Perturbation Theory of Pion-Pion Interaction. II. Two-Pion Approximation

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On the basis of the model of a four-particle direct interaction without derivative coupling, relations are obtained between the various renormalized quantities related to pion-pion interaction. By considering only two-pion intermediate states but preserving crossing symmetry and unitarity up to the production threshold, closed systems of nonlinear integral equations are obtained from these relations to describe approximately the low-energy pion-pion scattering.

1. INTRODUCTION

SOME time ago, Chew and Mandelstam[1] pointed out that some knowledge of the pion-pion interaction is a necessary prerequisite to the further understanding of other strong interactions. One of the reasons is that, because of the small mass of the pion as compared with other strongly interacting particles, the pion-pion interaction may be adequately described by a ϕ^4 interaction, at least at low energies, and hence the masses of these other particles do not appear. Since then a great deal of work[2] has been done on the problem of the pion-pion scattering. Most of this work is based on two assumptions: the Mandelstam conjecture of double dispersion relations and the possibility of using the unitarity relation without four-particle intermediate states even above their production threshold. It seems now that both assumptions are of doubtful validity, and furthermore the desire to obtain a p-wave dominant solution makes it necessary to introduce undesirable additional parameters into the problem.

On the basis of the ϕ^4 coupling, it has been shown in a previous paper[3] (hereafter designated as I), that finite results can be obtained for the pion problem using perturbation theory. Since the coupling constant is not small, truncating the perturbation series at an early stage cannot be expected to give accurate answers. On the other hand, in connection with quantum statistical mechanics, Lee and Yang[4] have shown that sometimes, even if a perturbation series is term by term meaningless, results may be obtained by properly summing the series. It is, therefore, the purpose of this paper to try to re-sum the perturbation series as a possible alternative to the Chew-Mandelstam program.

In order to obtain a closed system of equations for the description of the pion-pion system, it is necessary to make some approximations. Thus, the difficulty as to which Feynman graphs should be taken into account is encountered. When there is a small parameter, as in the case of a dilute system of hard-sphere bosons treated by Lee and Yang,[4] the problem as to which graphs to include is well defined. Since this cannot be the case for problems of strong interactions, it is necessary to look for an alternative and unavoidably less satisfactory criterion; this problem is, of course, not new and has been encountered, for example, in the problem of the nuclear matter. For the present problem, the validity of the approximation used by Chew and Mandelstam, (i.e., for low-energy pion-pion scattering, only two-particle intermediate states are of importance) is assumed. So far as the author is aware, there does not seem to be any convincing argument for this approximation. [It can only be stated that if the mass of the intermediate state does not determine its importance, then it seems rather difficult even to begin to justify the neglect of π-K interaction, for example, in the treatment of π-π scattering. In other words, it becomes hard to see why pion-pion scattering is related to a Langrangian with a ϕ^4 coupling in the first place.]

It remains to be decided how two-particle intermediate states are to be taken into account. Since crossing symmetry is always exactly satisfied in perturbation theory, the aim is to select a subset of vertex graphs such that the lowest-order graph is included and that the sum of their renormalized contributions satisfies the unitarity relation up to the production threshold. Making use of the terminology introduced in I, the superproper vertex graphs do not give any two-particle intermediate states, and hence no superproper vertex graph need be included in this subset except the lowest-order one. Accordingly, the smallest subset that satisfies the above conditions consists of those vertex graphs that do not have any self-energy insertion, and that lead to, after repeated splitting as described in Step 2 of Sec. 5 of I, only this lowest-order superproper graph. For example, all the graphs shown in Fig. 17 of I are to be included except the last one. In order to see how the graphs can be explicitly chosen, the problem is first considered in Sec. 2 under the pretense that there is no divergence. Then in Secs. 4–5, analogous exact relations are derived for the renormalized theory, after a study in Sec. 3 of the integrability conditions for the vertex functions. Closed systems of integral equations can be obtained readily from these relations under simple approximate assumptions.

Unless stated otherwise, the equations in this paper,

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[2] See, for example, G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **5**, 580 (1960); K. A. Ter-Martirosyan, Soviet Phys.—JETP **12**, 575 (1961).
[3] T. T. Wu, Phys. Rev. **125**, 1436 (1962).
[4] T. D. Lee and C. N. Yang, Phys. Rev. **117**, 897 (1960).

like those of I, are exact in the sense of being formally true to every finite order of the coupling constant. However, since the meaning of the assumptions used to get the closed equations is not clear, the final closed system of equations can only be understood to determine the pion-pion scattering at low energies, at best approximately in a sense not too well defined. This is perhaps unavoidable in problems of strong interactions.

2. APPROXIMATION FOR THE UNRENORMALIZED THEORY

This section is to be devoted to a formal discussion of the Feynman graphs, ignoring the divergences due to the integrations. The notation of I is to be used.

According to the rule of differentiation given in Sec. 5 of I, each vertex graph is first split into superproper pieces. Therefore, loosely each graph may be considered to be the result of connecting superproper graphs together. The unrenormalized vertex Γ_0 must be known once there is sufficient information about superproper graphs. It is the first step here to find this connection. For this purpose, let

$$\Gamma_{s0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4) = \sum_{\text{Superproper } V} \mathfrak{F}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4;\Delta_F{}^0,-2,-i\lambda;G) \quad (1)$$

and

$$\Gamma_{t0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4) = \sum \mathfrak{F}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4;\Delta_F{}^0,-2,-i\lambda;G), \quad (2)$$

where the sum is over those non-superproper vertex graphs where the external lines 1 and 2 can be disconnected from 3 and 4 by removing two internal lines. Then,

$$\begin{aligned}\Gamma_0(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4) =& \Gamma_{s0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4) \\ &+\Gamma_{t0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4) \\ &+\Gamma_{t0}(k_1,I_1;k_3,I_3;k_2,I_2;k_4,I_4) \\ &+\Gamma_{t0}(k_1,I_1;k_4,I_4;k_2,I_2;k_3,I_3). \quad (3)\end{aligned}$$

It remains to express Γ_{t0} in terms of Γ_0 and Γ_{s0}. It is convenient to define

$$\begin{aligned}\Gamma_{v0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4) =& \tfrac{1}{2}\Gamma_{s0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4) \\ &+\Gamma_{t0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4). \quad (4)\end{aligned}$$

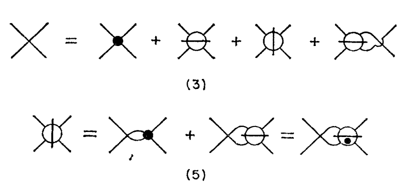

FIG. 1. Symbolic graphic representation of (3) and (5).

