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Lifetimes of Quasi-Particles and Phonons in a Superconductor at Zero Temperature*

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General expressions for decay rates of a quasi-particle and a phonon in a superconductor at zero temperature are obtained from Nambu's nonlocal self-consistency conditions. The valid coherence factors in the transition probabilities are encountered for the different processes we consider. The decay rate of a quasi-particle due to emission of single phonons is found to vanish with the 7/2 power of its excess excitation energy above the gap. The contribution to decay arising from emission of single pairs of particles via Coulomb interaction turns out to be at the most a few percent of that coming from phonon emission. The level width of a phonon caused by decay into single pairs of particles is shown to rise like a step function from zero to $\pi/2$ times the normal state value when the phonon energy reaches twice the gap energy. Additional contributions to particle decay arising from two-phonon emission or, from single pair emission, are obtained from the second-order self-energy diagram. The two-phonon emission effect is found to be completely negligible in comparison to that due to single phonon emission.

I. INTRODUCTION

WHEREAS at the outset of the Bardeen-Cooper-Schrieffer[1] theory of superconductivity an explicit construction of the ground state is made by using the pairing condition, Bogoliubov[2] starts his formulation by forming the elementary excitation or the "quasi-particle" as a coherent mixture of an added and a subtracted electron excitation both having the asme momentum and spin component along the axis of quantization. The amount of mixing can be determined self-consistently by a generalized Hartree-Fock method. This mixing leads to a gap in excitation energy above the ground state, and it amounts to the occurence of the "coherence factor"[1] in the transition probability of a quasi-particle when transitions are induced by an external field. The coherence factor applies also for the spontaneous transitions a quasi-particle undergoes by interaction with the phonons of the medium or by Coulomb interaction with the medium thus causing the particle to have a finite inverse decay time, or level width Γ.

The mechanism by which the electron-phonon and Coulomb interaction between the electrons act on the one hand to form an elementary excitation as an electron-hole mixture, and cause on the other hand this excitation to decay, can be displayed most clearly in a formulation based on a procedure developed by Nambu.[3] Two-component operators are introduced consisting of an electron and a hole component, and a matrix one-particle Green's function is defined in terms of these operators. From the spectral representation of this Green's function we shall see that the energy E_p and the level width Γ_p of a quasi-particle are determined by the real and imaginary part, respectively, of the poles of this function in its complex energy variable. Using infinite-order perturbation theory, the particle propagator can be expressed in terms of the propagator for the unperturbed system and a matrix self-energy part, $\Sigma(p)$. The nondiagonal component of $\Sigma(p)$ plays the role of the energy gap, and it determines also the amount of electron-hole mixture in a particle. $\Sigma(p)$ is determined self-consistently by equating an ansatz for $\Sigma(p)$ with the self-energy amplitude calculated from the diagrams in the dressed particle and interaction picture. In this way the energy E_p and the lifetime $(\Gamma_p)^{-1}$ of a quasi-particle are determined by the same self-consistency condition containing both the electron-phonon and the Coulomb interaction.

The various contributions to the decay rate of a quasi-particle correspond to the various orders of the self-energy diagrams and, for a particular diagram, to its different time orderings and to its two types of interaction, either phonon or Coulomb, inserted in its interaction lines. The processes taking part in the contribution from a specific diagram are described simply by the right half of this diagram (if time runs from right to left) when it is cut through one of its intermediate states. Since this contribution to the decay rate consists of the over-all transition probability for the processes allowed by energy and momentum conservation, we expect to encounter the coherence factor valid for the single process in question if the contribution is represented in integral form. One expects also to find correction terms to the anticipated expressions for transition probabilities, which have their origin in the nonlocal formulation of Nambu's theory. By nonlocal formulation we mean, that the ansatz for $\Sigma(p)$ is taken to depend both on the momentum and energy arbitrarily and that the self-consistency condition is extended to hold not only on the energy shell of the original particle but also on virtual intermediate states.

Actual calculation of contributions to the quasi-

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[1] J. Bardeen, L. N. Cooper and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[2] N. N. Bogliubov, V. V. Tolmachev, and D. V. Shirkov, *A New Method in the Thoory of Superconductivity* (Consultants Bureau, Inc., New York, 1959).
[3] Y. Nambu, Phys. Rev. **117**, 648 (1960).

particle decay rate will be carried out in the cases of single phonon emission, emission of pairs of particles by Coulomb interaction, and successive emission of two phonons. The latter contribution arises from the second-order self-energy diagram containing one crossing of interaction lines. For another time ordering of this diagram, and Coulomb instead of phonon interaction lines, one obtains a correction term to the decay rate arising from pair emission to lowest order. This correction term has been considered for the normal state by DuBois.[4] The Coulomb interaction induced decay rate to lowest order is determined by the imaginary part of the lowest-order polarization term; and this quantity also provides us immediately with the inverse decay time of a phonon into pairs of particles.

