predict the relative helicity of the protons in the Σ^+ and Λ decays, which depends on the sign $(g_{K\Sigma}/g_{K\Lambda})$ in the case considered by them in detail. In our model, on the other hand, the same helicity for the proton is almost definitely ruled out if the relative Σ-Λ parity is even, since a fit requires $g_\Sigma=2g_N$, $g_\Lambda=\pm 2g_N$ for $\Sigma^+\to n\pi^+$ going in s wave, and $g_\Sigma{}^2=(8.5\pm 2.9)g_N{}^2$, $g_\Lambda{}^2=(g_\Sigma+2g_N)^2$, for $\Sigma^+\to n\pi^+$ going in p wave. The choice is more difficult in the case of odd Σ-Λ parity, since the values of the coupling constants turn out to be practically the same as those which give rise to opposite proton helicities in the two decays.

Wolfenstein's model assumes that K decay is the more fundamental decay and that Σ and Λ decays are secondary. He, therefore, neglects baryon poles completely, but has to include two-particle intermediate states. His model, like that of Feldman *et al.*, also has (KYN) vertices, and his prediction regarding the angular-momentum states involved in $\Sigma^\pm$ decays into a neutron depends on the (KYN) and $(\Sigma\Lambda)$ parities assumed. Further, while in our model the fact that $\Sigma^+\to n\pi^+$ goes into s wave and $\Sigma^-\to n\pi^-$ into p wave is due to a dynamical cancellation between various diagrams, in the model of Wolfenstein, the Σ^+ goes into s wave only because a certain parity is assumed for the K meson and for $(\Sigma\Lambda)$, so that only a single diagram (K-pole diagram) contributes to it.[19]

We have already remarked about our omission of the K-pole diagrams. Apart from the fact that their inclusion would have increased the number of parameters in the problem, we were encouraged to neglect them by the frequently expressed conjecture that the K couplings are weak compared to the π couplings. It is, therefore, interesting that we are able to fit the experimental data without the inclusion of these diagrams. We have also omitted diagrams involving more than one-particle intermediate states, which would have to be included in a complete S-matrix approach. The lowest mass two-particle diagram has a pion and a nucleon in the intermediate state. Because the $J=\frac{1}{2}$ πN interaction is known to be small at the relevant energies, the contribution of this diagram may be expected to be small. We expect the πY intermediate-state diagrams to make an even smaller contribution since there is no strong $J=\frac{1}{2}$ interaction of the πY system either.[20]

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[19] Wolfenstein's expressions could be used to evaluate g_Σ/g_Λ from his model in the way we have done. In the case considered by him, if one assumes $x_0=-x_\Lambda$ and $|M_\Lambda|=|M_0|$, one is led to $g_\Lambda=g_\Sigma$ in his model too. It is curious that with both, our model which includes only baryon poles and excludes K poles, and Wolfenstein's model which excludes baryon poles, one is led to a global-symmetric solution.

[20] Note that the only charged Y^* known has a spin $>\frac{3}{2}$; see Robert P. Ely, Sun-Yiu Fung, George Gidal, Yu-Li Pan, Wilson M. Powell, and Howard S. White, Phys. Rev. Letters **7**, 461 (1961).

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N/D Method*

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A rigorous mathematical discussion of the N/D method is presented. It is shown in particular that we can always make sure that the N/D solution is the correct one in the sense that the D function has no redundant zeros, by examining the high-energy behavior of the phase shift in the solution obtained. It is also shown how the N/D method exhausts all the possible solutions of the original equation systematically according to the high-energy behavior of the phase shift. Virtually, the same arguments are shown to be applicable to the inverse method. It is, however, pointed out that the N/D method is preferable to the inverse method for both technical and physical reasons.