Consider the first step in splitting a vertex graph, where the result is an ordered sequence of vertex graphs. In the case where this splitting is nontrivial, consider the splitting to the extreme right. After these two internal lines are removed, what stands to the left can be any vertex graph, but what stands to the right must be either a superproper graph or a graph that can only be split in another direction. Accordingly, the following integral relation is formally valid:

$$\begin{aligned}\Gamma_{t0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4) =& \sum_{J_1,J_2}\int d^4p_1 d^4p_2\, \delta(p_1+p_2+k_3+k_4)\Delta_F(p_1{}^2)\Delta_F(p_2{}^2) \\ &\times\Gamma_0(k_1,I_1;k_2,I_2;-p_1,J_1;-p_2,J_2) \\ &\times\Gamma_{v0}(p_1,J_1;k_3,I_3;p_2,J_2;k_4,I_4). \quad (5)\end{aligned}$$

Equations (3)–(5) may be considered to give Γ_0 in terms of Γ_{s0}. For the sake of clarity, it is convenient to use a symbolic graphic representation for (5) and similar equations to appear later: denote Γ_0 by an ordinary intersection, Γ_{s0} by an intersection with a heavy dot, Γ_{t0} by a circle with a straight-line cut to separate 1 and 2 from 3 and 4, Γ_{v0} the same with an additional heavy dot, and Δ_F by a solid line. Thus, (3) and (5) are simply expressed as shown in Fig. 1.

So long as (3)–(5) are considered as equations for Γ_0 with Δ_F and Γ_{s0} given, Δ_F and Γ_{s0} may be considered to be arbitrary functions. Assume, however, that Δ_F is the sum of $\Delta_F{}^0$ together with a part whose spectrum function is zero below the three-pion threshold, then the solution of (3)–(5) satisfies the unitarity relation below the production threshold to every finite order of the unrenormalized coupling constant, no matter what superproper graphs are included in Γ_{s0}. The reason is that the subset of graphs taken into account by solving the integral equation has the property that if G_1 and G_2 are in the subset, then so is the graph shown in Fig. 2.

It, therefore, follows that crossing symmetry is satisfied exactly and unitarity up to the four-pion production threshold if (3)–(5) are solved with the following simple approximation:

$$\Delta_F \to \Delta_F{}^0$$

and

$$\Gamma_{s0} \to -i\lambda[\delta(I_1,I_2)\delta(I_3,I_4)+\delta(I_1,I_3)\delta(I_2,I_4)+\delta(I_1,I_4)\delta(I_2,I_3)]. \quad (6)$$

More explicitly, except for the complications arising from renormalization, the following equations give an approximate description of the pion-pion scattering at

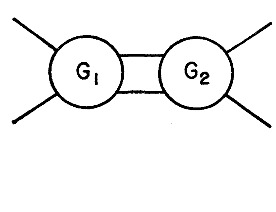

FIG. 2. A graph included in solving the integral equation.

low energies:

$$\begin{aligned}&\Gamma_0(k_1,I_1;\,k_2,I_2;\,k_3,I_3;\,k_4,I_4)\\&\quad=-i\lambda[\delta(I_1,I_2)\delta(I_3,I_4)+\delta(I_1,I_3)\delta(I_2,I_4)\\&\qquad+\delta(I_1,I_4)\delta(I_2,I_3)]+\Gamma_{t0}(k_1,I_1;\,k_2,I_2;\,k_3,I_3;\,k_4,I_4)\\&\qquad+\Gamma_{t0}(k_1,I_1;\,k_3,I_3;\,k_2,I_2;\,k_4,I_4)\\&\qquad\qquad+\Gamma_{t0}(k_1,I_1;\,k_4,I_4;\,k_2,I_2;\,k_3,I_3),\end{aligned}\tag{7}$$

and

$$\begin{aligned}&\Gamma_{t0}(k_1,I_1;\,k_2,I_2;\,k_3,I_3;\,k_4,I_4)\\&\quad=\sum_{J_1,J_2}\int d^4p_1d^4p_2\,\delta(p_1+p_2+k_3+k_4)(p_1^2+m^2-i\epsilon)^{-1}\\&\quad\times(p_2^2+m^2-i\epsilon)^{-1}\Gamma_0(k_1,I_1;\,k_2,I_2;\,-p_1,J_1;\,-p_2,J_2)\\&\quad\times\{-\tfrac{1}{2}i\lambda[\delta(J_1,J_2)\delta(I_3,I_4)+\delta(J_1,I_3)\delta(J_2,I_4)\\&\quad+\delta(J_1,I_4)\delta(J_2,I_3)]\\&\qquad\qquad+\Gamma_{t0}(p_1,J_1;\,k_3,I_3;\,p_2,J_2;\,k_4,I_4)\}.\end{aligned}\tag{8}$$

3. INTEGRABILITY

In I, it is remarked that integrability conditions need not be discussed so far as renormalization is concerned. However, in order to get a relation of the type (5) for the renormalized quantities, a discussion of the integrability conditions is required. Note that these conditions have not been considered before, because they are hard to satisfy *iteratively*, but here a definitive sum of graphs is to be considered noniteratively. In this section, these conditions on the vertex graphs are to be considered; the corresponding problem with the self-energy graphs seems more difficult, but fortunately need not be studied for the present purpose.

