thermal equilibrium has, except very close to the electrodes, uniform densities n_0 and n_{t0} of free and trapped majority carriers, respectively. Under an applied voltage, *majority* carriers are injected from the ohmic contact into the bulk, the position- and time-dependent free and trapped majority-carrier densities under these conditions being denoted by $n(x', t')$ and $n_t(x', t')$, respectively.

The carrier flow in the crystal is governed by the law of conduction, the continuity equation, Poisson's equation, and a system of equations relating the free and trapped carriers. For planar geometry, these equations are, respectively:

$$J_c(x', t') = q\mu n(x', t')E(x', t') - qD\partial n(x', t')/\partial x', \quad (1)$$

$$\partial J_c(x', t')/\partial x' = -q[\partial n(x', t')/\partial t' + \partial n_t(x', t')/\partial t'], \quad (2)$$

$$\partial E(x', t')/\partial x' = (4\pi q/\kappa)[n(x', t') - n_0 + n_t(x', t') - n_{t0}], \quad (3)$$

$$\partial n_t(x', t')/\partial t' = C_n\{n(x', t')[N_t - n_t(x', t')] - \theta_0(N_t - n_{t0})n_t(x', t')\}. \quad (4)$$

Here $J_c(x', t')$ is the conduction-current density, $E(x', t')$ the electric field, q the carrier charge, κ the dielectric constant, and μ and D the carrier mobility and diffusion constant, respectively (cgs units). The signs of q and μ should be taken positive for hole conduction and negative for electron conduction. Equation (4) expresses the trapping kinetics between the free carriers and a discrete set of localized levels present with a density N_t and having a capture probability C_n . The second term in the curly brackets represents the rate of carrier release from the traps into the conducting band; θ_0 as determined from the principle of detailed balance is given by

$$\theta_0 = n_0/n_{t0} = n_1/(N_t - n_{t0}), \quad (5)$$

¹¹ A. Many, M. Simhony, S. Z. Weisz, and J. Levinson, International Conference on Photoconductivity, Cornell University, August, 1961. [J. Phys. Chem. Solids, **22**, 285 (1961).]

¹² M. Silver, M. Swicord, R. C. Jarnagin, A. Many, S. Z. Weisz, and M. Simhony, J. Phys. Chem. Solids **24** (1962).

¹³ A. Many, S. Z. Weisz, and M. Simhony, following paper [Phys. Rev. **126**, 1989 (1962)].

where n_1 is the free-carrier concentration when the Fermi level coincides with the trapping level. For the general case of several sets of levels present, Eq. (4) should be replaced by a system of equations, each of identical form to (4).

Obviously, the solution of Eqs. (1)–(4) for the general case is a formidable task. A few simplifying assumptions will therefore be introduced. For Eq. (1), we shall neglect the diffusive contribution to the conduction current, as discussed above. Equation (1) is then replaced by

$$J_c(x', t') = q\mu n(x', t')E(x', t'). \quad (1')$$

Regarding Eq. (4), we shall require that the filling of the trapping levels at any time be small compared to the empty levels at thermal equilibrium:

$$n_t(x', t') - n_{t0} \ll N_t - n_{t0}. \quad (6)$$

This condition is certainly fulfilled up to a sufficiently small time interval following the onset of injection. It is often met in practice over most of the range of interest. The total space charge injected into the sample is proportional to the applied voltage. Since usually N_t is of the order of 10^{13} – 10^{15} cm⁻³ or larger, condition (6) for levels lying at or above the Fermi level will not be violated up to considerably large applied voltages. With the use of (6), Eq. (4) reduces to

$$\partial n_t(x', t')/\partial t' = (1/T_t)[n(x', t') - \theta_0 n_t(x', t')], \quad (4')$$

where T_t is the trapping time, given by

$$T_t = [C_n(N_t - n_{t0})]^{-1}. \quad (7)$$

It should be pointed out that Eq. (4') may easily be extended to cover the more general case, for which several sets of trapping levels are present in the crystal, provided that the trapping rate at each set is large compared to the release rate. The latter condition usually holds for deep traps for which $\theta_0 \ll 1$, and enables one to neglect the second term on the right-hand side of Eq. (4') over most of the decay range of $n(x', t')$. In this form, Eq. (4') would be valid for the general case, with $n_t(x', t')$ and n_{t0} representing the *total* trapped carrier densities, and T_t the trapping time due to *all* sets of levels.

It is convenient to obtain the solutions in terms of dimensionless distance, time (trapping time), field strength, charge density, and conduction-current density variables, x , t , (τ) , ϵ , ρ , and j_c , respectively, defined as follows:

$$x = x'/L, \quad t = t'/t_0, \quad \tau = T_t/t_0, \quad \epsilon = E/(V_0/L), \\ \rho = qn/(\kappa V_0/4\pi L^2), \quad j_c = J_c/(\kappa\mu V_0/4\pi L^3), \quad (8)$$

where L is the length of the sample and V_0 the applied voltage; $t_0 = L^2/\mu V_0$ is the transit time of a carrier in the absence of space-charge effects. The charge densities ρ_0 , ρ_{t0} , ρ_t are defined similarly in terms of n_0 , n_{t0} , and

n_t , respectively. Equations (1'), (2), (3), and (4') then become

$$j_c(x,t) = \rho(x,t)\epsilon(x,t), \quad (9)$$

$$\partial j_c(x,t)/\partial x = -(\partial/\partial t)[\rho(x,t) + \rho_t(x,t)], \quad (10)$$

$$\partial \epsilon(x,t)/\partial x = \rho(x,t) + \rho_t(x,t) - \rho_0 - \rho_{t0}, \quad (11)$$

$$\partial \rho_t(x,t)/\partial t = (1/\tau)[\rho(x,t) - \theta_0 \rho_t(x,t)]. \quad (12)$$

We shall now discuss the boundary conditions for Eqs. (9)–(12). At $x=0$ an ohmic contact is assumed, near which a large reservoir of majority carriers is available. Its magnitude is sufficiently large to ensure for any voltage and at all times, the presence of a “virtual cathode” near the interface where the electric field vanishes. The separation of the virtual cathode from the interface is of the order of the Debye length, characteristic to the sample material. For trap concentrations of the order of 10^{13} cm^{-3} or larger, this distance is very small (10^{-4} cm or smaller) and may usually be neglected compared to the sample length L . Hence, the virtual cathode will be taken to be at $x=0$. Similar conditions are assumed to hold at the opposite electrode so as not to obstruct the current flow.

