Upper Limits for Coupling Constants in Field Theories*

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Upper bounds to the magnitude of the coupling constant for the vertex $A+B\leftrightarrow C$ are explored for meson theories with spatially fixed sources and also for the full relativistic theory without approximation. Two types of limit are obtained which depend upon whether or not the vertex $A+\bar{B}\leftrightarrow D$ exists where $\bar{B}$ is the antiparticle to B and D is stable. Any greater value is inconsistent with unitarity and the mass spectrum of stable particles. In fixed-source theories the limits can be explicitly expressed in terms of simple properties of the given source function; in the nonapproximated relativistic theory they involve some knowledge of the number or position of the nodes of the single-partial wave absorptive amplitude on the nonphysical (left-hand) cut. When inelastic scattering of A by B is neglected and the mass of C is only slightly less than the sum of the masses of A and B, the upper bounds are equal to each other and to the coupling constant if C is a pure bound state of A and B.

I. INTRODUCTION

THE strength of the coupling among elementary particles is generally described by the renormalized coupling constant which specifies the magnitude of the interaction among almost free "clothed" particles. For the strongest interactions it is of order unity and is progressively orders of magnitude less for electromagnetic, weak, and gravitational couplings. In a previous paper[1] the question of upper limits for coupling constants was investigated for models in which the interaction vanished, when the particles were separated beyond some critical distance. Here, completeness and the masses of the interacting particles did imply upper bounds for the coupling constants, but the results were not applicable to relativistic field theories or even to fixed source meson theories other than those with spatial distributions which vanished beyond a fixed radius.

In this paper, the previous results are strengthened and also extended to general fixed source meson theories. The methods used are also applicable to relativistic field theories, if the Mandelstam representation is assumed. However, in this case the upper limits that exist for coupling constants turn out to be less definitive. For fixed source meson theories, the maximum value of the coupling constant depends only upon the mass spectrum and some simple properties of the known source distribution. In the full relativistic theory its upper limit involves, instead of a known source distribution, either the maximum number of nodes of the analytic continuation of the single partial wave absorptive amplitude on the left hand (nonphysical cut) or, if this is infinite, some knowledge about the spacing of these nodes. Although this is known for various models in which only a finite number of partial waves are retained, or in which the left-hand cut is approximated by a finite number of poles, in general, one has no *a priori* knowledge of it. Therefore, a rigorous maximum possible coupling constant for the fully relativistic nonapproximated theory is not determined. The maxima, which we shall determine, have a simple analog in potential scattering which also suggests the physical situation which obtains when the coupling constant approaches its maximum value: The renormalized coupling constant for the three-particle vertex $A+B\leftrightarrow C$ reaches the maxima obtained here, when the particle C can be represented purely as a bound state of A and B or at least contains as much amplitude for being just such a bound state as is compatible with other constraints on the nature of the interacting particles.

The most general upper limits found for coupling constants are of the type given by Eqs. (51), (52), (60), and (61). These are the chief results to be presented.

II. "CAUSAL" INELASTIC POTENTIALS AND BOUND STATES

We shall begin with a resume of some results for potential scattering which are an extension of those discussed in I. The results have an exact analog in the relativistic field theory problem.

Consider a particle described by a Klein-Gordon equation

$$(\omega^2+\nabla^2-\mu^2)\varphi(\mathbf{r},\omega)=V(r,\omega)\varphi(\mathbf{r},\omega),\tag{1}$$

where $V=0$ for $r\geqq a$. The positive- and negative-frequency solutions which have, in general, different interactions with the potential are interpreted as those for a positively and negatively charged particle, respectively. The ω dependence of the potential implies that the interaction is nonlocal in time.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{i\omega t}V(r,\omega)\varphi(\mathbf{r},\omega)=\int_{-\infty}^{\infty}\bar{V}(r,\,t-t')\varphi(\mathbf{r},t')dt'.\tag{2}$$

We assume that the potential is causal, i.e., that future

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[1] M. Ruderman and S. Gasiorowicz, Nuovo cimento 8, 860 (1958); hereafter referred to as I. See also: A. A. Ansel'm, V. N. Gribov, G. S. Danilov, I. T. Dyatlov, and V. M. Shekhter, Soviet Phys.—JETP 14, 444 (1962); [J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 619–628 (1961)].

and past are kept distinct:

$$\bar{V}(r, t-t')=0, \quad t-t'<0 \tag{3}$$

so that

$$V(r,\omega)=\int_0^\infty \bar{V}(r,\tau)e^{i\omega\tau}d\tau. \tag{4}$$

It follows that $V(r,\omega)$ is a regular function of ω for $\mathrm{Im}\omega\geqq 0$, except for possible poles at infinity, and we assume these are not present. Then

$$V(r,\omega)=V(r,0)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}V(r,\omega')d\omega'}{\omega'-\omega}, \quad \mathrm{Im}\omega>0. \tag{5}$$

The restriction that inelastic processes occur only for positive kinetic energies $|\omega|>\mu$ gives

$$\mathrm{Im}V(r,\omega')=0, \quad |\omega'|<\mu \tag{6}$$

so that $V(r,\omega)$ is regular in the entire ω plane except for cuts along the real axis, with a gap at least as wide as the line $-\mu<\omega<\mu$ in which the potential is real. The negatively charged particles may have a set of N s-wave bound states at energies ω_n, where $0<\omega_n<\mu$ whose wave functions are given by

$$\varphi_n(r)=A_ne^{-\alpha_nr}/r, \quad (r>a) \tag{7}$$

with

$$\alpha_n=+(\mu^2-\omega_n^2)^{\frac{1}{2}}; \tag{8}$$

similarly for the positively charged particles the bound state wave functions are given by

$$\varphi_m(r)=B_me^{-\beta_mr}/r \tag{9}$$

with

$$\beta_m=+(\mu^2-\omega_m^2)^{\frac{1}{2}}. \tag{10}$$

Let $F^-(k)$ and $F^+(k)$ be the s-wave scattering amplitude for the negatively and positively charged particles where $k=(\omega^2-\mu^2)^{\frac{1}{2}}$. If the potential $V(r,\omega)$ is less singular than r^{-2} at the origin, the scattering amplitudes $F^{\mp}(k)$ for real k are the boundary values of functions with specified singularities in the upper half plane $\mathrm{Im}k>0$. These analytic properties are contained in the statement that $R_1(k)$ and $R_2(k)$ defined below are both regular functions, including the point at infinity, for $\mathrm{Im}k>0$.