In Sec. 8 of I, renormalized contributions from individual Feynman graphs are defined. Let

$$\mathfrak{D}_\mu^{(ij)}=\frac{\partial}{\partial k_{i\mu}}-\frac{\partial}{\partial k_{j\mu}},\tag{9}$$

with $i,\ j=1,\ 2,\ 3,\ 4$;

$$\mathcal{G}'(k_1,k_2,k_3,k_4;\,G)=\sum_{G'}\mathcal{G}(k_1,k_2,k_3,k_4;\,G'),\tag{10}$$

where G is a V and the G' are obtained from G by $\mathfrak{D}$; and

$$\mathcal{G}_\mu^{(ij)}(k_1,k_2,k_3,k_4;\,G)=\sum_{G^{(ij)}}\mathcal{G}_\mu(k_1,k_2,k_3,k_4;\,G^{(ij)}),\tag{11}$$

where G is a V and the $G^{(ij)}$ are obtained from G by $\mathfrak{D}_\mu^{(ij)}$. Clearly,

$$\mathcal{G}'=\mathfrak{D}\mathcal{G},\tag{12}$$

and the question to be discussed is whether $\mathcal{G}_\mu^{(ij)}$ and $\mathfrak{D}_\mu^{(ij)}\mathcal{G}$ are the same or not.

If G is an irreducible vertex graph, then its unrenormalized contribution is logarithmically divergent. Since in this case the G' and $G^{(ij)}$ are all irreducible,

$$\mathcal{G}_\mu^{(ij)}=\mathfrak{D}_\mu^{(ij)}\mathcal{G}\tag{13}$$

for irreducible G. Next, assume that (13) holds also for all vertex graphs with at most $N-1$ four-vertices, and consider a reducible V with N four-vertices. Let $\mathfrak{D}$ be applied to the graphs $G^{(ij)}$, again using the rules given in Sec. 5 of I, and call the resulting graphs $H^{(ij)}$. Note that not all graphs $H^{(ij)}$ are admissible in the sense of I; but nevertheless, it is possible to define a value $\mathcal{G}_\mu(k_1,k_2,k_3,k_4;\,H^{(ij)})$ using SE, SE', V, and possibly one SE'' insertion. Let

$$\mathcal{G}_\mu'^{(ij)}(k_1,k_2,k_3,k_4;\,G)=\sum_{H^{(ij)}}\mathcal{G}_\mu(k_1,k_2,k_3,k_4;\,H^{(ij)}),\tag{14}$$

then it is clear by construction that

$$\mathfrak{D}\mathcal{G}_\mu^{(ij)}=\mathcal{G}_\mu'^{(ij)}.\tag{15}$$

Consider secondly $\mathfrak{D}_\mu^{(ij)}\mathcal{G}'$. There are a number of terms, due to the differentiation of the explicit weight factors, and the sum of these terms is $\mathfrak{D}_\mu^{(ij)}\mathcal{G}$, by the rule of momentum differentiation. The other terms come from the individual graphs G'; if (13) holds for all vertex graphs with less than N four-vertices, then these terms combine to give just $\mathcal{G}_\mu'^{(ij)}$. Thus,

$$\mathfrak{D}_\mu^{(ij)}\mathcal{G}'=\mathfrak{D}_\mu^{(ij)}\mathcal{G}+\mathcal{G}_\mu'^{(ij)},\tag{16}$$

or

$$\mathfrak{D}_\mu^{(ij)}\mathfrak{D}\mathcal{G}=\mathfrak{D}_\mu^{(ij)}\mathcal{G}+\mathfrak{D}\mathcal{G}_\mu^{(ij)}.\tag{17}$$

But

$$[\mathfrak{D}_\mu^{(ij)},\mathfrak{D}]=\mathfrak{D}_\mu^{(ij)};$$

thus

$$\mathfrak{D}[\mathfrak{D}_\mu^{(ij)}\mathcal{G}-\mathcal{G}_\mu^{(ij)}]=0,$$

or

$$\mathfrak{D}_\mu^{(ij)}\mathcal{G}-\mathcal{G}_\mu^{(ij)}=\text{const.}$$

But both terms on the left-hand side approach zero at infinity. So the constant must be zero and (13) holds for G. By induction, (13) always holds.

The conclusion is therefore reached that integrability conditions are satisfied by each vertex graph. In particular, each first derivative of the renormalized contribution of a V can be obtained correctly by applying the rule of momentum differentiation to the vertex first, and then computing directly from the resulting V' graphs. Note that the value of $\mathcal{G}$ has been computed from that of $\mathcal{G}'$, whereby a constant is introduced and then eliminated by the renormalization process. If $\mathcal{G}_\mu^{(ij)}$ had been used instead of $\mathcal{G}'$, then arbitrary functions would be introduced which were somewhat more complicated to eliminate. (See, however, Sec. 5.) This justifies the introduction of $\mathfrak{D}$ in I.

4. REDUCTION TO SUPERPROPER VERTEX GRAPHS

The manipulations in Sec. 2 are purely formal and strictly meaningless due to divergences. It is the next step to adapt the general idea to the renormalized theory; however, both the procedure and the result become more involved. The reason is that, since the

process of associating a function with a graph in the renormalized theory involves the detailed discussion of insertions, it is not possible to pick the splitting to the extreme right as done before. In this section, again through nonlinear integral equations, the renormalized vertex is to be expressed in terms of superproper graphs exactly in the sense used before.