An ohmic contact may be brought about by the special nature of the metal-crystal contact, as is the case, e.g., for an indium or gallium electrode on cadmium sulfide.¹⁴ Alternatively, the reservoir characteristic of an ohmic contact may be obtained in certain cases by photogeneration of hole-electron pairs near or at the surface.^{11–13} Such a generation is effected by an illumination of one surface, through a transparent electrode, by highly absorbed light. For sufficiently intense illumination, a majority- and minority-carrier charge separation takes place under an applied voltage until the field vanishes at some point near the interface (the “virtual cathode”). The charge separation is effected within a relaxation time associated with the photogenerated carrier conductance, and for the intense light assumed, takes place a very short time following the voltage pulse, i.e., effectively at $t=0$. At $t=0$ no charge redistribution has yet taken place, except very close to the interface ($x=0$), and hence in the bulk of the sample the field is given by V_0/L , and the free- and trapped-carrier concentrations still retain their values at thermal equilibrium. Finally,

$$\int_0^L E(x',t') dx' = V_0,$$

at all times following the pulse onset.

From the physical considerations discussed above, the boundary conditions may be expressed in terms of the dimensionless variables, defined by Eq. (8), as

follows:

$$\epsilon(0,t) = 0, \quad t \geq 0$$

$$\epsilon(x,0) = 1, \quad x > 0$$

$$\int_0^1 \epsilon(x,t) dx = 1, \quad t \geq 0 \quad (13)$$

$$\left. \begin{aligned} \rho(x,0) &= \rho_0 \\ \rho_t(x,0) &= \rho_{t0} \end{aligned} \right\}, \quad x > 0. \quad (14)$$

Equations (9) and (10) are expressed in terms of the conduction-current density j_c . A much more significant variable is the total-current density j which is the sum of the conduction- and displacement-current densities. According to Kirchhoff's law, j is a function of time only and is given by

$$j(t) = j_c(x,t) + \partial \epsilon(x,t)/\partial t. \quad (15)$$

This may also be verified by eliminating $\rho(x,t) + \rho_t(x,t)$ from Eqs. (10) and (11) and integrating with respect to x . By integrating Eq. (15) with respect to x and making use of Eq. (11) and the boundary conditions (13), we obtain

$$j(t) = \frac{1}{2} \epsilon^2(1,t) + \rho_0 + \rho_{t0} - \int_0^1 \rho_t(x,t) \epsilon(x,t) dx. \quad (16)$$

Equations (9)–(12) constitute a system of hyperbolic equations for the flow in the crystal. The characteristics of this system are the lines $x = \text{const}$ and the carrier-flow lines $x(t)$:

$$dx(t)/dt = \epsilon(x,t). \quad (17)$$

Equations (9)–(11) may be combined into one equation for ρ and another for ϵ containing derivatives along the flow lines $x(t)$:

$$d\rho(x(t),t)/dt = -\rho(\rho + \rho_t - \rho_0 - \rho_{t0}) - (1/\tau)(\rho - \theta_0 \rho_t), \quad (18)$$

$$d\epsilon(x(t),t)/dt = j(t) + (\rho_t - \rho_{t0} - \rho_0)\epsilon(x(t),t). \quad (19)$$

The derivation of Eq. (18) is as follows. From Eq. (17) it follows that:

$$d\rho(x(t),t)/dt = (\partial\rho/\partial x)\epsilon + \partial\rho/\partial t.$$

As $j(t)$ is independent of x , it follows by the use of Eqs. (9) and (15) that

$$\frac{\partial j(t)}{\partial x} = 0 = \frac{\partial \rho}{\partial x} + \rho \frac{\partial \epsilon}{\partial x} + \frac{\partial}{\partial t} \left(\frac{\partial \epsilon}{\partial x} \right).$$

Use of Eqs. (11) and (12) then leads to Eq. (18). Equation (19) is similarly derived by the use of Eqs. (9), (11), and (15).

Equations (17)–(19) have a simple physical meaning and can be written down from first principles. Equation (17) states that we are moving along the crystal with a front of advancing carriers. Equation (18) states that the rate of decay of free charge-density along such a flow line is composed of two terms, the first represents

¹⁴ R. W. Smith, Phys. Rev. **97**, 1525 (1955).

the attenuation with time along a flow line due to Coulomb repulsion between the free charge density (ρ) and the total charge density ($\rho + \rho_t - \rho_0 - \rho_{t0}$) at that point, while the second term represents the net rate of free-charge trapping. Equation (19) states that while moving along with a front of carriers the total current density $j(t)$ is composed of a displacement current ($d\epsilon/dt$) and a conduction current in the opposite direction due to the localized charge density $\rho_t - \rho_{t0} - \rho_0$, which moves in the opposite direction relative to the advancing front, with a velocity $-\epsilon$. Finally, Eq. (12) is already in normal form, only a derivative along the line $x = \text{const}$ appears.