$$R_1(k)=\left[\frac{F^+(k)+F^-(k)}{2}+\sum_n\frac{4\pi|A_n{}^2\omega_n|}{k^2+\alpha_n{}^2}+\sum_m\frac{4\pi|B_m{}^2\omega_m|}{k^2+\beta_m{}^2}\right]e^{2iak} \tag{11}$$

and

$$R_2(k)=\left[\mu\left(\frac{F^-(k)-F^+(k)}{2\omega}\right)-\sum_n\frac{4\pi\mu|B_n|^2}{k^2+\beta_n{}^2}+\sum_m\frac{4\pi\mu|A_n|^2}{k^2+\alpha_n{}^2}\right]e^{2iak}. \tag{12}$$

For real k, $R_1(k)=R_1^*(-k)$; $R_2(k)=R_2^*(-k)$.

One proof of the regularity of $R_{1,2}(k)$ follows closely along the lines of the analogous field theory derivation of Sec. III. This regularity is the relativistic analog of the well-known result for the analyticity of the partial wave amplitude for the Schrödinger equation which obtains when $F^-=F^+$ and $|\omega_n|\to\mu$.

From unitarity and the regularity of $R_{1,2}(k)$, upper bounds for the normalization constants $|A_n|^2$ and $|B_m|^2$ follow. To see this we note that

$$F_\pm(k)\equiv\left(\frac{F^+(k)+F^-(k)}{2}\pm\frac{\mu}{\omega}\frac{F^-(k)-F^+(k)}{2}\right)e^{2iak} \tag{13}$$

is regular in the upper half-plane except for poles at $k=i\alpha_n$, $n=0, 1, 2, \cdots$ and $k=i\beta_m$, $m=0, 1, 2, \cdots$. For real k, it satisfies $F_\pm(k)=F_\pm^*(-k)$, $\mathrm{Im}F_\pm(k)\geqq 0$, and $|F_\pm(k)|\leqq 1/k$. The function,

$$\mathfrak{F}_\pm(k)=\frac{i\alpha_0-k}{i\alpha_0+k}e^{2iak}\left[F_\pm(k)+\frac{1}{2ik}\right]-\frac{1}{2ik}, \tag{14}$$

has exactly the properties detailed above for $F_\pm(k)$:

$$\mathfrak{F}_\pm(k)=\mathfrak{F}_\pm^*(-k), \quad k \text{ real} \tag{15}$$

$$\mathrm{Im}\mathfrak{F}_\pm(k)\geqq 0, \quad k \text{ real and positive} \tag{16}$$

$$|\mathfrak{F}_\pm(k)|\leqq|1/k|, \quad k \text{ real.} \tag{17}$$

However, it does not have a pole at $k=i\alpha_0$, so that if $\beta_m\neq\alpha_0$ and $\alpha_n\neq\alpha_0$ for $n\neq 0$ then

$$\mathfrak{F}_\pm(i\alpha_0)=-\frac{|A_0|^2e^{-2a\alpha_0}\pi}{\alpha_0{}^2}(|\omega_0|\pm\mu)+\frac{1}{2\alpha_0}. \tag{18}$$

From the regularity and symmetry of $\mathfrak{F}_\pm(k)$ it can be expressed, for $\mathrm{Im}k\geqq 0$, as a Cauchy integral over real values of k together with terms which explicitly exhibit its singularities:

$$\mathfrak{F}_\pm(k)-2\pi i\left[\sum_{n\neq 0}\frac{|A_n|^2(|\omega_n|\pm\mu)(\alpha_0-\alpha_n)}{\alpha_n(k-i\alpha_n)(\alpha_0+\alpha_n)}e^{-2\alpha_na}+\sum_m\frac{|B_m|^2(|\omega_m|\mp\mu)(\alpha_0-\beta_m)}{\beta_m(k-i\beta_m)(\alpha_0+\beta_m)}e^{-2\beta_ma}\right]=\frac{2}{\pi}\int_0^\infty\frac{dk'\,k'\,\mathrm{Im}\mathfrak{F}_\pm(k')}{k'^2-k^2}. \tag{19}$$

From Eqs. (16), (17), (18), and (19)

$$0\leqq-|A_0|^2e^{-2a\alpha_0}\frac{\pi}{\alpha_0{}^2}(|\omega_0|\pm\mu)+\frac{1}{2\alpha}-2\pi\left[\sum_{n\neq 0}\frac{|A_n|^2e^{-2\alpha_na}}{\alpha_n(\alpha_0+\alpha_n)}(|\omega_n|\pm\mu)+\sum_m\frac{|B_m|^2e^{-2\beta_ma}}{\beta_m(\alpha_0+\beta_m)}(|\omega_m|\mp\mu)\right]\leqq 1/\alpha. \tag{20}$$

The following are immediate consequences of the above inequalities:

If there are no bound states of positive (negative) particles ($B_m=0$), then the maximum possible $|A_0|^2$ for a negative (positive) particle bound state satisfies the inequality

$$|A_0|^2 \leqq \alpha_0 e^{2a\alpha_0}/2\pi(|\omega_0|+\mu). \tag{21}$$

If there are bound state(s) of the positive (negative) particle as well as of negative (positive) ones or for a neutral particle which is identical to its antiparticle, then

$$|A_0|^2 \leqq \alpha_0 e^{2a\alpha_0}/2\pi|2\omega_0|. \tag{22}$$

These upper limits are to be compared to the bound imposed upon $|A_0|^2$ by the condition that the charge represented by the bound state wave function is unity, if all other possible components of this bound state other than the particle plus the potential are ignored. Then $\int d\mathbf{r}\, 2\omega_0\varphi_0{}^*\varphi_0=\pm 1$; in the limit $a\rightarrow 0$ a pure one-component bound state at ω_0 gives

$$|A_0|^2=\alpha_0/4\pi|\omega_0|. \tag{23}$$

For the neutral particle the upper limit can be reached only if there is no inelastic scattering at any energy. Inelasticity means that over part of the region of integration of the Cauchy integral in Eq. (19) $\mathrm{Im}\mathfrak{F}<1/k$ and $|A_0|^2$ is always less than the bound given in the inequality (22). The existence of channels other than the elastic one for energies above some threshold plus the causal property of the interaction imply that the bound state must have a nonzero amplitude for the states of these other channels: if in addition to the elastic channel $\alpha\rightarrow\alpha$ we have, for sufficiently high energies, $\alpha\rightarrow\beta$, γ, etc., then any bound state for α really includes a mixture of β, γ, etc. This implication does not necessarily follow for an energy-dependent absorptive potential which does not satisfy Eq. (5).