It can be expected that all the contributions to particle or phonon damping derived for the superconducting state will deviate markedly from their normal state values if the excitation energies are of the order of the gap, for in that energy range not only the dispersion laws of energies and density of states of particles are greatly modified, but also the coherence factors come into full play.

One practical reason for carrying out these investigations is to determine whether or not single phonon emission is overwhelmingly responsible for quasiparticle decay. This dominance of single phonon emission (and absorption at finite temperatures) has been tacitly assumed in the theory of the electronic component of thermal conductivity in case when impurity scattering is absent.[5] However, it is known that this part of the theory of thermal conductivity of superconductors fails to agree with the experimental findings.

II. GENERAL THEORY OF LIFETIMES IN NAMBU'S FORMULATION

The problem of superconductivity can be treated with a modified perturbation theory formalism. A convenient way introduced by Nambu[3] is to use two-component operators $\psi(\mathbf{p})$ and $\psi^\dagger(\mathbf{p})$ for the electrons defined by

$$\psi(\mathbf{p})=\begin{pmatrix} c_{p\frac{1}{2}} \\ c_{-p-\frac{1}{2}}^\dagger \end{pmatrix}, \quad \psi^\dagger(\mathbf{p})=(c_{p\frac{1}{2}}^\dagger, c_{-p-\frac{1}{2}}). \tag{2.1}$$

$c_{p\sigma}$ and $c_{p\sigma}^\dagger$ are annihilation and creation operators, respectively, for electrons with momentum $\mathbf{p}$ and spin component σ along the axis of quantization. The 2×2 Pauli matrices corresponding to this two-component notation are

$$\tau_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \tau_2=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \tau_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[4] D. F. DuBois, Ann. Phys. (New York) **8**, 24 (1959).

[5] J. Bardeen, G. Rickayzen and L. Tewordt, Phys. Rev. **113**, 982 (1959).

If H is the Hamiltonian of the electron-phonon system with electron-phonon and Coulomb interaction between electrons included, and if N is the particle number operator and μ the chemical potential of the system, then we obtain for $(H-\mu N)$ in terms of the new electron-hole operators

$$\begin{aligned} H=&\textstyle\sum_p \epsilon_p \psi^\dagger(\mathbf{p})\tau_3\psi(\mathbf{p})+\sum_q \omega_q(b_q^\dagger b_q+\frac{1}{2})+\sum_p \epsilon_p \\ &+\sum_{p,q} g_q\psi^\dagger(\mathbf{p}+\mathbf{q})\tau_3\psi(\mathbf{p})\frac{1}{\sqrt{2}}(b_q+b_{-q}^\dagger) \\ &+\tfrac{1}{2}\sum_{p,p',q\neq 0} v_q\psi^\dagger(\mathbf{p})\tau_3\psi(\mathbf{p}+\mathbf{q})\psi^\dagger(\mathbf{p}')\tau_3\psi(\mathbf{p}'-\mathbf{q}). \end{aligned} \tag{2.2}$$

We use units with $\hbar=1$, and we set the volume of the system equal to one. $v_q=4\pi e^2/q^2$ is the Fourier transform of the static Coulomb potential between the electrons. The one-electron energies are

$$\epsilon_p=(p^2/2m)-\mu-\tfrac{1}{2}\sum_{q\neq 0} v_q.$$

b_q and $b_q^\dagger$ are annihilation and creation operators, respectively, for longitudinal phonons with momentum $\mathbf{q}$ and unrenormalized frequency ω_q. If one treats the ions of the lattice as point charges Ze of mass M and density n, the electron-phonon coupling constant becomes $g_q=-(v_q/\omega_q)^{\frac{1}{2}}\omega_{\mathrm{pl}}$, where $\omega_{\mathrm{pl}}=(4\pi Z^2e^2n/M)^{\frac{1}{2}}$ is the classical ion plasma frequency.