Equations (12) and (17)–(19) are equivalent to Eqs. (9)–(12). They can be solved analytically for the two limiting cases of no trapping ($\tau \rightarrow \infty$) and fast trapping ($\tau \rightarrow 0$). Both cases lend themselves to the same mathematical treatment and will be discussed in detail in the next Section. In Sec. IV an approximate solution will be given for the case of flow under slow-trapping. The general case can be treated only numerically by finite difference methods. Almost any method suitable for one-dimensional flow of a compressible fluid in hydrodynamics can be applied to the problem at hand. Numerical solutions are presented in Sec. V for various values of the trapping time (τ) and equilibrium density (ρ_0).

III. CARRIER FLOW IN A CRYSTAL WITH NO TRAPPING OR WITH FAST TRAPPING

We shall first show that the two limiting cases of no trapping ($\tau \rightarrow \infty$) and fast trapping ($\tau \rightarrow 0$) are mathematically equivalent. For the former case $\rho_t = \rho_{t0} = \text{const}$ and Eqs. (18) and (19) reduce to

$$d\rho/dt = -\rho(\rho - \rho_0), \quad (20)$$

$$d\epsilon/dt = j(t) - \rho_0\epsilon, \quad (21)$$

the derivatives being taken, as before, along the flow lines defined by Eq. (17). For the case of fast trapping ($\tau \rightarrow 0$), all sets of trapping levels are assumed to be in very intimate contact with the conducting band, so that the condition $\rho = \theta_0 \rho_t$ is established in a time short compared to any significant time interval. This condition is effectively realized whenever τ is very short compared to the transit time, i.e., $\tau \ll 1$. If $\rho' = \rho + \rho_t = (1 + \theta_0^{-1})\rho$ is taken as an effective free charge density, then Eqs. (10) and (11) reduce to their form without trapping. Equation (9), however, should then be replaced by

$$j_e = (1 + \theta_0^{-1})^{-1} \rho' \epsilon. \quad (9')$$

If finally, an effective mobility $\mu' = (1 + \theta_0^{-1})^{-1} \mu$ is also defined, and the dimensionless variables are altered accordingly, the case of fast trapping reduces completely to that of no trapping. The latter case will accordingly be assumed to hold throughout this section.

Equation (20) may easily be integrated to give the

dispersion of charge along a flow line. For an insulating crystal ($\rho_0 = 0$), one obtains

$$\rho(t) = \rho_a [1 + \rho_a(t - t_a)]^{-1}, \quad (22)$$

while for a conducting crystal ($\rho_0 \neq 0$),

$$\rho(t) = \rho_0 \left[1 - \frac{\rho_a - \rho_0}{\rho_a} \exp[-\rho_0(t - t_a)] \right]^{-1}. \quad (23)$$

These equations express the decay (along a flow line) of ρ with time from its value ρ_a at any given prior time t_a .

(a) Small-Signal Case

Although we are primarily interested in this paper with carrier flow under space-charge limitation, it may be useful to discuss first the much simpler case of the flow of a *small* majority-carrier disturbance for which the external field applied to the sample is not appreciably distorted. Such a disturbance may be obtained, for example, by a low-intensity flash illumination of one electrode in the presence of a steady field of the proper polarity. Under these conditions, a short pulse of optically generated majority carriers is injected into the crystal. It is easy to see from Eqs. (22) and (23) how this pulse is propagated along the crystal. Since the disturbances in carrier concentration $\Delta\rho$ is assumed small, $\Delta\rho \ll 1$, it follows that the field is approximately constant $\epsilon(x, t) \approx 1$. To this approximation every carrier in the injected pulse drifts with the same uniform velocity [Eq. (17)]. For an insulator we then obtain from Eq. (22), neglecting $\Delta\rho_a$ compared to unity, that $\Delta\rho(t) = \Delta\rho_a = \text{const}$ and the pulse drifts solidly down the crystal as shown by the dashed curves in Fig. 1(a). The current arising from such a drift is given by the total space charge (the area of the dashed curves) multiplied by the average drift velocity (~ 1). It is therefore a constant up to the transit time ($t = 1$), thence it falls sharply to zero (the width of the pulse is

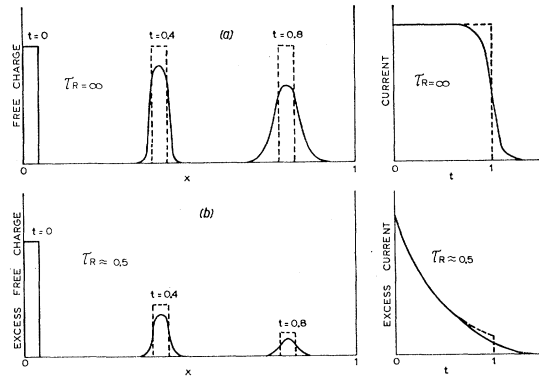


FIG. 1. Propagation of a *small* pulse of injected majority carriers and the resulting current, for an insulator (a) and for a conducting crystal (b), in the absence of trapping. The dashed curves correspond to the approximation which neglects dispersion; the solid curves illustrate the effect of charge dispersion.

taken as small compared to the sample's length). Actually, as seen from Eq. (22), some dispersion of charge takes place, as illustrated by the solid curves in Fig. 1(a). As long as the carriers remain in the sample, however, the total charge is constant, giving rise to an approximately constant current. The only effect of the dispersion on the current is a slight rounding off of the drop near $t=1$, as shown by the solid curve in Fig. 1(a).