In complete analogy with (1) and (2), define the following quantities again using the notations of I:

$$\bar{\Gamma}_{s0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4) = \sum_{\text{Superproper } V} \mathcal{G}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4;G) \quad (18)$$

and

$$\bar{\Gamma}_{t0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4) = \sum \mathcal{G}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4;G). \quad (19)$$

The same set of graphs is used in (19) as in (2). Equation (3) then holds for the renormalized quantities, and (4) with all Γ replaced by $\bar{\Gamma}$ may be used to define $\bar{\Gamma}_{v0}$. It again remains to express $\bar{\Gamma}_{t0}$ in terms of $\bar{\Gamma}_{s0}$, and this time it is necessary to analyze the V' graphs. The result is that after using (13), $\bar{\Gamma}_{t0}$ is determined by

$$\bar{\Gamma}_{t0}=0 \quad (20)$$

at the symmetry point, and

$$\begin{aligned}
&\mathfrak{D}\bar{\Gamma}_{t0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4)\\
&=\sum_{J_1,J_2}\int d^4p_1d^4p_2\,\delta(p_1+p_2+k_3+k_4)\bar{\Delta}_F(p_1^2)\sum_\mu \tfrac{1}{2}(k_{1\mu}+k_{2\mu})\frac{\partial}{\partial p_{2\mu}}\bar{\Delta}_F(p_2^2)\\
&\quad\times\bar{\Gamma}_0(k_1,I_1;k_2,I_2;-p_1,J_1;-p_2,J_2)\bar{\Gamma}_0(k_3,I_3;k_4,I_4;p_1,J_1;p_2,J_2)\\
&\quad+\sum_{J_1,J_2}\int d^4p_1d^4p_2\,\delta(p_1+p_2+k_3+k_4)\bar{\Delta}_F(p_1^2)\bar{\Delta}_F(p_2^2)\bar{\Gamma}_0(k_1,I_1;k_2,I_2;-p_1,J_1;-p_2,J_2)\\
&\quad\times\sum_\mu\left[k_{3\mu}\frac{\partial}{\partial k_{3\mu}}+k_{4\mu}\frac{\partial}{\partial k_{4\mu}}+\tfrac{1}{2}(k_{1\mu}+k_{2\mu})\left(\frac{\partial}{\partial p_{1\mu}}+\frac{\partial}{\partial p_{2\mu}}\right)\right]\bar{\Gamma}_{v0}(k_3,I_3;p_1,J_1;k_4,I_4;p_2,J_2)\\
&\quad+\sum_{J_1,J_2}\int d^4p_1d^4p_2\,\delta(p_1+p_2+k_1+k_2)\bar{\Delta}_F(p_1^2)\bar{\Delta}_F(p_2^2)\bar{\Gamma}_0(k_3,I_3;k_4,I_4;-p_1,J_1;-p_2,J_2)\\
&\quad\times\sum_\mu\left[k_{1\mu}\frac{\partial}{\partial k_{1\mu}}+k_{2\mu}\frac{\partial}{\partial k_{2\mu}}+\tfrac{1}{2}(k_{3\mu}+k_{4\mu})\left(\frac{\partial}{\partial p_{1\mu}}+\frac{\partial}{\partial p_{2\mu}}\right)\right]\bar{\Gamma}_{v0}(k_1,I_1;p_1,J_1;k_2,I_2;p_2,J_2)\\
&\quad+\tfrac{1}{2}\sum_{J_1,J_2,J_3,J_4}\int d^4p_1d^4p_2d^4p_3d^4p_4\,\delta(p_1+p_2-k_1-k_2)\delta(p_3+p_4-k_3-k_4)\bar{\Delta}_F(p_1^2)\bar{\Delta}_F(p_2^2)\bar{\Delta}_F(p_3^2)\bar{\Delta}(p_4^2)\\
&\quad\times\bar{\Gamma}_0(k_1,I_1;k_2,I_2;-p_1,J_1;-p_2,J_2)\bar{\Gamma}_0(k_3,I_3;k_4,I_4;-p_3,J_3;-p_4,J_4)\\
&\qquad\times\sum_\mu\tfrac{1}{2}(k_{1\mu}+k_{2\mu})\left(\frac{\partial}{\partial p_{1\mu}}+\frac{\partial}{\partial p_{2\mu}}-\frac{\partial}{\partial p_{3\mu}}-\frac{\partial}{\partial p_{4\mu}}\right)\bar{\Gamma}_{v0}(p_1,J_1;p_3,J_3;p_2,J_2;p_4,J_4). \quad (21)
\end{aligned}$$

The symbolic graphic representation introduced in the last section may be used here also with two modifications. The various portions of the graph now refer to the corresponding renormalized quantities, and a solid triangle is used to denote the appropriate momentum differentiations. Thus, for example, the rather complicated (21) is shown graphically in Fig. 3.

Equations (3), (4), (20), and (21) may be considered to give the renormalized vertex function $\bar{\Gamma}_0$ in terms of the renormalized propagator $\bar{\Delta}_F$ and the function $\bar{\Gamma}_{s0}$. The discussion preceding (6) then applies here also.

There is virtually no similarity between (21) for the renormalized theory and (5) for the unrenormalized theory. If the renormalized vertex function and the renormalized propagator both behave asymptotically the same as the bare vertex and propagator except possibly for logarithmic factors, then the right-hand side of (5) is logarithmically divergent, when the corresponding renormalized quantities are used. However, the following quantity is in general not finite and hence, meaningless:

$$\begin{aligned}
&\left[\sum_{J_1,J_2}\int d^4p_1d^4p_2\,\delta(p_1+p_2+k_3'+k_4')\bar{\Delta}_F(p_1^2)\bar{\Delta}_F(p_2^2)\right.\\
&\quad\times\bar{\Gamma}_0(k_1',I_1;k_2',I_2;-p_1,J_1;-p_2,J_2)\\
&\quad\left.\times\bar{\Gamma}_{v0}(p_1,J_1;k_3',I_3;p_2,J_2;k_4',\dot{I}_4)\right]\Bigg|_0^1, \quad (22)
\end{aligned}$$

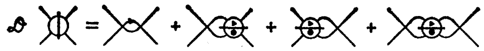

FIG. 3. Symbolic graphic representation of (21).

where 1 denotes the point $k_i'=k_i$ for all i and 0 the symmetry point. Indeed, the Salam[5] prescription of separating the divergent part from a Feynman graph with possibly overlap insertions gives a result that satisfies (21). Accordingly, the removal of divergences from the right-hand side of (5) is necessarily more complicated than that indicated by (22).