1. INTRODUCTION AND THE STATEMENT OF THE PROBLEM

THE N/D method[1] has been widely used in the dispersion theoretic approach to the scattering problem; the partial-wave scattering amplitude $F(z)$ (z stands for the complex c.m. energy) is represented as

$$F(z)=N(z)/D(z), \tag{1}$$

where $N(z)$ and $D(z)$ are individually analytic everywhere except for certain poles and cuts, and the coupled equations for $N(z)$ and $D(z)$ are solved. The inverse method,[2] which deals with the equation for the inverse amplitude has also been used.

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[1] See, for example, J. S. Ball and D. Y. Wong, Phys. Rev. Letters **7**, 390 (1961), which quotes virtually all other references.

[2] B. H. Bransden and J. W. Moffat, Phys. Rev. Letters **6**, 708 (1961) **8**, 145 (1962).

The major difficulty common in all the previous works[1,2] is that the equivalence of the equations for $N(z)$ and $D(z)$ and/or the inverse amplitude to the original equation has never been fully established; it even happened that the solutions obtained did not satisfy the original equation.[1,3] This is entirely due to the ambiguous mathematics employed, especially in locating zeros of $D(z)$. These zeros, though completely hidden in the dispersion relation, give rise to poles of $F(z)$, as is evident from (1), unless $N(z)$ happens to have zeros at the positions of these poles. Therefore, the N/D solution can be considered correct only after we locate all the zeros of $D(z)$. The same remark applies also to the inverse method. Because of the inability to locate the zeros of $D(z)$, no one up to now has been sure of the correctness of his solution.

It is, therefore, essential for the sake of completeness of the N/D method to make sure that $D(z)$ has no redundant zeros. This is also of practical importance because the recent evidence[1] is that even a far-away pole of $F(z)$ has a major effect on the low-energy behavior of the phase shift.

The purpose of the present paper is to show that one can really make sure rigorously that $D(z)$ has no redundant zeros by examining the high-energy behavior of the phase shift in the solution obtained. The argument is based upon the mathematical properties of $D(z)$ which are analyzed in Sec. 2 and in Appendixes 1 and 2. The specific recipe for checking the correctness of the solution is given in Sec. 3.

The N/D method outlined in Sec. 3 is not only exact in the above sense but also complete in the sense that all the possible solutions for $F(z)$ can be exhausted systematically according to the high-energy behavior of the phase shift. Therefore, the N/D method presented in Sec. 3 becomes ideal from the point of view of selecting the physically meaningful solution out of many[4] (possibly infinitely many) solutions for $F(z)$, on the basis of the high-energy behavior of the phase shift.[5]

The inverse method is discussed in Sec. 4. We show that the inverse method can be analyzed in quite the same manner. It is shown, however, that the inverse solutions are characterized by a parameter which is not convenient technically and whose physical significance is not clear.

2. MATHEMATICAL PROPERTIES OF D FUNCTIONS

To proceed further, we have to state a few of the assumptions about $F(z)$: $F(z)$ is analytic except for poles in a cut z plane with the positive cut (x_0 to ∞) and the negative cut ($-\infty$ to y_0) both on the real axis. Our conventions are that x and y are both real and $\infty > x \geqslant x_0$ and $-\infty < y \leqslant y_0$, and $F(x)$, etc. stand for $F(z\rightarrow x+i\epsilon)$, etc. where ϵ is an infinitesimal positive real number. Then $F(x)$ must assume an expression

$$F(x)=g(x)\sin\delta(x)e^{i\delta(x)}, \tag{2}$$

where $g(x)$ is a known, real function and positive-definite, and $\delta(x)$ is the real, partial-wave phase shift. We normalize $\delta(x)$ such that $\delta(x_0)=0$. We assume that $\delta(x)$ is finite, smooth[6] and has a limit $\delta(\infty)$ as $x\rightarrow\infty$. We assume further that $F(z)$ is real in the sense that

$$F^*(z)=F(z^*), \tag{3}$$

and that $F(z)$ satisfies a dispersion relation. We do not have to know the exact number of subtractions, because such an analytic function can always be made at most finite at infinity by dividing it by some real polynomial of z,[7] which affects only $g(x)$ and the poles of $F(z)$ the details of which are completely irrelevant for our present discussion.