The natural generalization of the usual one-particle Green's function is a matrix Green's function defined as

$$G_{ij}(\mathbf{p},t)=-i\langle\Psi_0|T(\psi_i(\mathbf{p},t)\psi_j^\dagger(\mathbf{p},0))|\Psi_0\rangle. \tag{2.3}$$

$|\Psi_0\rangle$ is the exact Heisenberg ground state of the interacting system. The time variable arguments in the ψ operators denote Heisenberg operators. If $|\Psi_0\rangle$ is constructed from the ground state of the noninteracting system, $|\Phi_0\rangle$, by perturbation theoretic method, the quasi-particle propagator takes the form

$$G_{ij}(\mathbf{p},t)=-i\langle\Phi_0|T(\psi_i^I(\mathbf{p},t)\psi_j^{I\dagger}(\mathbf{p},0)S)|\Phi_0\rangle_c. \tag{2.4}$$

The superscript I on the ψ operators denotes the interaction representation, the S-matrix S contains the electron-phonon and the Coulomb interaction in this representation, and the subscript c indicates that only connected Green function diagrams should be taken into account in a diagrammatic analysis of this expression. The Fourier transform of $G(\mathbf{p},t)$ in the time variable, $G(\mathbf{p},p_0)\equiv G(p)$, can be expressed with the help of Dyson's equation by the irreducible proper self-energy part $\Sigma(p)$ and the bare particle propagator $G_0(p)$:

$$G(p)=[G_0^{-1}(p)-\Sigma(p)]^{-1}. \tag{2.5}$$

$G_0(p)$ is easily calculated from Eq. (2.4) by setting $S=1$ and performing a Fourier transform in the time

variable. The result is

$$G_0(p)=\frac{p_0+\epsilon_p\tau_3}{p_0^2-\epsilon_p^2+i\delta}, \tag{2.6}$$

where δ is a positive infinitesimally small quantity.

For the complete propagator of phonon and Coulomb interaction, $D(\mathbf{q},q_0)\equiv D(q)$, we will use a general form which is based on derivations given by Schultz,[6] i.e.,

$$D(q)=\left(\frac{v_q^{\frac{1}{2}}\omega_{\mathrm{pl}}}{K(q)}\right)^2\frac{1}{[q_0^2-\omega_q^2+\omega_q\Pi(q)]}+\frac{v_q}{K(q)}. \tag{2.7}$$

$K(q)\equiv K(\mathbf{q},q_0)$ is the longitudinal dynamical dielectric constant of the medium, and $\Pi(q)$ is the proper irreducible self-energy part of the phonon. $K(q)$ is determined by the proper irreducible polarization part $\Lambda(q)$,

$$K(q)=1-v_q\Lambda(q), \tag{2.8}$$

and the $\Pi(q)$ is expressable in terms of the dielectric constant through

$$\Pi(q)=-\frac{\omega_{\mathrm{pl}}^2}{\omega_q}\left(\frac{1}{K(q)}-1\right). \tag{2.9}$$

It should be noted that the termination "irreducible polarization part" for $\Lambda(q)$ is not exact because $\Lambda(q)$ must be phonon irreducible, while the complete polarization part would include also phonon reducible parts. This is true for both the $\Lambda(q)$ determining the screening of the electron-phonon interaction and the phonon self-energy part $\Pi(q)$. Since all interaction lines containing a phonon line are already included in the phonon propagator [first term in Eq. (2.7)], the $\Lambda(q)$ appearing in the screened Coulomb interaction [second term in Eq. (2.7)] must also be phonon irreducible.

The rules for calculating contributions from four-momentum space diagrams in the dressed particle and interaction picture are then essentially the following: Each quasi-particle line (solid line) carries a matrix factor $iG(p)$. Each phonon-Coulomb interaction line (wavy line) carries a contribution $-iD(q)$. Each vertex carries a matrix factor τ_3. All diagrams containing parts which are connected to the rest of the diagram by only two interaction or two particle lines, respectively, have to be omitted.

The problem is to determine the self-energy part $\Sigma(p)$ and the polarization part $\Lambda(q)$. In principle these quantities can be obtained in a self-consistent way by equating $-i\Sigma(p)$ and $i\Lambda(q)$ to the sum of all contributions from proper irreducible self-energy and polarization diagrams, respectively, in the dressed particle and interaction picture. This is represented in Fig. 1, where the dots indicate the higher-order diagrams.

[6] T. D. Schultz, *Quantum Field Theory and the Many-Body Problem* (Space Technology Laboratories, Inc., Los Angeles, 1960).

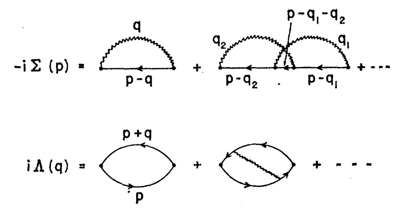

FIG. 1. Schematic representation of the self-consistency condition for the self-energy part $\Sigma(p)$ of a particle and the polarization part $\Lambda(q)$ of the interaction. The solid and wavy lines denote dressed particle and interaction lines, respectively. The dots represent higher than second order diagrams.