5. ALTERNATIVE INTEGRAL EQUATIONS

Although (22) is meaningless, it is still possible to obtain for the renormalized theory an integral equation similar to (5). This again makes use of the integrability condition in (13). Consider Feynman graphs for the quantity

$$\left(\frac{\partial}{\partial k_{3\mu}}-\frac{\partial}{\partial k_{4\mu}}\right)\mathcal{G}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4;G) \\ =\mathcal{G}_\mu{}^{(34)}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4;G), \quad (23)$$

where G is a vertex graph included in the sum of (2). Suppose the process of splitting a vertex graph discussed after (4) is applied to $G^{(34)}$. Then, what stands to the left is a vertex graph, and what stands to the right is a V' which is either superproper or can only be split in another direction. Accordingly, for the renormalized theory the following analog of (5) is valid:

$$\left(\frac{\partial}{\partial k_{3\mu}}-\frac{\partial}{\partial k_{4\mu}}\right)\bar{\Gamma}_{t0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4) \\ =\sum_{J_1,J_2}\int d^4p_1d^4p_2\,\delta(p_1+p_2+k_3+k_4)\bar{\Delta}_F(p_1^2)\bar{\Delta}_F(p_2^2) \\ \times\bar{\Gamma}_0(k_1,I_1;k_2,I_2;-p_1,J_1;p_2,J_2) \\ \times\left(\frac{\partial}{\partial k_{3\mu}}-\frac{\partial}{\partial k_{4\mu}}\right)\bar{\Gamma}_{v0}(p_1,J_1;k_3,I_3;p_2,J_2;k_4,I_4). \quad (24)$$

The complication arises from the boundary conditions. Since $\bar{\Gamma}_{t0}$ has the symmetry

$$\bar{\Gamma}_{t0}(1,2,3,4)=\bar{\Gamma}_{t0}(2,1,3,4)=\bar{\Gamma}_{t0}(3,4,1,2), \quad (25)$$

integration of the left-hand side of (24) leaves arbitrary a function of k_1+k_2. In order to avoid getting another complicated integral equation for this function of one variable, it is necessary to consider the asymptotic behavior of the vertex function. It is perhaps not unreasonable to assume that

$$\mathfrak{D}_\mu{}^{(ij)}\bar{\Gamma}_{t0}\to 0, \quad (26)$$

when some of the invariants approach infinity. If so, (24) and (25) may be used together with

$$\left(\frac{\partial}{\partial k_{1\mu}}+\frac{\partial}{\partial k_{2\mu}}-\frac{\partial}{\partial k_{3\mu}}-\frac{\partial}{\partial k_{4\mu}}\right) \\ \times\bar{\Gamma}_{t0}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4)\to 0 \quad (27)$$

for each fixed k_1+k_2 but with $k_1^2\to\infty$ and/or $k_3^2\to\infty$. This leaves one constant to be determined by (20).

The system of equations (3), (4), (20), and (21) is to be referred to as (A), while the system (3), (4), (20), (24), (25), and possibly (27) as (B). Both (A) and (B) are nonlinear integro-differential equations for the determination of $\bar{\Gamma}_0$ for given $\bar{\Delta}_F$ and $\bar{\Gamma}_{s0}$, and they are obtained by summing the same set of functions. However, the author does not know whether a solution of (A) is necessarily a solution of (B), and/or vice versa. On the one hand, since the series of renormalized contributions cannot be absolutely and uniformly convergent, two different methods of summing can in principle give different answers. On the other hand, if (A) and (B) do give different answers, there seems to be no criterion to prefer one over the other. If one takes the optimistic view that (A) and (B) do give the same solution, then the choice between them must be on the basis of convenience of numerical analysis.

6. INTEGRAL EQUATIONS IN TERMS OF INVARIANTS

In order to rewrite (A) and (B) in terms of the Lorentz invariants formed with the four-momenta, let

$$\bar{\Gamma}(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4) \\ =\bar{\Gamma}(k_1,I_1;k_2,I_2;k_3,I_3;k_4,I_4) \quad (28)$$

for various subscripts of $\bar{\Gamma}$. Here, for $i,\ j=1,\cdots 4$,

$$s_i=k_i^2 \quad\text{and}\quad s_{ij}=(k_i+k_j)^2; \quad (29)$$

and the seven s-variables are related by

$$s_1+s_2+s_3+s_4=s_{12}+s_{13}+s_{14}. \quad (30)$$

For this notation, it is convenient to replace p_i and J_i by k_{i+4} and I_{i+4}, respectively. Let $V(5,6)$ be the volume formed by k_1, k_3, k_5, and k_6; i.e.,

$$V(5,6)=\text{absolute value of}\begin{vmatrix}k_{10}&k_{30}&k_{50}&k_{60}\\k_{11}&k_{31}&k_{51}&k_{61}\\k_{12}&k_{32}&k_{52}&k_{62}\\k_{13}&k_{33}&k_{53}&k_{63}\end{vmatrix}. \quad (31)$$

In view of the δ functions in (21), the following convention is to be used:

$$k_1+k_2=-k_3-k_4=k_5+k_6=-k_7-k_8. \quad (32)$$

The s variables are to be defined with the proper signs; for example,

$$s_{15}=(k_1-k_5)^2 \quad\text{and}\quad s_{37}=(k_3-k_7)^2,$$

[5] A. Salam, Phys. Rev. **82**, 217 (1951).

but

$$s_{57}=(k_5+k_7)^2. \tag{33}$$

In terms of these variables, $V(5,6)$ may be expressed as follows:

$$V(5,6)=[-\bar{V}(5,6)]^{\frac{1}{2}}, \tag{34}$$

where

$$\bar{V}(5,6)=\frac{1}{64}\begin{vmatrix} s_{12} & s_1-s_2 & -s_3+s_4 & s_5-s_6 \\ s_1-s_2 & 2(s_1+s_2)-s_{12} & s_{13}-s_{14} & -s_{15}+s_{16} \\ -s_3+s_4 & s_{13}-s_{14} & 2(s_3+s_4)-s_{12} & s_{35}-s_{36} \\ s_5-s_6 & -s_{15}+s_{16} & s_{35}-s_{36} & 2(s_5+s_6)-s_{12} \end{vmatrix}. \tag{35}$$

More generally, the notation $V(i_1,i_2,i_3,i_4)$ is to be used to denote the volume formed by the k's corresponding to the four i's; e.g., $V(1,2,3,4)=0$ and $V(1,3,5,6)=V(5,6)$. Similar to (34), also let

$$\bar{V}(i_1,i_2,i_3,i_4)=-[V(i_1,i_2,i_3,i_4)]^2, \tag{36}$$

which can be expressed in terms of the s variables.