The situation is markedly different for a conducting crystal. We shall assume that the excess charge density $\Delta\rho = \rho - \rho_0$ is not only small compared to unity but also compared to ρ_0 . Equation (23) then reduces to

$$\Delta\rho = \Delta\rho_a \exp[-\rho_0(t-t_a)].$$

To the approximation that $\epsilon(x,t) = \text{const}$, the injected pulse preserves its shape but its area (the total space charge) decays with the relaxation time $\tau_R = 1/\rho_0$. This is illustrated by the dashed curves in Fig. 1(b) where τ_R has been taken as 0.5. As for the insulator case, a slight dispersion of charge will actually take place (solid curves). The excess current will be a decaying exponential with a sharp drop at $t=1$. The effect of the charge dispersion would again be to round off the drop near $t=1$. This situation illustrates well the physical meaning of the relaxation time τ_R . It is the average time required to expel from a crystal a small space charge introduced into it. The process of charge expulsion is effected via the free carriers present in the neutral crystal at thermal equilibrium, and hence is faster the larger ρ_0 is. One can immediately see, also, why it is pointless to inject majority carriers into a *highly* conducting sample. Consider, for example, an extrinsic germanium sample at room temperature. Here τ_R is of the order of 10^{-12} sec. Hence an injected pulse of majority carriers and the resulting excess current would disappear (by carriers leaving the other end of the sample) in a time short compared to the experimental resolution time.

(b) Flow in an Insulating Crystal

From now on we return to the original problem, as formulated in Sec. II, in which, for $t \geq 0$, the cathode *continuously* injects carriers into the crystals, in amounts prescribed by space-charge limitation only.

For $\rho_0 = 0$ and $\rho_i(x,t) = \rho_{i0}$, Eqs. (16) and (19) reduce to

$$j(t) = \frac{1}{2}\epsilon^2(1,t), \quad (24)$$

and

$$d\epsilon(x(t),t)/dt = j(t) = \frac{1}{2}\epsilon^2(1,t). \quad (25)$$

By the use of Eqs. (11) and (17) we have

$$\begin{aligned} \frac{d\epsilon(x(t),t)}{dt} &= \frac{\partial\epsilon(x,t)}{\partial x} \frac{dx}{dt} + \frac{\partial\epsilon(x,t)}{\partial t} \\ &= \rho(x,t)\epsilon(x,t) + \frac{\partial\epsilon(x,t)}{\partial t} = \frac{1}{2}\epsilon^2(1,t), \end{aligned}$$

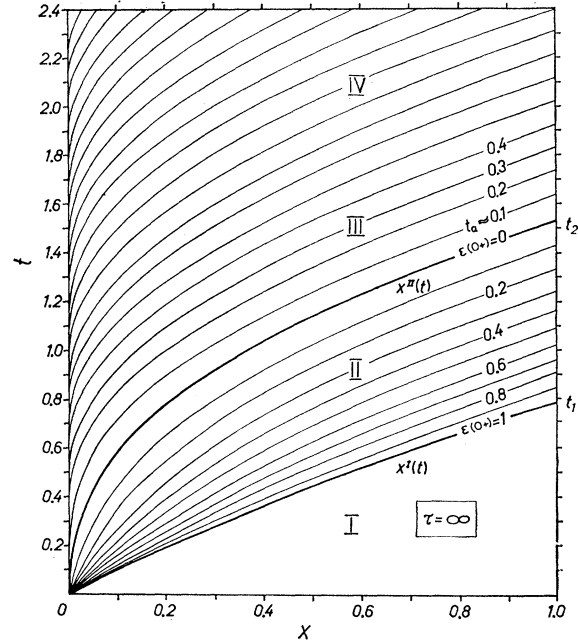


FIG. 2. Flow lines for a trap-free insulator.

and, at $x=1$,

$$d\epsilon_1(t)/dt = \frac{1}{2}\epsilon_1^2(t) - \rho_1(t)\epsilon_1(t), \quad (26)$$

where $\epsilon_1(t) \equiv \epsilon(1,t)$ and $\rho_1(t) \equiv \rho(1,t)$. Now $\rho(x(t),t)$ is known explicitly along a flow line [Eq. (22)]. Hence, if we know which flow line arrives at $x=1$ at time t we can calculate $\rho_1(t)$. We then have in Eq. (26) a simple ordinary differential equation for $\epsilon_1(t)$ from which $j(t)$ may be calculated.

The first step is thus to obtain a characterization of each flow line. The flow lines defined by Eq. (17) and calculated below are plotted in Fig. 2. The diagram is divided into four zones which will be discussed separately.

Zone I. This zone includes all flow lines originating from $x > 0$ at $t=0$. For an insulator, no carriers flow along these lines since at $t=0$ carriers are present only at the interface $x=0$. Zone I is bounded by the flow line originating from $x=0$ at $t=0$ which corresponds to the *first* front of injected carriers. Let t_1 denote the transit time for this leading front. Then for $t < t_1$, $\rho_1(t) = 0$ and the solution of Eq. (26) subject to the boundary conditions (13) is

$$\left. \begin{aligned} \epsilon_1(t) &= (1 - \frac{1}{2}t)^{-1}, \\ j(t) &= \frac{1}{2}(1 - \frac{1}{2}t)^{-2}, \end{aligned} \right\} \quad 0 \leq t \leq t_1. \quad (27)$$

From Poisson's equation it is evident that at any time in this interval the field in the region between the position of the leading front and $x=1$ is constant and given by Eq. (27). This is also just the field under which the leading front drifts. Hence, introducing the value of $\epsilon_1(t)$ from Eq. (27) into Eq. (17) we obtain

for the flow line of the leading front,

$$x^I(t) = \int_0^t \epsilon_1(t') dt' = -2 \ln(1 - \frac{1}{2}t), \quad 0 \leq t \leq t_1. \quad (28)$$