We now state the exact definition of $N(z)$ and $D(z)$ in (1). These are defined as individually real in the sense of (3) and satisfying dispersion relations with, however, only one cut-integral; the x cut is assigned to $D(z)$ and $N(z)$ can have only the y cut. We do not usually specify the poles of $N(z)$ and $D(z)$.

According to Chew and Mandelstam,[8] an $N_0(z)$ and $D_0(z)$ set which meets all the requirements is given by

$$D_0(z)=\exp\left[-\frac{z}{\pi}\int_{x_0}^{\infty}\frac{\delta(x)dx}{x(x-z)}\right], \tag{4}$$

and $N_0(z)=F(z)D_0(z)$. We can confirm this statement by observing that all these functions are individually real and analytic and $D_0(x)=|D_0(x)|e^{-i\delta(x)}$. It is important to observe that $D_0(z)$, which is usually called the Omnes D function, has no zeros and no poles anywhere except at infinity since $\delta(x)$ is normalized such that $\delta(x_0)=0$ and is assumed as finite and smooth.[9]

Let $D(z)$ be a general D function for the same $F(z)$. The function $D(z)/D_0(z)$ is real and analytic everywhere except for poles and the x cut. However, we see that both $D(z)$ and $D_0(z)$ have the same phase on the x cut, implying that $D(z)/D_0(z)$ must be analytic except for poles. Thus, if we introduce two real polynomials of z, $P_1(z)$ and $P_2(z)$, with orders of n_1 and n_2, respectively, the most general $D(z)$ must appear as

$$D(z)=D_0(z)P_1(z)/P_2(z). \tag{5}$$

We see that $D(z)$ has n_1 zeros and n_2 poles except the one at infinity due to $D_0(z)$.

[3] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).

[4] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956); referred to as CDD.

[5] G. F. Chew and S. C. Frautschi, Phys. Rev. **124**, 264 (1961).

[6] We can show that continuity alone is sufficient for the present purpose. We assume smoothness to avoid mathematical complications. See reference 9.

[7] M. Sugawara and A. Kanazawa, Phys. Rev. **123**, 1895 (1961).

[8] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[9] We can show that continuity alone is sufficient to assure that $D_0(z)$ has no zeros or poles anywhere except at infinity. See reference 6.

According to (5), the behavior of $D(z)$ at infinity is essentially determined by that of $D_0(z)$. In order to examine this behavior, we look at the exponent of (4), which can be written as

$$-\frac{z}{\pi}\int_{x_0}^{\infty}\frac{\delta(x)dx}{x(x-z)}=\frac{\delta(\infty)}{\pi}\int_{x_0}^{\infty}\left[\frac{1}{x}-\frac{1}{x-z}\right]dx+\frac{1}{\pi}\int_{x_0}^{\infty}\left[\frac{1}{x}-\frac{1}{x-z}\right][\delta(x)-\delta(\infty)]dx. \quad (6)$$

The first term can be directly evaluated and becomes $[\delta(\infty)/\pi]\ln z$ except for terms which remain at most finite at infinity. Aside from terms going to zero at infinity, the second term can be put equal to

$$\frac{1}{\pi}\int_{x_0}^{\infty}\left[\frac{1}{x}-\frac{1}{x+z}\right][\delta(x)-\delta(\infty)]dx, \quad (7)$$

according to the theorem proved in the Appendix of reference 7.