The bound on $|A_0|^2$ for a potential which gives a bound state for a particle but not its antiparticle can be much more stringent than that for a particle which is identical to its antiparticle. (We consider only energy-dependent potentials which are bounded as the energy becomes infinite.) For a weakly bound particle $|\omega_0|\sim\mu$ and both inequalities give the same limit, which is exactly what obtains from the nonrelativistic Schrödinger equation. However, for very tight binding $|\omega_0|\rightarrow 0$ and the limit on the bound-state amplitude is much more severe for the case which gives the inequality (21). This comes about in the following way: If the antiparticle (negative frequency) is not bound but the particle (positive frequency) is, then the potential is clearly ω dependent. From Eq. (5) we recall that causality, energy dependence, and boundedness for infinite energies imply inelasticity. If the potential is very different for $\pm|\omega_0|$ even as $|\omega_0|$ approaches zero, the inelastic part of the potential must become dominant and $|A_0|^2$ cannot grow arbitrarily large, since the elastic channel contributes only a small part of it. For p-wave interactions we exploit the fact that $f_\pm{}^p(k)$ now vanishes at $k=0$. Instead of Eq. (14) we follow I and define

$$\mathfrak{F}_\pm{}^p(k)=\left(\frac{k-i\alpha_0}{k+i\alpha_0}\right)\left(\frac{\alpha_0 ak+k+i\alpha}{\alpha_0 ak+k-i\alpha}\right) \times e^{2iak}\left(f_\pm(k)+\frac{1}{2ik}\right)-\frac{1}{2ik}. \tag{24}$$

Because $f_\pm(k)=0$, it follows that

$$\mathfrak{F}_\pm{}^p(0)=0. \tag{25}$$

The analytic and symmetry properties of $\mathfrak{F}^p$ lead to Eq. (19) with an additional term $(-iK_\pm)(k-i\gamma)^{-1}$ on the right-hand side with $\beta(a+\gamma^{-1})=1$. From Eq. (25) the sign of $K_\pm$ can be evaluated for several special cases and, instead of the previous results, we obtain for a p-wave bound state the more restrictive condition for those cases which gave the inequalities (21) and (22) for a s-wave residues.

$$|A_0{}^p|^2\leqq\frac{\alpha_0 e^{2a\alpha_0}}{2\pi(|\omega_0|+\mu)}\frac{a\alpha_0}{(2+a\alpha_0)}, \tag{21'}$$

$$|A_0{}^p|^2=\frac{\alpha_0 e^{2a\alpha_0}}{2\pi 2|\omega_0|}\frac{a\alpha_0}{(2+a\alpha_0)}. \tag{22'}$$

III. ANALYTIC PROPERTIES AND FIXED-SOURCE MODELS

The limits on normalization constants for the finite range "causal" potential bound state depended only upon analyticity properties of the scattering amplitude and inequalities which followed from unitarity. In field theory the renormalized coupling constants play the role of the normalization constants in the potential problem. The main difference in the analytic structure of the scattering amplitudes in the energy-dependent potential problem of Sec. II and the amplitudes of field theory lies in the absence of a finite range a. Instead of an essential singularity at infinity (which is easily canceled by a simple exponential) branch cuts are present. However analyses similar to that of Sec. II are still feasible. The analytic behavior of a single partial wave elastic scattering amplitude, considered as a function of center-of-mass momentum, is well known for many special models. From the Mandelstam representation it is also known for any local relativistic field theory, once the mass spectrum is given and may follow even if the Mandelstam representation is not valid.[2] In the upper half k plane ($k\equiv$c.m. momentum) all of the models which we shall consider give scattering amplitudes which have one or more poles at $k=i\alpha_0$, $i\alpha_1$, etc. and branch cuts along the imaginary axis from some

[2] J. G. Taylor, Nuovo cimento 22, 92 (1961).

im_0 to $i\infty$. Generally, in the fully relativistic models the cut may begin below the isolated poles; in fixed-source models it proves possible to discuss linear combinations of amplitudes for which there is always in a gap between the beginning of the cut and the poles.

We consider the elastic scattering of a particle $(-)$ and its antiparticle $(+)$ by a scatterer in the notation of I, For a boson described by a field operator $\varphi(\mathbf{r},t)$ whose equation of motion is

$$(\Box-\mu^2)\varphi(\mathbf{r},t)=j(\mathbf{r},t), \tag{26}$$

the elastic scattering amplitude $M^{(-)}$ for a boson of frequency ω is conveniently expressed by the reduction formula:

$$\begin{aligned}M^{(-)}&(\omega,\mathbf{k},\mathbf{k}')\\ &=-i\int dt\int d\mathbf{r}\int d\mathbf{r}'\exp(i\omega t+i\mathbf{k}\cdot\mathbf{r}-i\mathbf{k}'\cdot\mathbf{r}')\\ &\quad\times\langle I|\theta(t)[j(\mathbf{r}',t),j^*(\mathbf{r},0)]\\ &\qquad-\delta(t)[j(\mathbf{r}',t),\dot{\varphi}^*(\mathbf{r},0)]|I\rangle. \end{aligned}\tag{27}$$

Here $\mathbf{k}$ and $\mathbf{k}'$ are the incident and final momenta of the scattered boson, $\omega^2-\mu^2=k^2=k'^2$, and $|I\rangle$ is the initial state function of the scatterer. Because we consider a spatially fixed source the initial (and final) state of the scatterer are independent of $\mathbf{k}$ and $\mathbf{k}'$. The charge conjugate boson has an amplitude $M^{(+)}$ related to $M^{(-)}$ by crossing symmetry

$$M^{(-)}(\omega,\mathbf{k},\mathbf{k}')=M^{(+)}(-\omega,-\mathbf{k},-\mathbf{k}').$$