In terms of the s variables, (A) and (B) may be written as follows. The equations common to (A) and (B) are

$$\begin{aligned}\bar{\Gamma}_0(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4) &=\bar{\Gamma}_{s0}(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4)\\ &\quad+\bar{\Gamma}_{t0}(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4)\\ &\quad+\bar{\Gamma}_{t0}(s_1,s_3,s_2,s_4,s_{13},s_{12},s_{14};I_1,I_3,I_2,I_4)\\ &\quad+\bar{\Gamma}_{t0}(s_1,s_4,s_3,s_2,s_{14},s_{13},s_{12};I_1,I_4,I_3,I_2),\end{aligned} \tag{37}$$

$$\begin{aligned}\bar{\Gamma}_{v0}(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4) &=\tfrac{1}{2}\bar{\Gamma}_{s0}(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4)\\ &\quad+\bar{\Gamma}_{t0}(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4),\end{aligned} \tag{38}$$

$$\bar{\Gamma}_{t0}(-m^2,-m^2,-m^2,-m^2,-\tfrac{4}{3}m^2,-\tfrac{4}{3}m^2,-\tfrac{4}{3}m^2;I_1,I_2,I_3,I_4)=0, \tag{39}$$

and

$$\begin{aligned}\bar{\Gamma}_{t0}(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4) &=\bar{\Gamma}_{t0}(s_2,s_1,s_3,s_4,s_{12},s_{14},s_{13};I_2,I_1,I_3,I_4)\\ &=\bar{\Gamma}_{t0}(s_3,s_4,s_1,s_2,s_{12},s_{13},s_{14};I_3,I_4,I_1,I_2).\end{aligned} \tag{40}$$

In addition to (37)–(40), (A) consists of the following equation:

$$\begin{aligned}&2\left(s_1\frac{\partial}{\partial s_1}+s_2\frac{\partial}{\partial s_2}+s_3\frac{\partial}{\partial s_3}+s_4\frac{\partial}{\partial s_4}+s_{12}\frac{\partial}{\partial s_{12}}+s_{13}\frac{\partial}{\partial s_{13}}+s_{14}\frac{\partial}{\partial s_{14}}\right)\bar{\Gamma}_{t0}(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4)\\
&=\sum_{I_5,I_6}\int ds_5ds_6ds_{15}ds_{16}ds_{35}ds_{36}[V(5,6)]^{-1}\delta(s_1+s_2+s_5+s_6-s_{12}-s_{15}-s_{16})\delta(s_3+s_4+s_5+s_6-s_{12}-s_{35}-s_{36})\\
&\quad\times\bar{\Delta}_F(s_5)\tfrac{1}{2}(s_5-s_6-s_{12})\frac{\partial}{\partial s_6}\bar{\Delta}_F(s_6)\bar{\Gamma}_0(s_1,s_2,s_5,s_6,s_{12},s_{15},s_{16};I_1,I_2,I_5,I_6)\bar{\Gamma}_0(s_3,s_4,s_5,s_6,s_{12},s_{35},s_{36};I_3,I_4,I_5,I_6)\\
&\quad+\sum_{I_5,I_6}\int ds_5ds_6ds_{15}ds_{16}ds_{35}ds_{36}[V(5,6)]^{-1}\delta(s_1+s_2+s_5+s_6-s_{12}-s_{15}-s_{16})\delta(s_3+s_4+s_5+s_6-s_{12}-s_{35}-s_{36})\bar{\Delta}_F(s_5)\\
&\quad\times\bar{\Delta}_F(s_6)\bar{\Gamma}_0(s_1,s_2,s_5,s_6,s_{12},s_{15},s_{16};I_1,I_2,I_5,I_6)\left[2s_3\frac{\partial}{\partial s_3}+2s_4\frac{\partial}{\partial s_4}+\tfrac{1}{2}(s_{12}+s_5-s_6)\frac{\partial}{\partial s_5}+\tfrac{1}{2}(s_{12}-s_5+s_6)\frac{\partial}{\partial s_6}+2s_{12}\frac{\partial}{\partial s_{34}}\right.\\
&\quad\left.+(s_{35}+\tfrac{1}{2}s_3+\tfrac{1}{2}s_4-\tfrac{1}{2}s_5-\tfrac{1}{2}s_6)\frac{\partial}{\partial s_{35}}+(s_{36}+\tfrac{1}{2}s_3+\tfrac{1}{2}s_4-\tfrac{1}{2}s_5-\tfrac{1}{2}s_6)\frac{\partial}{\partial s_{36}}\right]\bar{\Gamma}_{v0}(s_3,s_5,s_4,s_6,s_{35},s_{12},s_{36};I_3,I_5,I_4,I_6)\\
&\quad+(1,2)\leftrightarrow(3,4)\\
&\quad+\tfrac{1}{2}\sum_{I_5,I_6,I_7,I_8}\int ds_7ds_8ds_{17}ds_{18}ds_{37}ds_{38}[V(1,3,7,8)]^{-1}\delta(s_1+s_2+s_7+s_8-s_{12}-s_{17}-s_{18})\\
&\quad\times\delta(s_3+s_4+s_7+s_8-s_{12}-s_{37}-s_{38})\int ds_5ds_6ds_{15}ds_{16}ds_{57}ds_{58}[V(1,5,6,7)]^{-1}\\
&\quad\times\delta(s_1+s_2+s_5+s_6-s_{12}-s_{15}-s_{16})\delta(s_5+s_6+s_7+s_8-s_{12}-s_{57}-s_{58})\bar{\Delta}_F(s_5)\bar{\Delta}_F(s_6)\bar{\Delta}_F(s_7)\bar{\Delta}_F(s_8)\\
&\quad\times\bar{\Gamma}_0(s_1,s_2,s_5,s_6,s_{12},s_{15},s_{16};I_1,I_2,I_5,I_6)\bar{\Gamma}_0(s_3,s_4,s_7,s_8,s_{12},s_{37},s_{38};I_3,I_4,I_7,I_8)\\
&\quad\times\left[\tfrac{1}{2}(s_{12}+s_5-s_6)\frac{\partial}{\partial s_5}+\tfrac{1}{2}(s_{12}-s_5+s_6)\frac{\partial}{\partial s_6}+\tfrac{1}{2}(s_{12}+s_7-s_8)\frac{\partial}{\partial s_7}+\tfrac{1}{2}(s_{12}-s_7+s_8)\frac{\partial}{\partial s_8}+2s_{12}\frac{\partial}{\partial s_{12}}\right]\\
&\quad\times\bar{\Gamma}_{v0}(s_5,s_7,s_6,s_8,s_{57},s_{12},s_{58};I_5,I_7,I_6,I_8),\end{aligned} \tag{41}$$