In the following, we let z be real and approach positive real infinity (denoted by $z\to\infty$). This is sufficient not only for determining the number of necessary subtractions in the dispersion relation[7] for $D(z)$, but also for the purpose of making sure that $D(z)$ has no zeros. According to Appendix 1, the integral (7) remains finite as $z\to\infty$ and, therefore,

$$D_0(z)\underset{z\to\infty}{\longrightarrow}\propto z^{\delta(\infty)/\pi}, \quad (8)$$

if and only if $\delta(x)$ satisfies the condition that

$$\frac{1}{\pi}\int_{x_0}^{\infty}\frac{\delta(x)-\delta(\infty)}{x}dx \text{ is convergent.} \quad (9)$$

We further show in Appendix 2 that if (9) is not imposed, the divergence of $D_0(z)$ may be smeared out but only as much as implied by the following notation:

$$D_0(z)\underset{z\to\infty}{\longrightarrow}\propto\langle z^{\delta(\infty)/\pi}\rangle, \quad (10)$$

where $\langle z^\alpha\rangle$ is defined as

$$\lim_{z\to\infty}\frac{\langle z^\alpha\rangle}{z^{\alpha+\epsilon}}=0,\quad \lim_{z\to\infty}\frac{\langle z^\alpha\rangle}{z^{\alpha-\epsilon}}=\infty,\quad \lim_{z\to\infty}\frac{\langle z^\alpha\rangle}{z^\alpha}=\text{anything including }\infty, \quad (11)$$

for ϵ which is an arbitrarily small (but nonzero) positive number. In other words, $\langle z^\alpha\rangle$ may be different from z^α by factors like $\ln z$ or $1/\ln z$, but not so much as by factors $z^{\pm\epsilon}$.

The behavior (8) was obtained by Chew and Frautschi,[5] who, however, failed to realize the condition (9). This condition implies that the approach of $\delta(x)$ to $\delta(\infty)$ cannot be so slow as, say, $1/\ln x$ but the approach as $1/(\ln x)^2$ is already within the permissible limit.[10]

In summary, the most general $D(z)$ behaves as $z\to\infty$ as

$$D(z)\underset{z\to\infty}{\longrightarrow}\propto\langle z^{\delta(\infty)/\pi}\rangle z^{n_1}z^{-n_2}, \quad (12)$$

where $\delta(\infty)$ is the high-energy limit of $\delta(x)$, and n_1 and n_2 are, respectively, the numbers of zeros and poles of $D(z)$. If the high-energy behavior of $\delta(x)$ satisfies (9), $\langle z^\alpha\rangle$ is nothing but z^α, but otherwise is smeared out by as much as defined by (11).

3. THE *N/D* METHOD WITHOUT ZEROS IN *D* FUNCTIONS

We can always make manifest in the dispersion relation, the number of poles, and the divergence at infinity. In the N/D method presented here, $D(z)$ always assumes no poles and behaves at infinity as

$$D(z)/z^n\underset{z\to\infty}{\longrightarrow}0, \quad (13)$$

where $n=0, 1, 2, \cdots$. This obviously does not exclude any solutions for $F(z)$. We presuppose that $D(z)$ has no zeros (this will be checked later), and we assign all poles of $F(z)$ to $N(z)$. According to the argument after (3), $F(z)$ can be assumed to remain at most finite at infinity. Therefore, $N(z)$ must also satisfy (13). We are thus led[7] to the following set of dispersion relations:

$$\begin{aligned}N(z)&=N_0+N_1z+\cdots+N_{n-1}z^{n-1}+\text{“poles”}+\frac{z^n}{\pi}\int_{-\infty}^{y_0}\frac{\operatorname{Im}N(y)dy}{y^n(y-z)},\\ D(z)&=D_0+D_1z+\cdots+D_{n-1}z^{n-1}+\frac{z^n}{\pi}\int_{x_0}^{\infty}\frac{\operatorname{Im}D(x)dx}{x^n(x-z)},\end{aligned} \quad (14)$$

where $2n$ subtraction terms all have real coefficients and "poles" stands for a group of pole-terms with known structure. The set of Eqs. (14) with $n=0, 1, 2, \cdots$ is completely equivalent to the original equation for $F(z)$ if and only if $D(z)$ has no zeros.