In fixed-source theories the spatial dependence of $j(\mathbf{r},t)$ factors so that

$$j(\mathbf{r},t)=g_0\rho(r)O(t), \tag{28}$$

where O involves spin, I-spin, etc., but not $\mathbf{r}$, and $\rho(r)$ is a fixed, given function of r. For p-wave coupling $O(t)$ is proportional to $\boldsymbol{\sigma}\cdot\hat{r}$ and thus involves the direction but not the magnitude of $\mathbf{r}$; only trivial changes in the following argument result. With Eq. (28) it is possible to perform the integrations over $\mathbf{r}$ and $\mathbf{r}'$ in Eq. (27) explicitly. The resulting integral is proportional to $[g(k)]^2$, where

$$g(k)=g_0\int\rho(r)\exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{r}. \tag{29}$$

The singularities of the s-wave scattering amplitude are most simply exhibited in the Low[3] equations which are obtained from Eq. (27) for the s-wave scattering amplitudes $F^{\pm}(k)$. The combinations $F^-(k)+F^+(k)$ and $(\mu/\omega)[F^-(k)-F^+(k)]$ do not have the branch point at $k=i\mu$, which exists for F^- and F^+ separately. Explicitly, we have

$$\begin{aligned}\frac{F^-(k)+F^+(k)}{2}&=g^2(k)\sum_m\frac{(E_m-E_I)}{(E_m-E_I)^2-\mu^2-k^2}\\ &\quad\times(|\langle I|O^*|m\rangle|^2+|\langle I|O|m\rangle|^2)\end{aligned}\tag{30}$$

$$\begin{aligned}\frac{\mu}{2\omega}[F^-(k)-F^+(k)]&=g^2(k)\sum_m\frac{\mu}{(E_m-E_I)^2-\mu^2-k^2}\\ &\quad\times(|\langle I|O|m\rangle|^2-|\langle I|O^*|m\rangle|^2).\end{aligned}\tag{31}$$

The singularities in Eqs. (30) and (31) occur wherever the form factor $g(k)$ is not regular, on the real axis from plus to minus infinity, and, if there exist possible states m' of the scatterer whose mass is less than $E_I+\mu$, at $k=\pm i\alpha$ where

$$\alpha=[\mu^2-(E_{m'}-E_I)^2]^{\frac{1}{2}}. \tag{32}$$

Such states contribute poles along the imaginary k axis $|\mathrm{Im}k|<\mu$; the residues at these poles are, except for arbitrary numerical factors, the renormalized coupling constants for the vertices corresponding to boson $+I\leftrightarrow I'$. If the scatterer is a proton and the scattered boson a scalar charged "pion," the residue in Eq. (31) is the conventional renormalized coupling constant g^2 defined by

$$g\equiv g^2(i\mu)|\langle n|\tau_-|p\rangle|^2. \tag{33}$$

For sufficiently strong coupling a bound state of π^+ and p (p^{++}) is also expected which contributes another pole at $k=i\mu'$ and a residue $2g'^2=2g^2(i\mu')|\langle p^{++}|\tau_+|p\rangle|^2$. It will be significant for the derivation of upper bounds that the residues add in Eq. (30) but enter with opposite signs into Eq. (31).

The magnitudes of the combination of elastic scattering amplitudes in Eq. (31) and also the sum $(F^-+F^+)/2\pm(\mu/2\omega)(F^--F^+)$ are never greater than $|1/k|$ on the real axis because of unitarity. If it were not for the singularities contributed by $g(k)$, it would be very simple to calculate upper bounds for the renormalized coupling constants by using exactly the methods of Sec. II. Instead we shall construct a function $s(k)$ which has prescribed singularities and has unit magnitude on the real axis. It will then prove possible to introduce $s(k)$ as a multiplier of combinations of scattering amplitudes in such a way that the resulting functions can be treated by previously discussed methods with similar results. We wish to construct an $s(k)$ such that $|s(k)|=1$ and $s(k)=s^*(-k)$ for real k and $s(k)[f_\pm(k)-i/2k]$ is regular in the upper half-plane of k space except for discrete bound-state poles with

$$f_\pm(k)=F^-+F^+\pm(\mu/\omega)(F^--F^+). \tag{34}$$

We assume the function $\rho(r)$ to be a distribution of

[3] F. Low, Phys. Rev. 97, 1392 (1955).

Yukawa functions of arbitrary masses. Then

$$g(k)=\int_{m_0}^{\infty}\frac{\sigma(m)dm}{m^2+k^2}, \tag{35}$$

where m_0 is the lightest mass contributing to the form factor. Exactly such a spectral resolution of $g(k)$ will also be relevant for the full relativistic theory. We shall first find an $s_1(k)$ such that $s_1(k)g(k)$ is regular in the upper half-plane except for the isolated poles whose residues are the renormalized coupling constants. The function $g(k)$ is regular in the cut upper half-plane with a cut on the imaginary axis from im_0 to $i\infty$. The real part of g is continuous across the cut but the imaginary part is $\mp i\pi\sigma(m)/2m$ for $k=im\pm\epsilon$. The phase jump of $g(k)$ is then

$$\Delta\varphi=2\tan^{-1}\left(\pi\sigma(m)\Big/2P\int\frac{\sigma(m)dm}{|k|^2-m^2}\right). \tag{36}$$

A function $s_1(k)$ which satisfies the prescribed conditions can have the form

$$s_1(k)=\exp\left(2ik\int_0^{\infty}\frac{dm\bar{\sigma}(m)}{m^2+k^2}\right). \tag{37}$$

The function $s(k)$ is regular except for the cut on the imaginary axis. For real k, $|s(k)|=1$. The phase discontinuity across the imaginary axis at $k=im$ is given by

$$s_1(im+\epsilon)/s_1(im-\epsilon)=\exp[2\pi i\bar{\sigma}(m)]. \tag{38}$$

Therefore, we take

$$\bar{\sigma}(m)=-\frac{1}{\pi}\tan^{-1}\left(\pi\sigma(m)\Big/2P\int\frac{\sigma(\kappa)d\kappa}{\kappa^2-m^2}\right) \tag{39}$$

and

$$s_1(k)=\exp\left[\frac{2ik}{\pi}\int_{m_0}^{\infty}\frac{dm}{m^2+k^2}\times\tan^{-1}\left(\pi\sigma(m)\Big/2P\int\frac{\sigma(\kappa)d\kappa}{\kappa^2-m^2}\right)\right]. \tag{40}$$

Only the value of the function $s_1(k)$ or its lower bound at the discrete poles of the scattering amplitude will be relevant to putting limits on the coupling constant. A lower bound for $s_1(i\alpha)$ can be obtained which depends only upon the number of nodes of the spectral function $\sigma(m)$ or their spacing. We assume that the minimum mass of the source distribution is larger than that of the free boson:

$$\mu<m_0. \tag{41}$$

Then in Eq. (40) that part of the integrand which multiplies $\tan^{-1}$ is positive over the entire range of integration.