while (B) consists of the following equations

$$\left[\tfrac{1}{2}(s_{12}+s_1-s_2)\frac{\partial}{\partial s_1}+\tfrac{1}{2}(s_{12}-s_1+s_2)\frac{\partial}{\partial s_2}+\tfrac{1}{2}(s_{12}+s_3-s_4)\frac{\partial}{\partial s_3}+\tfrac{1}{2}(s_{12}-s_3+s_4)\frac{\partial}{\partial s_4}+2s_{12}\frac{\partial}{\partial s_{12}}\right]$$
$$\times\bar{\Gamma}_{t0}(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4)\rightarrow 0 \quad (42)$$

for fixed s_{12} but with $s_1\rightarrow\infty$ and/or $s_3\rightarrow\infty$, and

$$\mathbf{D}_i\bar{\Gamma}_{t0}(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4)$$
$$=\sum_{I_5,I_6}\int ds_5ds_6ds_{15}ds_{16}ds_{35}ds_{36}[V(5,6)]^{-1}\delta(s_1+s_2+s_5+s_6-s_{12}-s_{15}-s_{16})\delta(s_3+s_4+s_5+s_6-s_{12}-s_{35}-s_{36})$$
$$\times\bar{\Delta}_F(s_5)\bar{\Delta}_F(s_6)\bar{\Gamma}_0(s_1,s_2,s_5,s_6,s_{12},s_{15},s_{16};I_1,I_2,I_3,I_4)\mathbf{D}_i'\bar{\Gamma}_{v0}(s_3,s_5,s_4,s_6,s_{35},s_{12},s_{36};I_3,I_5,I_4,I_6), \quad (43)$$

where the three pairs $(\mathbf{D}_i,\mathbf{D}_i')$ are

$$\mathbf{D}_1=-(s_{12}+s_3-s_4)\frac{\partial}{\partial s_3}+(s_{12}-s_3+s_4)\frac{\partial}{\partial s_4}+(s_1-s_2-s_3+s_4)\frac{\partial}{\partial s_{13}}-(s_1-s_2+s_3-s_4)\frac{\partial}{\partial s_{14}}, \quad (44)$$

$$\mathbf{D}_1'=-(s_{12}+s_3-s_4)\frac{\partial}{\partial s_3}+(s_{12}-s_3+s_4)\frac{\partial}{\partial s_4}-(s_3-s_4-s_5+s_6)\frac{\partial}{\partial s_{35}}-(s_3-s_4+s_5-s_6)\frac{\partial}{\partial s_{36}}, \quad (45)$$

$$\mathbf{D}_2=(-s_1+s_2+s_{13}-s_{14})\frac{\partial}{\partial s_3}+(s_1-s_2+s_{13}-s_{14})\frac{\partial}{\partial s_4}+(2s_{13}+s_1+s_2-s_3-s_4)\frac{\partial}{\partial s_{13}}-(2s_{14}+s_1+s_2-s_3-s_4)\frac{\partial}{\partial s_{14}}, \quad (46)$$

$$\mathbf{D}_2'=(-s_1+s_2+s_{13}-s_{14})\frac{\partial}{\partial s_3}+(s_1-s_2+s_{13}-s_{14})\frac{\partial}{\partial s_4}+(s_{13}-s_{14}-s_{15}+s_{16})\frac{\partial}{\partial s_{35}}+(s_{13}-s_{14}+s_{15}-s_{16})\frac{\partial}{\partial s_{36}}, \quad (47)$$

$$\mathbf{D}_3=(3s_3+s_4-s_{12})\frac{\partial}{\partial s_3}+(s_3+3s_4-s_{12})\frac{\partial}{\partial s_4}+(2s_{13}-s_1-s_2+s_3+s_4)\frac{\partial}{\partial s_{13}}+(2s_{14}-s_1-s_2+s_3+s_4)\frac{\partial}{\partial s_{14}}, \quad (48)$$

and

$$\mathbf{D}_3'=(3s_3+s_4-s_{12})\frac{\partial}{\partial s_3}+(s_3+3s_4-s_{12})\frac{\partial}{\partial s_4}+(2s_{35}+s_3+s_4-s_5-s_6)\frac{\partial}{\partial s_{35}}+(2s_{36}+s_3+s_4-s_5-s_6)\frac{\partial}{\partial s_{36}}. \quad (49)$$

In (41) and (43), the limits of integration are given by the condition that all V's that appear explicitly in the integrand must be real.