Since $D(z)$ in (14) has no poles, we are here bound, according to (12) and (13), to such solutions in which $\delta(\infty)$ and n_1 satisfy

$$\langle z^{\delta(\infty)/\pi}\rangle z^{n_1}/z^n\underset{z\to\infty}{\longrightarrow}0. \quad (15)$$

We see immediately that n_1 must be zero [or $D(z)$ has no zeros] if, for example, $\delta(\infty)>(n-1)\pi$. After careful

[10] The condition (9) is assumed also tacitly by R. Omnes, Nuovo cimento **21**, 525 (1961).

examination of various possibilities, we find that we are assured of no zeros in $D(z)$ except for one exceptional case if we make sure that the phase shift of the solution tends to a limit $\delta(\infty)\geqslant(n-1)\pi$. The exceptional case concerns the solution for which $\delta(\infty)=(n-1)\pi$, the condition (9) is not satisfied, and $\delta(x)-\delta(\infty)$ approaches zero as $x\rightarrow\infty$ from the negative side.[11] In case of this odd solution, both $n_1=0$ and 1 are permissible, since $\langle z^{\delta(\infty)/\pi}\rangle$ in (15) does diverge weaker than z^{n-1}.

This odd solution, however, does not cause any difficulty. We have only to observe that this odd solution and all the solutions with $\delta(\infty)<(n-1)\pi$ do correspond to $D(z)$ which diverges as (13) only with $n-1$ instead of n if $D(z)$ has really no zeros. Therefore, this odd solution is the correct one if it remains a solution of the new set of equations (14) with n replaced by $n-1$. Besides, we can make sure, in this way, of the correctness of all solutions with $(n-1)\pi>\delta(\infty)\geqslant(n-2)\pi$, with the exception of the odd solution in which $\delta(\infty)=(n-2)\pi$. We thus can check the correctness of all the solutions with $\delta(\infty)\geqslant-\pi$, with the exception of the odd solution with $\delta(\infty)=-\pi$.

We can apply the same criterion to those solutions with $\delta(\infty)<-\pi$ and the last odd solution, if we make the additional check on the solution that $z^mD(z)\rightarrow_{z\rightarrow\infty}0$ with $m=1,2,3,\cdots$. If we make sure that $z^mD(z)\rightarrow_{z\rightarrow\infty}0$, then all solutions with $-m\pi>\delta(\infty)\geqslant-(m+1)\pi$ and the odd solution with $\delta(\infty)=-m\pi$ are correct with the exception of the odd solution with $\delta(\infty)=-(m+1)\pi$.

We now realize that the N/D method presented here is not only exact in the sense that we can always make sure that the solutions are all correct, but also this N/D method exhausts all the possible solutions of $F(z)$ systematically according to the high-energy behavior of the phase shift. The set of equations (14) gives us the odd solution with $\delta(\infty)=n\pi$ and all the solutions whose $\delta(\infty)$ fall between the limits

$$n\pi>\delta(\infty)\geqslant(n-1)\pi, \tag{16}$$

with the exception of the odd solution with $\delta(\infty)=(n-1)\pi$. There is a complication due to occurrence of the odd solution which is caused by the behavior (10) other than (8). There is, however, no ambiguity or difficulty.

This is exactly what was suggested by Chew and Frautschi,[5] but now it is expressed in an exact and complete way.

It is also seen that those solutions with CDD poles[4] in the usual term are those solutions of (14) with higher n. Instead of poles and residues of these poles, Eq. (14) contains N_i's and D_i's which become in general adjustable parameters. They are identified as the genuine parameters if the solutions remain correct in the above sense under variation with respect to these N_i's and D_i's.

[11] This last condition is not stated in the most general form. More precisely, it should be that the integral in (9) diverges to negative infinity.

4. THE INVERSE METHOD

The inverse method can be analyzed similarly if we introduce the phase shift $\delta(y)$ along the y cut by putting

$$F(y)=f(y)e^{i\delta(y)}, \tag{17}$$

where both $f(y)$ and $\delta(y)$ are real; $\delta(y)$ is normalized as $\delta(y_0)=0$; and $\delta(y)$ is finite, smooth and has a limit $\delta(-\infty)$ as $y\rightarrow-\infty$.