The principal part integral in Eq. (36) will be infinite at discontinuities of $\sigma(m)$ but does not change sign at these points; therefore, at zeros associated with infinite values of the denominator at discontinuities of $\sigma(m)$, $\Delta\varphi/2$ touches but does not pass through an integral multiple of π. Rather a passing through $n\pi$ occurs only where $\sigma(m)$ passes through zero. [Delta-function singularities in $\sigma(m)$, when considered as the limit of a continuous function, also give this change in $\Delta\varphi/2$.] The maximum value of $\Delta\varphi/2$ is then $n\pi$, where n is the number of nodes of $\sigma(m)$ between $m=m_0$ and $m=\infty$. Then from Eq. (40), after replacing the $\tan^{-1}$ by $(1+n)\pi$, we have the lower bound at $k=i\alpha$:

$$s_1(i\alpha)\geqq\left(\frac{m_0-\alpha}{m_0+\alpha}\right)^{n+1}. \tag{42}$$

If the nodes of $\sigma(m)$ are at $m=m_i$, then

$$s_1(i\alpha)\geqq\left(\frac{m_0-\alpha}{m_0+\alpha}\right)\prod_i\left(\frac{m_i-\alpha}{m_i+\alpha}\right). \tag{43}$$

The function $s_2(k)\equiv s_1^2(k)$ then has precisely the discontinuity across the cut so that $s_2(k)g^2(k)$ is regular for $\mathrm{Im}k\geqq0$. We now wish to construct the $s(k)$, such that $s(k)[f_\pm(k)+1/2ik]$ is regular except for poles when $\mathrm{Im}k>0$. Although this $s(k)$ cannot be expressed as explicitly as $s_2(k)$, we can construct an upper bound exactly analogous to that for $s_1(i\alpha)$. The combination $f_\pm(k)+(2ik)^{-1}$ has an imaginary part for k pure imaginary, which is exactly the same as that of $f_\pm(k)$ alone. However, the real part is different because of the additional $1/2ik$ and so the phase jump $\Delta\varphi'$ is no longer that of $g^2(k)$.

To calculate $\Delta\varphi'$ exactly it would now be necessary to know the exact eigenstates of the Hamiltonian. But again the phase jump can pass through $n\pi$ only when $\mathrm{Im}f_\pm(i|k|)-\mathrm{Im}(2|k|)^{-1}=\mathrm{Im}f_\pm(i|k|)$ vanishes. From Eqs. (34), (31), and (30) this can occur if $g^2(i|k|)=0$ or from the possible vanishing of the sum. Because $E_m\geqq E_I+\mu$ for all states m except m', which contributes the pole, each term in the sum is positive except $m=m'$ which has opposite sign for f_+ and f_-. We can always choose one for which the entire sum is guaranteed positive and this choice will also turn out to be the one which gives the smallest upper bound to the coupling constant. Then $\Delta\varphi'$ can equal $n\pi$ only at the nodes of $g^2(i|k|)$. Therefore, $s(i\alpha)$ can be chosen to satisfy the same lower bound as $[s_1(i\alpha)]^2$ or

$$s(i\alpha)\geqq\left(\frac{m_0-\alpha}{m_0+\alpha}\right)^{2n+2}. \tag{44}$$

If the nodes of $\sigma(m)$ are at known positions $m=m_i$, then we have again the stronger limit

$$s(i\alpha)\geqq\left(\frac{m_0-\alpha}{m_0+\alpha}\right)^2\prod_i\left(\frac{m_i-\alpha}{m_i+\alpha}\right)^2. \tag{45}$$

For an infinite number of nodes of σ the product con-

verges if the node spacing increase at least logarithmically as $m \to \infty$. For application to higher partial waves it is significant that the above $s(k)$ is not unique. One can always multiply $s(k)$ by a finite product which has only zeros in the upper half-plane and does not contribute any singularities there. We define

$$\bar{s}(k) \equiv s(k) \prod_{j=1}^{n} \left(\frac{\gamma_j + ik}{\gamma_j - ik}\right). \tag{46}$$

This $\bar{s}(k)$ is another possible solution to the problem of constructing an $s(k)$ which has absolute magnitude unity on the real axis and a prescribed phase jump across the imaginary axes (mod 2π).

We may exploit this arbitrariness to construct an $\bar{s}(k)$ which satisfies the inequality (45) and also $i\bar{s}'(0) \leqq 0$. The product can be expressed as

$$\bar{s}(k) = \exp\left\{\frac{2ik}{\pi}\int_{m_0}^{\infty} \frac{dm}{m^2+k^2}[\theta(m)+\Delta\varphi'(m)]\right\}, \tag{47}$$

where $\theta(m)$ is a multistep function: $\theta(m)=0$ for $m \leqq \gamma_1$, $\theta = 2\pi$ for $\gamma_1 < m \leqq \gamma_2$, $\theta = 4\pi$ for $\gamma_2 < m \leqq \gamma_3$, etc. Now $\Delta\varphi'$ can go through an integral multiple of 2π only at one of the nodes of $\sigma(m)$ and at $m = m_0$ and has the maximum of $2\pi(n+1)$ only if at each such node $\Delta\varphi(m)$ is increasing. If we choose the set γ_j to coincide with those nodes of $\sigma(m)$ where $\Delta\varphi'$ is not increasing, then $\theta(m)+\Delta\varphi'(m)$ is a positive definite function with upper bound $2\pi(n+1)$. Thus, $\bar{s}(i\alpha)$ satisfies the inequalities (44) and (45) and

$$i\bar{s}'(0) = -\frac{2}{\pi}\int_{m_0}^{\infty} \frac{dm}{m^2}[\theta(m)+\Delta\varphi'(m)], \tag{48}$$