Nambu makes an ansatz for $\Sigma(p)$ given by

$$\Sigma(p)=p_0\zeta(p)+\chi(p)\tau_3+\phi(p)\tau_1, \tag{2.10}$$

where $\zeta(p)$, $\chi(p)$, and $\phi(p)$ are assumed to be even functions in p_0. Then one obtains from Eqs. (2.5) and (2.6)

$$G(p)=\frac{p_0Z(p)+\bar{\epsilon}(p)\tau_3+\phi(p)\tau_1}{p_0^2[Z(p)]^2-[E(p)]^2}, \tag{2.11}$$

where

$$Z(p)=1-\zeta(p),\quad \bar{\epsilon}(p)=\epsilon_p+\chi(p),$$
$$[E(p)]^2=[\bar{\epsilon}(p)]^2+[\phi(p)]^2. \tag{2.12}$$

While Nambu considers only the contribution to $\Sigma(p)$ from the lowest-order diagram in Fig. 1, we shall take into account also the contribution arising from the next higher order diagram containing one crossing of interaction lines. For $\Lambda(q)$ we shall make the so-called pair-approximation in which only the lowest order diagram in Fig. 1 is considered. In these approximations the self-consistency conditions are

$$\Sigma(p)=i\int\frac{d^4q}{(2\pi)^4}\tau_3G(p-q)\tau_3D(q)$$
$$-\iint\frac{d^4q_1d^4q_2}{(2\pi)^8}\tau_3G(p-q_2)\tau_3G(p-q_1-q_2)$$
$$\times\tau_3G(p-q_1)\tau_3D(q_1)D(q_2), \tag{2.13}$$

$$\Lambda(q)=-i\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}[\tau_3G(p+q)\tau_3G(p)]. \tag{2.14}$$

For an essentially attractive interaction between the electrons there exists a solution to the self-consistency condition Eq. (2.13) with $\mathrm{Re}\phi(p)$ different from zero near the Fermi surface, and this leads, as we shall see, to an energy gap for elementary excitations above the ground state and thus to the superconducting state. However, nonzero off-diagonal elements of the matrix propagator $G_{ij}(\mathbf{p},t)$ are not consistent with the original definition of this propagator in the Heisenberg picture, Eq. (2.3), if $|\psi_0\rangle$ is taken to be an eigenstate of the electron number operator. Therefore, it seems to be more satisfactory to adopt a modified definition of the matrix Green's function, which is due to Kadanoff and

Martin.[7] An annihilation operator for two electrons with total momentum and spin equal to zero is introduced by

$$a(t)=C\sum_p c_{p\frac{1}{2}}(t)c_{-p-\frac{1}{2}}(t).$$

If the constant C is determined from the condition $\langle\psi_0|aa^\dagger|\psi_0\rangle=1$, it is found to go to zero as $1/N$ (N = number of electrons) in the case of the BCS solution. The operators $\psi(\mathbf{p},t)$ and $\psi^\dagger(\mathbf{p},t)$, and thus $G(\mathbf{p},t)$, are then redefined by inserting $a(t)c_{-p-\frac{1}{2}}^\dagger(t)$ and $a^\dagger(t)c_{-p-\frac{1}{2}}(t)$, respectively, instead of $c_{-p-\frac{1}{2}}^\dagger(t)$ or $c_{-p-\frac{1}{2}}(t)$ in the second components of Eqs. (2.1).

The general analytic properties of the matrix propagator $G(\mathbf{p},p_0)$ in the complex p_0 plane and the asymptotic behavior of the Fourier transform $G(\mathbf{p},t)$ for large values of the time variable can be derived in a way closely analogous to the work of Galitskii and Migdal.[8] This is done by using Lehmann's spectral representation in a complete set $|\Psi_n{}^N\rangle$ of energy and particle number eigenstates of the interacting system with excitation energies $\epsilon_n{}^N$ with respect to the ground state of an N-electron system. If one neglects the difference in the chemical potential of an N- and $(N+1)$-electron system and sets the chemical potential equal to μ, and if one takes $(H-\mu N)$ as the Hamiltonian of the system, then one finds the following spectral representation for the matrix Green's function:

$$G_{ij}(\mathbf{p},p_0)=\int_0^\infty\frac{\rho_{ij}(\mathbf{p},\epsilon)d\epsilon}{p_0-\epsilon+i\eta}+\int_{-\infty}^0\frac{\rho_{ij}'(\mathbf{p},-\epsilon)d\epsilon}{p_0-\epsilon-i\eta}. \quad (2.15)$$