This flow line is represented in Fig. 2 by the heavy curve separating zones I and II. The transit time t_1 is obtained from Eq. (28) by putting $x^I(t_1) = 1$:

$$t_1 = 2(1 - e^{-\frac{1}{2}}) = 0.787. \quad (29)$$

Zone II. The diverging flow lines in this zone all originate from $x=0$ at $t=0$. Each line may be characterized by the field $\epsilon(0+)$, which the corresponding front feels just as it emerges from the injecting electrode, i.e., at a vanishingly short time following $t=0$. For the leading-front flow line $x^I(t)$, $\epsilon(0+)$ is unity while for the following lines it diminishes progressively, until for the last flow line to leave the injecting electrode at $t=0$, $\epsilon(0+)=0$, as shown in Fig. 2. If we define the functions:

$$K(t) = \int_0^t j(t') dt', \quad (30)$$

$$L(t) = \int_0^t K(t') dt', \quad (31)$$

then we obtain from Eqs. (17) and (25)

$$\epsilon(x(t), t) = \epsilon(0+) + K(t), \quad (32)$$

$$x(t) = \epsilon(0+)t + L(t). \quad (33)$$

As the charge density at the interface is infinite, we obtain from Eq. (22) for ρ along any flow line in zone II

$$\rho(x(t), t) = 1/t. \quad (34)$$

Now at any time t in the interval $t_1 \leq t \leq t_2$, where t_2 is the arrival time of the last flow line in zone II, $\rho_1(t)$ originates from a front moving along that flow line in zone II whose transit time is t . Hence, we have also

$$\rho_1(t) = 1/t, \quad t_1 \leq t \leq t_2. \quad (35)$$

Equation (26) now becomes

$$d\epsilon_1(t)/dt = \frac{1}{2}\epsilon_1^2 - \epsilon_1/t. \quad (36)$$

By substituting for ϵ_1 :

$$\epsilon_1(t) = -2\omega(t) / \int \omega(t) dt,$$

Eq. (36) is transformed into

$$d\omega/dt + (1/t)\omega = 0,$$

which may be integrated to give

$$\epsilon_1(t) = [(\beta - \frac{1}{2} \ln t)t]^{-1},$$

where β is a constant of integration. Its value can be determined from the requirement that $\epsilon_1(t)$ be con-

tinuous at $t=t_1$. With β thus determined [using Eqs. (27) and (29)] we have

$$\left. \begin{aligned} \epsilon_1(t) &= (e^{\frac{1}{2}t_1}/t) [1 - (e^{\frac{1}{2}} - 1) \ln(t/t_1)]^{-1}, \\ j(t) &= \frac{1}{2}e(t_1/t)^2 [1 - (e^{\frac{1}{2}} - 1) \ln(t/t_1)]^{-2}, \end{aligned} \right\} t_1 \leq t \leq t_2. \quad (37)$$

Using Eqs. (36) and (37) one can evaluate the functions $K(t)$ and $L(t)$, and thus the flow lines in zone II:

$$\begin{aligned} K(t) &= K(t_1) + \int_{t_1}^t j(t') dt' \\ &= K(t_1) + \epsilon_1(t) - \epsilon_1(t_1) + \int_{t_1}^t \frac{\epsilon_1(t')}{t'} dt', \end{aligned} \quad (38)$$

and by further integration it may be shown that

$$\begin{aligned} x(t) &= 1 + [\epsilon(0+) - 1]t + 2(t/t_1) \exp[-1/(e^{\frac{1}{2}} - 1)] \\ &\quad \times [F\{1/(e^{\frac{1}{2}} - 1)\} - F\{1/(e^{\frac{1}{2}} - 1) - \ln(t/t_1)\}], \end{aligned} \quad (39)$$

where

$$F\{u\} = \int_1^u \frac{\exp u'}{u'} du'.$$

This function is available in tabular form.¹⁵

The time of arrival of the *last* flow line in zone II, $x^{II}(t)$, may be calculated either from the condition that $x(t_2)=1$ for $\epsilon(0+)=0$, or from the requirement that the field $\epsilon(x(t), t)$ along $x^{II}(t)$ be equal to $\epsilon_1(t)$ at the instant of arrival t_2 . After some manipulation it is found that

$$t_2 = 1.915t_1 = 1.51. \quad (40)$$

Zone III. The flow lines sweeping this zone originate at $x=0$ at various times t_a following the onset of injection ($t=0$). Each line may accordingly be characterized by the value of t_a (Fig. 2). From Eqs. (22) and (26) we obtain

$$d\epsilon_1(t)/dt = \frac{1}{2}\epsilon_1^2(t) - \epsilon_1(t)/[t - t_a(t)], \quad (41)$$

where $t_a(t)$ corresponds to that flow line which arrives at $x=1$ in time t . In other words, $t - t_a(t)$ is the transit time of a carrier arriving at the anode at time t . From the condition:

$$x(t) = 1 = L(t) - L(t_a) - (t - t_a)K(t_a),$$

one obtains by differentiation [using Eqs. (25), (30), and (31)]:

$$dt_a/dt = (t - t_a)^{-1} [K(t) - K(t_a)] / \frac{1}{2}\epsilon_1^2(t_a). \quad (42)$$

$K(t)$ is defined by Eq. (30) or by

$$dK(t)/dt = \frac{1}{2}\epsilon_1^2(t). \quad (43)$$

Thus the problem in zone III reduces to the solution of three ordinary differential equations (41)–(43) for $\epsilon_1(t)$, $t_a(t)$, and $K(t)$. No analytical solution could be

¹⁵ E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1945).

found for these equations. Fortunately, however, such a solution is of little practical interest since at $t=t_2$ (treated for zone II), $\epsilon_1(t)$, $\rho(x,t)$, and $j(t)$ are already very close to their steady-state values. For example, $j(t_2)=1.11$ whereas the asymptotic value is 1.125 [Eq. (44)].