The dependence on isotopic spin can be explicitly written as

$$\bar{\Gamma}_{s0}(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4)$$
$$=\bar{\Gamma}_s(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14})[\delta(I_1,I_2)\delta(I_3,I_4)+\delta(I_1,I_3)\delta(I_2,I_4)+\delta(I_1,I_4)\delta(I_2,I_3)], \quad (50)$$

and two functions are needed for $\bar{\Gamma}_{t0}$

$$\bar{\Gamma}_{t0}(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14};I_1,I_2,I_3,I_4)$$
$$=\bar{\Gamma}_t(s_1,s_2,s_3,s_4,s_{12},s_{13},s_{14})\delta(I_1,I_2)\delta(I_3,I_4)+\bar{\Gamma}_u(s_1,s_3,s_2,s_4,s_{13},s_{12},s_{14})\delta(I_1,I_3)\delta(I_2,I_4)+\bar{\Gamma}_u(s_1,s_4,s_3,s_2,s_{14},s_{13},s_{12})\delta(I_1,I_4)\delta(I_3,I_2). \quad (51)$$

Both (A) and (B) can be written as integral equations for $\bar{\Gamma}_t$ and $\bar{\Gamma}_u$.

Finally, the approximation that corresponds to (6) is

$$\Delta_F\rightarrow\Delta_F^0 \quad\text{and}\quad \bar{\Gamma}_s\rightarrow -i\bar{\lambda}. \quad (52)$$

7. DISCUSSIONS

Two systems of integral equations have been obtained to give the renormalized vertex function in terms of the renormalized propagator and the vertex function arising from the superproper graphs. Under the replacement (52), it is in principle possible to get the renormalized vertex function approximately for low energies. It is not known whether the systems (A) and (B) give the same answer or not. From the point of view of numerical computation, each has its own serious disadvantage: (A) is extremely complicated in the explicit form (41), which involves an eight-fold integration; on the other hand, in the system (B), use is made of the behavior at infinity through (42), and hence it is essential to have, for large arguments, quite accurate results which may be hard to get numerically. In any case, qualitative or semiquantitative properties of these equations would have to be studied in some detail before any direct numerical computation is attempted.

Under the replacement (52), each approximate system of integral equations contains only one param-

eter, namely the renormalized coupling constant. There are no further parameters such as cutoffs. Of course it is by no means simple to solve either system of equations, but, from the point of view of a theory that does not stay on the mass shell, it is hardly conceivable that simpler equations can be obtained, except possibly a replacement of (42). In particular, the nonlinearity of the equations is perhaps to be expected as a consequence of unitarity.

It goes without saying that, once the renormalized vertex function is found, a number of other quantities can be computed in terms of it. For example, the electromagnetic form factor of the pion satisfies linear integral equations somewhat similar to (41) and (43) with the renormalized vertex function appearing in the kernel, at least to the lowest order in the fine structure constant.

Except for the complications arising from the momentum differentiations, the reduction to superproper graphs is similar to the so-called "parquet" problem of Pomeranchuk *et al.*[6] Note, however, that the coupling constant in the consideration of Pomeranchuk *et al.* is bare, while here the renormalized coupling constant appears. Whether this leads to any connection between the low-energy and high-energy behaviors for pion-pion scattering is not understood by the author.

The problem may also be considered of the possibility of improving upon the approximation used here. Spurious inclusion of additional graphs is always possible, but to achieve any systematic improvement it seems necessary to include all four-particle intermediate states. In Fig. 23 of I all superproper graphs up to the seventh order are shown. It is possible to enumerate, for these graphs, the minimum number of particles in intermediate state, a quantity that is related to the asymptote of the Mandelstam spectrum function. If a dash is used to indicate the absence of an asymptote, then the results are: $a(4,4,4)$; $b(4,4,-)$; $c(4,6,6)$; $d(4,4,4)$; $e(4,4,6)$; $f(6,6,6)$; $g(4,4,6)$; $h(4,4,6)$; $i(4,6,6)$; $j(4,6,6)$; $k(4,4,-)$; $l(4,4,-)$; $m(4,4,6)$; $n(4,6,6,)$; and $o(4,4,4)$. From these examples, it is seen that extremely complicated superproper graphs must be considered even under the moderate desire to include all graphs that have four-particle intermediate states in each of the three channels. From this point of view, it seems rather difficult to improve systematically on the approximation used here.

It is an extremely interesting but difficult problem whether either system (A) or system (B) of the approximate equations can give any information about the renormalized coupling constant itself, i.e., whether a solution exists for every value of the coupling constant or not. This point invites a great deal of speculation. At the one extreme, a solution, or maybe even a number of solutions, exists for each value of the renormalized coupling constant. Then, the approximation must break down when there is a sufficiently tight binding between the pions, and the reason for the observed value of the coupling constant must be sought elsewhere. Alternatively, solutions may exist for certain ranges of values of the coupling constant. Then, it may be tempting to identify the maximum permissible value of the coupling constant with the one that actually occurs in nature, as postulated by Chew and Frautschi.[2] This is still not satisfactory because no argument for such a proposition is known. There is also the possibility that a solution, or solutions, can exist for only one value, or maybe for a finite number of values, of the renormalized coupling constant different from zero. This would be most satisfactory, but there is at present no basis whatever for this hope. At the other extreme, there is also the possible disaster that neither (A) nor (B) can have any solution at all for nonzero values of $\bar{\lambda}$. This is possible because these approximate integral equations seem to imply a rather complicated asymptotic behavior for $\bar{\Gamma}$, which may even be incompatible with (42). In this case, it is natural to question the physical relevance of the present model and also the validity of the approximation of considering two-particle intermediate states only.

One may even speculate further and ask about the place of the recently discovered resonances in the present scheme. If the approximation is valid up to the rest mass of the ϕ meson ($\sim$420 Mev) or perhaps even up to the rest mass of the better established ζ meson ($\sim$575 Mev),[7] then presumably $\bar{\Gamma}_t$ should take on relatively large values at these masses. If this point of view is correct, then it is readily explained why, in the calculation of Zachariasen,[8] crossing symmetry should be neglected.

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