so that

$$0 > i\bar{s}'(0) \geqq -2(n+1)/m_0. \tag{49}$$

IV. COUPLING CONSTANT LIMITS IN FIXED-SOURCE MESON THEORY

The functions $s(k)$ and $\bar{s}(k)$ defined in Sec. III play exactly the role of $\exp(2ia k)$ in Sec. II in establishing upper bounds of coupling constants. We consider a negatively charged boson θ^- of mass μ incident upon a particle P. The renormalized coupling constant for $\theta^- + P \leftrightarrow Q$ is defined by

$$g^2(i\mu)|\langle Q|O\tau_-|P\rangle|^2 = g^2{}_{\theta PQ}, \tag{50}$$

in analogy with the pion-nucleon convention of Eq. (33). Then the results of Sec. II hold with the association $2\pi|A_0|^2 \to g^2{}_{\theta PQ}$, $e^{2iak} \to s(k)$ or $\bar{s}(k)$, and $|\omega_0| = |\Delta M|$, the absolute value of the mass difference between P and Q. Then from the definition

$$\alpha = [\mu^2 - (\Delta M)^2]^{\frac{1}{2}} \tag{32'}$$

and the inequalities (21), (22), and (44), we have the following theorems for the s-wave interaction of a boson with a source in fixed meson theory:

If g^2 is the renormalized coupling constant for the vertex $\theta^- + P \leftrightarrow N$ and $\theta^+ + P$ possesses no bound state then

$$g^2 \leqq \frac{\alpha}{\mu+\Delta M}\left(\frac{m_0+\alpha}{m_0-\alpha}\right)^{2n+2}. \tag{51}$$

A rigorous upper limit without restriction is

$$g^2 \leqq \frac{\alpha}{2\Delta M}\left(\frac{m_0+\alpha}{m_0-\alpha}\right)^{2n+2}. \tag{52}$$

If θ^- and θ^+ have identical interactions with P (viz., a π^- and π^+ with a Λ^0) then the inequality (52) is also valid.

If the nodes of $\sigma(m)$ are at known $m = m_i$ then the parentheses in (51) and (52) may be replaced by the right-hand side of the inequality (45).[4] The coupling constant g^2 can approach the limit of (52) only if inelastic scattering is negligible for $\theta^- + P$. If $\alpha \ll m_0$ (loose binding or short-range interaction) then in analogy with the potential case of Sec. II, $2g^2\Delta M/\alpha$ is the probability that the particle Q is a pure bound state of $\theta^- + P$, i.e., with no admixture of other particle states.[5,6] If both inelastic scattering is small and $\alpha \ll m_0$, g^2 can be much less than $\alpha/2\Delta M$ only if Q is treated in the model as, in large part, an additional "elementary particle" which exists independently of the $\theta^- + P$ interaction.

For loose binding $\Delta M \sim \mu$ and the two limits on g^2 are the same. When M_Q is much less than $M_P + \mu$, the conditions of (51) are much more restrictive than those of (52) and, as in the potential case, imply the existence of inelastic channels for $\theta^- + P$.

The inequality (51) can be exceeded if θ^+ has a bound state, so that there exists a vertex $\theta^+ + P \leftrightarrow P^{++}$ with P^{++} stable. Suppose this bound state corresponds to $k = i\alpha^{++}$. Then we define

$$s^{++} = \frac{i\alpha^{++} - k}{i\alpha^{++} + k}s, \tag{53}$$

so that $s^{++}f_\pm$ possesses neither the $\theta^- + P \leftrightarrow Q$ pole at $k = i\alpha$ nor the $\theta^+ = P \leftrightarrow P^{++}$ pole at $k = i\alpha^{++}$. Limits on g^2 can then be found which depend upon the four masses M_P, M_Q, μ, and $M_{P^{++}}$. For the case of the charged scalar mesons $\pi^\mp$, $P = p$, $Q = n$, and $P^{++} = p^{++}$, the doubly charged proton isobar. The resulting in-

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[4] The limit of a point source is obtained by taking m_0 infinite. For charged scalar meson theory with $\Delta M = 0$, the necessary condition for no isobars is then $g^2 \leqq 1$. This same general result has also been obtained by C. Goebel (private communication). The point nucleon has also been discussed by B. M. Barbashov and G. V. Efimov (to be published).

[5] V. N. Gribov, Ya. B. Zel'dovitch, and A. M. Perelomov, Soviet Phys.—JETP 13, 836 (1960); see also L. D. Landau, *ibid.* 12, 1294 (1960).

[6] Y. Nambu and J. J. Sakurai, Phys. Rev. Letters 6, 377 (1961).

equality is

$$g_{\pi pn}{}^2 \leqq \frac{\mu+[\mu^2-(M_{p^{++}}-M_p)^2]^{\frac{1}{2}}}{\mu-[\mu^2-(M_{p^{++}}-M_p)^2]^{\frac{1}{2}}}\left(\frac{m_0+\mu}{m_0-\mu}\right)^{2n+2}. \quad (54)$$

An implication is that as $g^2 \to \infty$ the isobar-proton mass difference goes to zero at least as fast as $2\mu/|g|$.

For p-state interactions the limits on g^2 are more stringent. The argument closely follows that for the square cutoff. We exploit the possibility of choosing instead of s an $\bar{s}$ which satisfies the inequality (49). For notational simplicity we assume a single pole at $k=i\alpha$ and define

$$\mathfrak{F}_\pm{}^p(k)=\frac{(k-i\alpha)(k+i\beta)}{(k+i\alpha)(k-i\beta)}\bar{s}(k)\left[f_\pm(k)+\frac{1}{2ik}\right]-\frac{1}{2ik}, \quad (55)$$

with β defined by

$$2\beta-2\alpha-i\alpha\beta\bar{s}'(0)=0, \quad (56)$$

so that

$$\mathfrak{F}_\pm{}^p(0)=0. \quad (57)$$

Then if we assume $\mathfrak{F}_\pm{}^p \to 0$ as $|k| \to \infty$ in the upper half-plane

$$\mathfrak{F}_\pm{}^p(k)=\frac{K}{k^2+\beta^2}+\frac{2}{\pi}\int_0^\infty dk' \frac{k' \operatorname{Im}\mathfrak{F}_\pm{}^p(k')}{k'^2-k^2}. \quad (58)$$

For $k=0$, the left-hand side vanishes and $K<0$. Since $\alpha>\beta$ from Eqs. (56) and (49), we have finally

$$\mathfrak{F}_\pm{}^p(i\alpha) \geqq 0. \quad (59)$$

The p-wave coupling constant f^2 is conventionally defined, so that instead of g^2 we have $-f^2/3$. From Eqs. (59), (55), (44), (34), and (30), we have finally

$$f^2 \leqq \left(\frac{3\alpha}{\Delta M+\mu}\right)\left(\frac{n\alpha+\alpha}{n\alpha+\alpha+2m_0}\right)\left(\frac{m_0+\alpha}{m_0-\alpha}\right)^{2n+2} \quad (60)$$

instead of the inequality (51). The last parentheses can again be replaced by the right-hand side of inequality (45).