Zone IV. The flow lines in this zone correspond to steady-state conditions and are accordingly all parallel. These conditions are established within a few units of time. We then have from Eqs. (9) and (11)

$$j(\infty) = j_c(\infty) = \rho(x, \infty) \epsilon(x, \infty) = \frac{1}{2} [d\epsilon^2(x, \infty)/dx],$$

and by integration

$$\epsilon(x, \infty) = (2j)^{1/2} x^{1/2}.$$

Using the condition $\int_0^1 \epsilon dx = 1$ we obtain

$$j(\infty) = 9/8, \quad \epsilon(x, \infty) = (3/2)x^{1/2}, \quad \rho(x, \infty) = (3/4)x^{-1/2}, \quad (44)$$

which are the well known steady-state expressions.¹

(c) Flow in a Conducting Crystal

The treatment in this case is similar to that for an insulating crystal. Analytical solutions have been obtained only for zone I and the asymptotic zone IV.

For $\rho_0 \neq 0$ and $\rho_i(x, t) = \rho_{i0}$, Eqs. (16) and (19) reduce to

$$j(t) = \frac{1}{2} \epsilon_1^2(t) + \rho_0, \quad (45)$$

$$d\epsilon(x(t), t)/dt = j(t) - \rho_0 \epsilon(x(t), t).$$

By the use of Eqs. (11) and (17) we obtain for the field at $x=1$:

$$d\epsilon_1(t)/dt = \frac{1}{2} \epsilon_1^2(t) + \rho_0 - \rho_1(t) \epsilon_1(t). \quad (46)$$

In zone I ($t \leq t_1$), $\rho_1(t) = \rho_0$, and we obtain by integration

$$\begin{aligned} \epsilon_1(t) &= \rho_0 - \alpha \cot(\frac{1}{2}\alpha t - \varphi), \quad \cos \varphi \equiv 1 - \rho_0, \\ &\quad \alpha \equiv \sin \varphi, \quad \rho_0 \leq 2; \\ &= \rho_0 - \beta \coth(\frac{1}{2}\beta t + \psi), \quad \cosh \psi \equiv \rho_0 - 1, \\ &\quad \beta \equiv \sinh \psi, \quad \rho_0 \geq 2. \end{aligned} \quad (47)$$

The flow line of the first front of injected carriers is given by

$$\begin{aligned} x^1(t) &= \rho_0 t - 2 \ln \left(\frac{\sin(\frac{1}{2}\alpha t - \varphi)}{-\sin \varphi} \right), \quad \rho_0 \leq 2; \\ &= \rho_0 t - 2 \ln \left(\frac{\sinh(\frac{1}{2}\beta t + \psi)}{\sinh \psi} \right), \quad \rho_0 \geq 2. \end{aligned} \quad (48)$$

The transit time is obtained by solving the transcendental equation $x^1(t_1) = 1$.

For zone II, Eq. (46) may be linearized by the transformation

$$\epsilon_1(t) = (du/dt)/u.$$

One obtains, using Eq. (23),

$$\frac{d^2 u}{dt^2} + \frac{\rho_0}{1 - \exp(-\rho_0 t)} \frac{du}{dt} + \frac{1}{2} \rho_0 u = 0. \quad (49)$$

This equation does not seem to have a solution in terms of elementary functions.

For the flow in zone III, a set of ordinary differential equations may be obtained as for the insulator case. This zone is again of little interest. The steady-state behavior (zone IV) can be obtained by analytic methods and has been treated by Lampert.⁷

IV. FLOW IN AN INSULATING CRYSTAL UNDER SLOW-TRAPPING

As has been shown in Sec. III(b), $j(t)$ and $\rho(t)$ reach their steady-state values in about $t = 2t_1$. In the presence of *slow* trapping ($\tau \gg t_1$) we have a quasi-stationary behavior, with $j(t)$ and $\rho(t)$ decaying slowly with time. For $t \gtrsim 2t_1$ we may accordingly neglect $\partial \epsilon(x, t)/\partial t$ and $\partial j_c(x, t)/\partial x$. To this approximation we have from Eqs. (9), (10), and (15):

$$\begin{aligned} \epsilon(x, t) &= \epsilon(x), \\ \rho(x, t) + \rho_i(x, t) &= f(x), \end{aligned} \quad (50)$$

$$j(t) = j_c(t) = \rho(x, t) \epsilon(x), \quad (51)$$

where $f(x)$ is a function of position only. It represents the free-charge distribution at $t \approx 2t_1$ [as given by Eq. (44)], before any appreciable trapping takes place. The solution of Eq. (12) is then:

$$\rho(x, t) = f(x) [(1 - \eta) \exp(-t/\tau') + \eta], \quad (52)$$

$$\rho_i(x, t) = g(t) \rho(x, t), \quad (53)$$

where

$$\begin{aligned} \eta &\equiv \theta_0 / (1 + \theta_0), \quad \tau' \equiv \tau / (1 + \theta_0), \\ g(t) &= [(1 - \eta) \exp(-t/\tau') + \eta]^{-1} - 1. \end{aligned} \quad (54)$$

Equation (11) can now be written as

$$d\epsilon(x)/dx = [1 + g(t)] \rho(x, t),$$

and thus [Eq. (9)]:

$$j(t) = \frac{1}{2} [1 + g(t)]^{-1} [d\epsilon^2(x)/dx]. \quad (55)$$

Integration with respect to x and the use of the condition $\int_0^1 \epsilon(x) dx = 1$ yields:

$$j(t) = (9/8) [(1 - \eta) \exp(-t/\tau') + \eta]. \quad (56)$$

It is thus seen that $j(t)$ decays exponentially from 9/8 to the steady-state value of $(9/8)\theta_0/(1+\theta_0)$. Although this approximation has been derived for $t \gtrsim 2t_1$, it may be extended to shorter times by taking into account the exact shape of $j_\infty(t)$ (the current in the absence of trapping) in this region as given by Eqs. (27) and (37). Equation (56) should then be replaced by

$$j(t) = j_\infty(t) - (9/8)(1 - \eta)[1 - \exp(-t/\tau')]. \quad (57)$$

For $t \gtrsim 2t_1$, $j_\infty(t) = 9/8$, and Eq. (57) coincides with (56). As will be shown in the next section, Eq. (57) is found to be a very good approximation for $t \geq t_1$ even for a trapping time as short as $\tau = 2$.