We then put $F(z)=P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are individually real and analytic, however, allowing no branch cuts for $P(z)$ and both x and y cuts for $Q(z)$. We can always construct a set $P_0(z)/Q_0(z)$ which meets these requirements by putting

$$Q_0(z)=\exp\left[-\frac{z}{\pi}\int_{x_0}^{\infty}\frac{\delta(x)dx}{x(x-z)}-\frac{z}{\pi}\int_{-\infty}^{y_0}\frac{\delta(y)dy}{y(y-z)}\right] \tag{18}$$

and $P_0(z)=F(z)Q_0(z)$. Since $P_0(z)$ must be a real, rational function of z, $[F(z)]^{-1}$ can be written as

$$[F(z)]^{-1}=Q_0(z)P_1(z)/P_2(z), \tag{19}$$

where $P_1(z)$ and $P_2(z)$ are two real polynomials of z with orders n_1 and n_2, respectively. Because $Q_0(z)$ defined by (18) has no zeros and no poles anywhere except at infinity, n_1 and n_2 must be the numbers of zeros and poles of $[F(z)]^{-1}$, respectively.

In the inverse method, we know the correct number and location of the zeros since they are the known poles of $F(z)$. It is, therefore, technically advisable to multiply $F(z)$ by a known real polynomial of z so that the modified $F(z)$ has no longer poles. This modification does not affect the previous argument since the phases are not affected. If our $F(z)$ is already modified this way, we have only to make sure that $[F(z)]^{-1}$ has no zeros.

We first observe that $[F(z)]^{-1}$ behaves as $z\rightarrow\infty$, according to (19), as

$$[F(z)]^{-1}\underset{z\rightarrow\infty}{\longrightarrow}\propto\langle z^{[\delta(\infty)-\delta(-\infty)]/\pi}\rangle z^{n_1}z^{-n_2}. \tag{20}$$

We also recall that the over-all divergence at infinity and n_2 are made manifest in the dispersion relation for $[F(z)]^{-1}$. We are, therefore, assured of no zeros (or $n_1=0$) by making sure that $\delta(\infty)-\delta(-\infty)$ falls in the correct region, except the minor complication due to occurrence of the odd solution.

Thus, the inverse method can also be developed in a way that is as rigorous and also as complete as the N/D method presented in the previous section. It is seen, however, that the inverse solutions are not classified in any simple manner by the high-energy behavior of the phase shift: They are characterized rather by the difference in the two limits of the phase shifts, or more suitably by the total number of zeros of $F(z)$ in the entire z plane; this is not a useful parameter, not only because the physical significance is obscure but also since the amplitude is likely to have an arbitrary

number of zeros even along the x cut. Therefore, the N/D method is preferable to the inverse method from both physical and technical grounds.

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APPENDIX 1

We prove here that the integral (7) has a finite limit as $z \to \infty$ if and only if the integral in (9) converges.

If the limit of (7) as $z \to \infty$ is finite, then this limit must be equal to the integral in (9), according to the theorem quoted by Amati *et al.*[12] This proves that (9) is the necessary condition.

We can see that (9) is also sufficient if we observe the following lemma: Suppose $h(x)$ defined by

$$h(x)=\int_b^\infty f(x,y)dy \tag{21}$$

converges for all $\infty > x \geqslant a$, where $f(x,y)$ is monotonic with respect to x for all $\infty > y \geqslant b$ and has a limit $f(\infty,y)$ as $x \to \infty$, the integral of which with respect to y also converges. Then

$$\lim_{x\to\infty} h(x)=\int_b^\infty f(\infty,y)dy. \tag{22}$$

The proof of this lemma is as follows: Define $h'(x)$ by

$$h'(x)=\int_b^\infty [f(x,y)-f(\infty,y)]dy, \tag{23}$$

which obviously converges for all $\infty > x \geqslant a$. We prove below that the limit of $h'(x)$ as $x \to \infty$ is zero, which proves the lemma.