For the pion-nucleon system with $n=0$, $\Delta M=0$, and m_0 the nucleon mass, this gives $f^2 \leqq 0.4$ compared with the observed $f^2 \cong 0.08$. Whether or not bound isobars exist, we have the rigorous upper bound

$$f^2 \leqq \frac{3\alpha}{2\Delta M}\left(\frac{n\alpha+\alpha}{n\alpha+\alpha+2m_0}\right)\left(\frac{m_0+\alpha}{m_0-\alpha}\right)^{2n+2}, \quad (61)$$

which is the p-wave equivalent of inequality (52). Again the inequality may be strengthened if the positions of the nodes of σ are known.

V. APPLICATIONS TO RELATIVISTIC FIELD THEORIES

The previous analyses which lead to coupling constant bounds depended upon the singularities of the scattering amplitude for a single partial wave in momentum space. For relativistic field theories the relevant analytic structure is known for weaker assumptions than the Mandelstam representation.[2] It is then possible to derive results analogous to these for fixed-source meson theory. There is, however, one significant difference: in fixed-source theory the position and number of nodes of Eqs. (52) and (60) are given *a priori* in specifying the source function, while in relativistic field theory its analog is determined from the dynamical properties obtained from the scattering solution itself. It is only within the framework of various common approximation techniques that an *a priori* restriction can be given for the coupling constant. The situation is quite analogous to that in potential scattering with a "causal" potential of the form

$$V(r,\omega)=\int_{m_0}^\infty \rho(\sigma,\omega)\frac{e^{-\sigma r}}{r}d\sigma. \quad (62)$$

In k space the number of nodes of the discontinuity in the scattering amplitude across the imaginary axis from im_0 to $i\infty$ depends upon the solution of the scattering problem. Except for certain regions, viz., im_0 to $2im_0$, it is not known *a priori*.[7]

The simplest case in the relativistic field theory is that of four equal-mass mesons with no three vertex; this is the pseudoscalar meson system treated by Mandelstam and Chew.[8] We shall avoid some notational complication by considering neutral pseudoscalar mesons only. The postulated analytic structure of the s-wave elastic scattering amplitude is conveniently expressed[8] in terms of

$$A(q^2,\cos\theta)=(q^2+\mu^2)^{\frac{1}{2}}F(q^2,\cos\theta), \quad (63)$$

where q is the center-of-mass momentum and F is the conventional scattering amplitude. $A^l(q^2)$ is the single partial-wave amplitude related to $A(q^2,\cos\theta)$ (for positive q^2 and real θ) by

$$A(q^2,\cos\theta)=\textstyle\sum_l(2l+1)A^l(q^2)P_l(\cos\theta). \quad (64)$$

In the absence of bound states the function $A^l(q^2)$ is regular everywhere in the finite q^2 plane, except for cuts on the real axis from 0 to $+\infty$ and from $-\mu^2$ to $-\infty$. The imaginary part of $A(q^2)$ has opposite sign when the cuts are approached from below or above and vanishes for real q^2 between $-\mu^2$ and 0. Following Chew and Mandelstam, we have for q^2 real and less than $-\mu^2$

$$\operatorname{Im}A^0(q^2)=\int_0^{-q^2-\mu^2}\frac{dq'^2}{q^2}\operatorname{Im}A\left(q'^2, 1+2\frac{q^2+\mu^2}{q'^2}\right). \quad (65)$$

It is the number and position of the nodes of $\operatorname{Im}A^0(q^2)$ on the left-hand cut that determines the upper bound for the coupling constant in this and related systems. Here,

[7] L. A. Khalfin, Soviet Phys.—JETP 14, 880 (1962).
[8] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

the renormalized four-vertex coupling constant λ is canonically defined by the equation

$$\lambda \equiv -A^0(-\tfrac{2}{3}\mu^2). \tag{66}$$

To determine an upper bound for $|\lambda|$ it is convenient to consider instead of the q^2 plane the upper half of the q plane, so that the analytic region is identical to that discussed in previous sections. Then we define

$$\frac{A^0(q^2)}{(\mu-iq)(\gamma+iq)} \equiv \mathcal{A}^0(q), \tag{67}$$

where

$$\gamma=\surd(2/3)\mu. \tag{68}$$

The function $\mathcal{A}^0(q)$ satisfies $\mathcal{A}^0(q)=\mathcal{A}^{0*}(-q)$ for real q and obeys the inequality

$$|q\mathcal{A}^0(q)| \leqq (\gamma^2+q^2)^{-\frac{1}{2}} \tag{69}$$

for real q. On the cut from $q=i\mu$ to $i\infty$ $\mathrm{Im}\mathcal{A}^0(q)$ has a node only where $\mathrm{Im}A^0(q^2)$ does. If $\mathcal{A}^0(q)$ does not vanish at infinity along every ray in the upper half-plane, it is sufficient to consider $\mathcal{A}^0(q)[\mu-iq]^{-m}$ which has the same analytic properties as $\mathcal{A}^0(q)$ and leads to a quantitatively similar upper bound for λ as long as m is finite.