For the region $0 \leq t \leq t_1$ (zone I), one can estimate $j(t)$ by assuming that the field is not yet appreciably changed by trapping. It can then be shown that

$$j_\infty(t) - j(t) = (1/\tau) \{ 0.47t + 0.15t^2 + 0.22t^3 + 0.58[t(1 - \frac{1}{2}t)^{-1} + 2 \ln(1 - \frac{1}{2}t)] \}. \quad (58)$$

The approximations derived here are for one set of trapping levels. If several such sets are present, they are usually characterized by markedly different values of τ and θ_0 . Under these conditions the analysis can fairly easily be extended to cover this more general case.¹³

V. PRESENTATION OF ANALYTIC AND NUMERICAL SOLUTIONS

In this section, we shall present in graphical form some of the results of the preceding analysis together with numerical solutions for cases which could not be treated analytically.

Figure 2 shows flow lines for an insulator in the absence of trapping, and has been discussed previously. Figure 3 is a similar diagram, but for an insulator in the presence of trapping, with $\tau=0.5$ and $\theta_0=0$. The flow lines here have been computed numerically. The diagram is again divided into zones, with $\epsilon(0+)$ and t_a characterizing the flow lines in zones II and III,

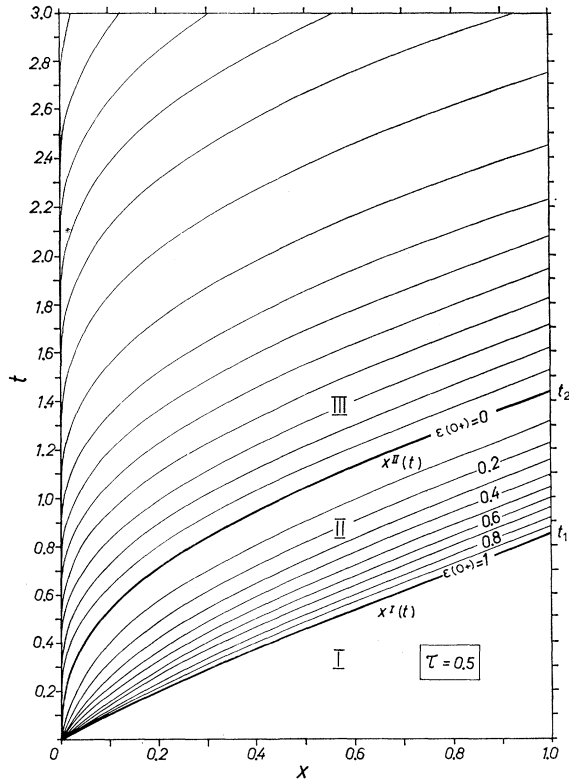


FIG. 3. Flow lines for an insulator in the presence of trapping, with $\tau=0.5$ and $\theta_0=0$.

respectively. In contrast to the case of no-trapping, the separation of the flow lines increases with time, until finally no flow lines leave the injecting electrode. This is obviously due to the accumulation of charge in traps which, for the case of $\theta_0=0$ being considered, eventually blocks the current completely. For $\theta_0>0$ (not shown here), the flow lines in the steady-state region (zone IV) would be equidistant as for the case of no trapping, but with a separation $(1/\theta_0)$ as large.

Figure 4 illustrates the injected charge-density distribution along an insulating crystal at two times following the onset of injection: $t=0.4$ (a) and $t=0.787$ (b). The dashed curves are for the trap-free case, and are calculated from Eqs. (28), (34), and (39). For $t=0.4$ the leading front reaches approximately half the length of the sample, while for $t=t_1=0.787$ it arrives at the opposite electrode. The solid curves are computed numerically and represent the free and trapped charge, in the presence of trapping, with $\tau=0.5$ and $\theta_0=0$. It is seen that the leading front slightly lags behind that for no trapping, and that due to

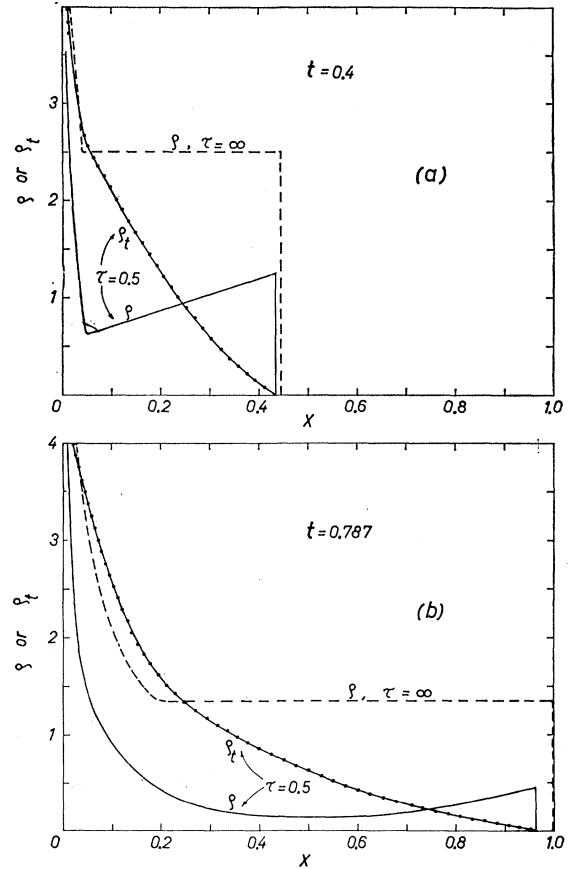


FIG. 4. Space-charge distribution along an insulating crystal at two instants following the onset of injection; (a) $t=0.4$, (b) $t=0.787$. The dashed curves are for a trap-free insulator. The solid curves correspond to the free and trapped charge-densities for an insulator with traps ($\tau=0.5$, $\theta_0=0$).

