Chamber's Solution of the Boltzmann Equation*

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It is shown that Chamber's kinetic integral is an exact solution of the Boltzmann equation in the relaxation time approximation.

STARTING from a kinetic argument, Chambers[1] has suggested that the following is a solution of the time-independent Boltzmann equation:

$$f=\int_{-\infty}^{t} f_0(E-\Delta E(t'))\exp\left(-\int_{t'}^{t}\frac{ds}{\tau(\mathbf{p}(s))}\right)\frac{dt'}{\tau(\mathbf{p}(t'))},\quad (1)$$

where

$$\Delta E=\int_{t'}^{t}\mathbf{F}\cdot\mathbf{V}(t'')dt''$$

is the energy gain of an electron between time t' and t in the absence of collisions, τ is the relaxation time, and $f_0=\exp(-E/kT)$ is the equilibrium distribution. Expanding the integrand to first order in E and integrating by parts gives

$$f=f_0-\frac{df_0}{dE}\int_{-\infty}^{t}\mathbf{F}\cdot\mathbf{V}(t')\exp\left(-\int_{t'}^{t}\frac{ds}{\tau(\mathbf{p}(s))}\right)dt'.\quad (2)$$

Equation (2) has been derived from the linearized Boltzmann equation by Suzuki,[2] while Heine[3] has argued that this is a valid solution to all orders in applied fields.

It will be shown that (1) is an exact solution of the Boltzmann equation and that (2) solves only the linearized equation.[4]

Let us consider the simple case of a spherical band, $\tau=\tau(p)$ and a constant force field $\mathbf{F}=F\mathbf{I}_z$. In this case:

$$\mathbf{V}(t')=[\mathbf{p}+\mathbf{F}(t'-t)]/m.\quad (3)$$

Integrating (1) by parts, using (3) and setting $t'-t=y$,

$$f=f_0\left[1+\int_0^{\infty}\exp\left(\frac{\mathbf{F}\cdot\mathbf{p}y-F^2y^2/2}{mkT}\right)\times\exp\left(-\int_0^{y}\frac{ds}{\tau(\mathbf{p}-\mathbf{F}s)}\right)\left(\frac{\mathbf{F}}{m}\cdot\left(\frac{\mathbf{p}-\mathbf{F}y}{kT}\right)\right)dy\right].\quad (4)$$

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[1] R. Chambers, Proc. Phys. Soc. (London) **A65**, 458 (1952).
[2] H. Suzuki, J. Phys. Soc. Japan **16**, 2347 (1961).
[3] V. Heine, Phys. Rev. **107**, 431 (1957).
[4] In disagreement with Heine.

The Boltzmann equation in the relaxation time approximation is simply

$$\mathbf{F}\cdot\nabla_p f=-(f-f_0)/\tau.\quad (5)$$

Putting (4) into (5) and noticing that

$$\mathbf{F}\cdot\nabla_p\frac{1}{\tau(\mathbf{p}-\mathbf{F}s)}=-\frac{d}{ds}\left(\frac{1}{\tau(\mathbf{p}-\mathbf{F}s)}\right),$$

we have

$$-\frac{\mathbf{F}\cdot\mathbf{p}}{mkT}+\int_0^{\infty}\exp\left(\frac{\mathbf{F}\cdot\mathbf{p}y-F^2y^2/2}{mkT}\right)\times\exp\left(-\int_0^{y}\frac{ds}{\tau(\mathbf{p}-\mathbf{F}s)}\right)\left[-\frac{\mathbf{F}\cdot(\mathbf{p}-\mathbf{F}y)\mathbf{F}\cdot\mathbf{p}}{(mkT)^2}+\frac{F^2}{mkT}+\frac{F^2y\mathbf{F}\cdot(\mathbf{p}-\mathbf{F}y)}{(mkT)^2}+\frac{\mathbf{F}\cdot(\mathbf{p}-\mathbf{F}y)}{mkT}\frac{1}{\tau(\mathbf{p}-\mathbf{F}y)}\right]dy=0.\quad (6)$$

Integrating the last term of (6) by parts, one finds an exact cancellation of all terms; thus, (1) is an exact solution of (5).

We can do a similar thing with (2). In this case, one finds in place of (6)

$$-\frac{\mathbf{F}\cdot\mathbf{p}}{mkT}+\int_0^{\infty}\exp\left(-\int_0^{y}\frac{ds}{\tau(\mathbf{p}-\mathbf{F}s)}\right)\left[-\frac{\mathbf{F}\cdot\mathbf{p}\mathbf{F}\cdot(\mathbf{p}-\mathbf{F}y)}{(mkT)^2}+\frac{F^2}{mkT}+\frac{\mathbf{F}\cdot(\mathbf{p}-\mathbf{F}y)}{\tau(\mathbf{p}-\mathbf{F}y)mkT}\right]dy=0.\quad (7)$$

Integrating the last term by parts, one finds that all terms cancel except

$$\frac{\mathbf{F}\cdot\mathbf{p}}{mkT}\int_0^{\infty}\exp\left(-\int_0^{y}\frac{ds}{\tau(\mathbf{p}-\mathbf{F}s)}\right)\frac{\mathbf{F}\cdot(\mathbf{p}-\mathbf{F}y)}{mkT}dy,\quad (8)$$

which is at least a second-order term.

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