Since the integral (23) converges, we can choose $Y(x)$, a finite but sufficiently large number which can depend upon x, such that the magnitude of the second term of

$$h'(x)=\int_b^{Y(x)} [f(x,y)-f(\infty,y)]dy + \int_{Y(x)}^\infty [f(x,y)-f(\infty,y)]dy \tag{24}$$

does not exceed ϵ, an arbitrarily small positive number. Since $f(x,y)-f(\infty,y)$ approaches zero as $x \to \infty$ monotonically, the division in (24) can be made x independent by choosing $Y \equiv Y(a)$ as the boundary. Because of the fact that $f(\infty,y)$ is the limit of $f(x,y)$ as $x \to \infty$, we can choose $X(y)$ such that for all $x > X(y)$

$$|f(x,y)-f(\infty,y)| < \epsilon'/(Y-b), \tag{25}$$

where ϵ' is another arbitrarily small positive number. It then follows from (25) that

$$\left|\int_b^Y [f(x,y)-f(\infty,y)]dy\right| < \epsilon', \tag{26}$$

for all $x > X$ that is the maximum of $X(y)$ for $Y \geqslant y \geqslant b$. We thus have shown that $|h'(x)| < \epsilon + \epsilon'$ for all $x > X$, which is equivalent to saying that $h'(x)$ approaches zero as $x \to \infty$.

APPENDIX 2

The proof in Appendix 1 says that the integral (7) must diverge as $z \to \infty$ if the integral (9) diverges. We prove here that this divergence is weaker than the logarithmic, more specifically that

$$\lim_{z\to\infty} [I(z)/\ln z]=0, \tag{27}$$

where $I(z)$ stands for the integral (7). When $I(z)$ tends to infinity, the limit (27) is equal to

$$\lim_{z\to\infty} z\frac{dI(z)}{dz}=\lim_{z\to\infty} \frac{z}{\pi}\int_{x_0}^\infty \frac{[\delta(x)-\delta(\infty)]}{(x+z)^2}dx. \tag{28}$$

In order to prove that this limit is zero, we split the integral in (28) into two parts, one from x_0 to $z^{\frac{1}{2}}$ and the other from $z^{\frac{1}{2}}$ to ∞. We can prove that the magnitudes of these two individually approach zero as $z \to \infty$:

$$\left|z\int_{x_0}^{z^{\frac{1}{2}}} \frac{[\delta(x)-\delta(\infty)]}{(x+z)^2}dx\right| \leqslant \max\int_{x_0}^{z^{\frac{1}{2}}} \frac{zdx}{(x+z)^2} = \max\left[\frac{z}{x_0+z}-\frac{z}{z^{\frac{1}{2}}+z}\right] \underset{z\to\infty}{\longrightarrow} 0, \tag{29}$$

where "max" stands for the maximum of $|\delta(x)-\delta(\infty)|$ for $z^{\frac{1}{2}} \geqslant x \geqslant x_0$; max is finite in (29), while it approaches zero as $z \to \infty$ in case of the second term of (28) where the rest of the expression remains finite.

Therefore, the behavior of $D_0(z)$ as $z \to \infty$ should now be given by

$$D_0(z) \underset{z\to\infty}{\longrightarrow} \propto z^{\delta(\infty)/\pi} \exp[I(z)], \tag{30}$$

where $I(z)$ behaves according to (27). This rather subtle behavior can be equivalently stated by our notion in (10) and (11). The equivalence can be checked directly if we observe that (27) is nothing but

$$|I(z)| < \epsilon \ln z, \tag{31}$$

for all z greater than a sufficiently large number to be chosen for an arbitrarily small positive number, ϵ.

[12] D. Amati, M. Fierz, and V. Glaser, Phys. Rev. Letters 4, 89 (1960).