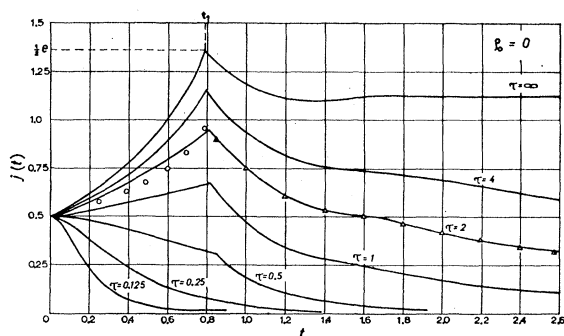


FIG. 5. The time dependence of the SCLC density for insulating crystals characterized by various trapping times, and $\theta_0 = 0$.

trapping more of the total charge is concentrated near the injecting electrode.

Figure 5 represents the SCLC density vs time for an insulating crystal characterized by various trapping times τ . The upper curve corresponds to the case of no trapping ($\tau = \infty$) and is given by Eqs. (27) and (37). Its general shape can be understood from the following physical considerations. At the onset of the injecting voltage pulse ($t=0$), a certain amount of charge $\rho(0+)$, flowing along the lines in zone II, is allowed to enter the crystal, and gives rise to the initial current density $j(0)$. It is easy to see why $j(0) = \frac{1}{2}\epsilon^2(1,0) = \frac{1}{2}$. The amount of space charge allowed to enter the crystal at $t=0$ is just that required to maintain $\epsilon(0,0)=0$ under the applied voltage-pulse. From Poisson's equation, we

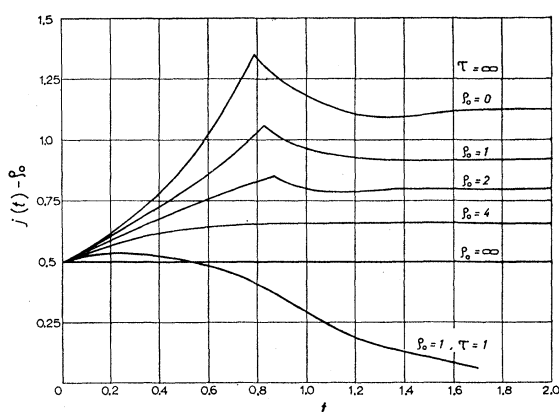


FIG. 6. The time dependence of the space-charge-limited excess current density in conducting crystals characterized by various equilibrium densities (ρ_0). All curves are for $\tau = \infty$ except the bottom curve for which $\rho_0 = 1$, $\tau = 1$, and $\theta_0 = 0$.

thus have $\rho(0+) = \epsilon(1,0) = 1$. Now the field $\epsilon(0+)$ acting on the carriers just after they have emerged from the interface, varies from 0 to 1, hence its average value is $\frac{1}{2}$. Thus $j(0) = \rho(0+)\frac{1}{2}\epsilon(1,0) = \frac{1}{2}$. For $t > 0$, more space-charge is being continually injected (along flow lines in zone III) and the current rises [Eq. (27)]. This process continues up to $t=t_1$, when the leading front reaches the opposite electrode ($x=1$). At this point the crystal contains the maximum amount of space charge and $j(t)$ attains its peak value. Thereafter $j(t)$ decays toward its steady-state value, as more space charge leaves the crystal than is being allowed to enter it, and the space-charge distribution at $t=t_1$ relaxes towards its steady-state configuration. The relaxation time is that associated with the conductance of the total injected charge and is therefore proportional to $1/V_0$, i.e., it is of the order of t_1 . Finally, one can also understand the slight dip in $j(t)$ below the asymptotic value, before the latter is attained. This is due to the fact that at $t=t_1$ the crystal contains more space charge than it can hold under steady-state conditions, since the conditions at $0 \leq t < t_1$ are the most favorable ones for injection. Hence, in the vicinity of $t=t_1$, correspondingly less space charge is allowed to be admitted into the crystal. This situation gives rise to an undershoot in $j(t)$ before it finally attains its asymptotic value. It is probable that the current exhibits an oscillatory behavior, but the amplitude is so strongly damped that only the first cycle is of significance.

The lower curves in Fig. 5 correspond to flow under trapping, with $\theta_0 = 0$, and have been computed numerically. The current decays faster the lower τ is, as is to be expected. At the same time, the transit time t_1 becomes slightly longer. This follows from the fact that for decreasing τ , increasingly more charge is trapped near $x=0$, giving rise (since $\int_0^1 \epsilon dx = \text{const} = 1$) to a correspondingly lower field $\epsilon(1,t)$ under which the leading front moves. The points (triangles) on the curve for $\tau=2$ at $t > t_1$ have been computed from the approximation given by Eq. (57), while the circles at $t < t_1$ have been obtained from Eq. (58). It is seen that the fit is quite good so that these approximations may be employed for $\tau \gtrsim 2$.

Finally, in Fig. (6) is plotted the excess current density $j(t) - \rho_0$ in a conducting crystal for various values of ρ_0 . The upper curves correspond to no-trapping. The portions of the curves up to t_1 (zone I) have been computed by use of Eqs. (45), (47), and (48), the rest numerically. The bottom curve is for flow under trapping, with $\rho_0 = 1$, $\tau = 1$ and $\theta_0 = 0$.