As in Secs. III and IV we introduce $s_1(q)$

$$s_1(q)=\exp\left[\frac{2iq}{\pi}\int_\mu^\infty \frac{dm}{m^2+q^2}\tan^{-1}\varphi(m^2)\right], \tag{70}$$

where $\varphi(m^2)$ is the phase of $\mathcal{A}^0(q)$ at $q=im$ or the phase of $\mathcal{A}^0(q)(\mu-iq)^{-m}$ if subtractions are necessary. The function $\mathcal{A}^0(q)s_1(q)$ is now regular in the upper half-plane except for the pole at $q=i\gamma$ and the manipulations of Secs. III and IV lead to the inequality

$$|\lambda| < \frac{2}{\pi}[1+(\tfrac{3}{2})^{\frac{1}{2}}]\left(\frac{\mu+\gamma}{\mu-\gamma}\right)\prod_j\left(\frac{m_j+\gamma}{m_j-\gamma}\right) \equiv |\lambda_{\max}|, \tag{71}$$

where $A^0(q^2)$ has nodes at $q=im_j$. If $\lambda<-|\lambda_{\max}|$, the violation of the inequality (71) is made possible by additional particles (bound states) of mass less than 2μ. However, λ cannot be greater than $|\lambda_{\max}|$ unless a ghost state of negative norm is present. For $q^2<-\mu^2$, $A^0(q^2)$ can be expanded in a Legendre series whose radius is at least $q^2=-9\mu^2$:

$$\mathrm{Im}A^0(q^2)=\int_0^{-q^2-\mu^2}\frac{dq'^2}{q^2} \times\sum_l \mathrm{Im}A^l(q'^2)P_l\left(1+2\frac{q^2+\mu^2}{q'^2}\right). \tag{72}$$

Since only even partial waves enter into the partial wave expansion (neutral scalar particles) with $P_{2l}(x)>0$ for $|x|>1$ and $\mathrm{Im}A^l(q'^2)\geqq 0$, it follows that $\mathrm{Im}A(q^2)\geqq 0$ wherever the Legendre series converges. Thus,

$$m_j \geqq 3\mu. \tag{73}$$

Various approximation schemes such as the replacement of the left-hand cut by a finite number of poles or the inclusion of a finite number of partial waves (with subtractions) will give an upper bound to $|\lambda|$ with n known *a priori*. In the unapproximated theory, only if the spacing increases at least logarithmically as $q\rightarrow i\infty$, will Eq. (71) result in an upper bound for λ.

In the case of the nonrelativistic elastic potential, for which Levinson's theorem[9] is valid, nodes on the imaginary axis are related to the existence of resonances. In this case, a complementary theorem to that of Levinson can be proved: the total change of phase along the cut on the imaginary axis in the upper half-plane of q space equals π times the number of poles in the lower half-plane ("resonances"). Therefore, in this case there is at least one node in the absorptive amplitude on the nonphysical cut for each "resonance." A similar situation might obtain in the field theoretic case where Regge poles[10] and nodes on the nonphysical cut may be similarly related. But the number of nodes and their spacing would, even then, be unknown *a priori*.

If an s-wave three-vertex is possible with the third particle of mass m, then the amplitude $A^0(q^2)$ has an additional term

$$G(q^2)\equiv g^2\left[\frac{2}{M^2-4\mu^2-4q^2}+\frac{1}{q^2}\ln\left(\frac{4q^2}{M^2}+1\right)\right], \tag{74}$$

where the third particle is assumed to be stable ($m^2<4\mu^2$) and may be the meson itself ($m^2=\mu^2$). In addition to the pole at $-q^2=\mu^2-M^2/4\equiv\beta^2$, $G(q^2)$ contributes a cut along the negative q^2 axis from $q^2=-M^2/4$ to $-\infty$. For $2\mu>M>\surd\mu$, there is a gap between the pole and the cut, and an upper bound to g^2 follows directly. Consider

$$C(q)\equiv A^0(q)/[q^2+\mu^2]^{\frac{1}{2}}, \tag{75}$$

which is a regular function of q in the upper half-plane except for the cut along the imaginary axis from $iM/2$ to $i\infty$. [If $C(q)$ does not approach zero for $q\rightarrow i\infty$ then the exponent of the denominator in Eq. (75) may be raised.] On the real axis $C(q)=C^*(-q)$, $q\,\mathrm{Im}C(q)\geqq 0$, and $|qC(q)|\leqq 1$, so that the arguments of Secs. III and IV may be applied directly. The cut which is canceled by s_1 extends down to $iM/2$, and the phase jump of $-\pi/2$ at $q=i\mu$ introduced by the denominator of Eq. (75) should be included. Then

$$g^2\leqq\frac{4\beta(\mu^2-\beta^2)^{\frac{1}{2}}}{\mu^2}\left(\frac{M+2\beta}{M-2\beta}\right)\prod_j\left(\frac{m_j+\beta}{m_j-\beta}\right). \tag{76}$$

This upper bound cannot be exceeded for a positive metric.

[9] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **25**, No. 9 (1949).
[10] T. Regge, Nuovo cimento **18**, 947 (1960).

For the three-vertex of equal-mass particles $M^2=\mu^2$, the pole term lies on the cut and no limit is obtained by the above method. Instead we define

$$D(q)=\frac{qA^0(q)}{(\mu-iq)^3}\frac{(\beta+iq)}{(\beta-iq)}s_1(q). \tag{77}$$

Here, $\beta=(3/4)^{\frac{1}{2}}\mu$. The function $s_1(q)$ is that of Eq. (70), so that it cancels the phase jump across the upper half imaginary axis from $q=i\mu$ to $i\infty$ only; $D(q)$ has a cut from $i\mu/2$ to $i\mu$. A contour integration in the q plane around the cut from $i\mu/2$ to $i\mu$ and just above the entire real axis gives

$$2g^2\left|\int_{\mu/2}^{\mu}\pi\frac{dy}{y}\frac{s_1(iy)}{(y+\mu)^3}\left(\frac{y-\beta}{y+\beta}\right)\right|=\left|\int_{-\infty}^{\infty}D(x)dx\right|\leqq\pi. \tag{78}$$

If $D(g)$ does not vanish at infinity in the upper half-plane, a similar result obtains with $(\mu-iq)$ in Eq. (77) raised to a higher power. By following the argument of Sec. III, it is always possible to choose an $\bar{s}_1(q)$ which is positive and monotonically decreasing along the imaginary q axis from $q=0$ to $q=i\mu$:

$$1>\bar{s}_1(iy)>\prod\left(\frac{m_j-y}{m_j+y}\right),\quad y\text{ real}<\mu. \tag{79}$$

The combination of the inequalities (78) and (79) give an upper bound for g^2 in terms of the location m_j of the nodes of $\mathrm{Im}A^0(q^2)$ on the nonphysical cut $q^2<-\mu^2$. If $A^0(q^2)$ does not have a zero at infinity, only minor and straightforward changes are required in the computation of a slightly larger limit.

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