If terms of order $1/N$ are neglected, the components of the spectral weight functions in Eq. (2.15) turn out to be

$$\begin{aligned}
&\rho_{11}(\mathbf{p},\epsilon)d\epsilon=\sum_n|\langle n|c_{p\frac{1}{2}}^\dagger|0\rangle|^2; && \rho_{22}(\mathbf{p},\epsilon)d\epsilon=\sum_n|\langle n|a^\dagger c_{-p-\frac{1}{2}}|0\rangle|^2;\\
&\rho_{21}(\mathbf{p},\epsilon)d\epsilon=\sum_n\langle 0|ac_{-p-\frac{1}{2}}^\dagger|n\rangle\langle n|c_{p\frac{1}{2}}^\dagger|0\rangle; && \rho_{12}(\mathbf{p},\epsilon)=\rho_{21}^*(\mathbf{p},\epsilon);\\
&\rho_{11}'(\mathbf{p},\epsilon)d\epsilon=\sum_n|\langle n|c_{p\frac{1}{2}}|0\rangle|^2; && \rho_{22}'(\mathbf{p},\epsilon)d\epsilon=\sum_n|\langle n|ac_{-p-\frac{1}{2}}^\dagger|0\rangle|^2;\\
&\rho_{21}'(\mathbf{p},\epsilon)d\epsilon=-\sum_n\langle 0|c_{p\frac{1}{2}}^\dagger|n\rangle\langle n|ac_{-p-\frac{1}{2}}^\dagger|0\rangle; && \rho_{12}'(\mathbf{p},\epsilon)=\rho_{21}'^*(\mathbf{p},\epsilon).
\end{aligned} \quad (2.16)$$

Here the sums over n are restricted by the conditions

$$\epsilon<\epsilon_n{}^{N\pm1}<\epsilon+d\epsilon.$$

By using the relations $\mathrm{Im}G_{ij}(\mathbf{p},\epsilon)=-\pi\rho_{ij}(\mathbf{p},\epsilon)$ for $\epsilon>0$, and $=\pi\rho_{ij}'(\mathbf{p},-\epsilon)$ for $\epsilon<0$, one can write the spectral representation of $G_{ij}(\mathbf{p},t)$ in the form

$$\begin{aligned}
G_{ij}(\mathbf{p},t)&=\frac{i}{\pi}\int_0^\infty d\epsilon\, e^{-i\epsilon t}\,\mathrm{Im}G_{ij}(\mathbf{p},\epsilon),\quad \text{for}\quad t>0\\
&=\frac{i}{\pi}\int_{-\infty}^0 d\epsilon\, e^{-i\epsilon t}\,\mathrm{Im}G_{ij}(\mathbf{p},\epsilon),\quad \text{for}\quad t<0.
\end{aligned}\quad (2.17)$$

We conclude from Eq. (2.15) that the analytic continuations of $G_{ij}(\mathbf{p},p_0)$ for $p_0>0$ and $p_0<0$ lie above and below the branch cut, respectively, which runs slightly above the negative real axis and slightly below the positive real axis. If p_0 crosses the branch cut, $G(\mathbf{p},p_0)$ goes over into its complex conjugate value.

A simple physical interpretation of the propagator $G_{ij}(\mathbf{p},t)$ is possible only if all four components of the spectral weight function $\rho_{ij}(\mathbf{p},\epsilon)$ [$\rho_{ij}'(\mathbf{p},-\epsilon)$], are peaked strongly on the real axis around an excitation energy, let's say $\epsilon=E_{rp}$ ($\epsilon=-E_{rp}'$), with the same half width, Γ_p (Γ_p'), where $\Gamma_p\ll E_{rp}$. If the peak of $\rho_{ij}(\mathbf{p},\epsilon)$ [$\rho_{ij}'(p,-\epsilon)$] arises from a simple pole at $E_{rp}-i\Gamma_p$ ($-E_{rp}'+i\Gamma_p'$) with almost purely real residue of the analytic continuation of $G_{ij}(\mathbf{p},p_0)$ for $p_0>0$ into the lower right p_0 plane (for $p_0<0$ into the upper left p_0 plane), then one finds an asymptotic behavior of $G(\mathbf{p},t)$ for large values of $|t|$, which is given by

$$\begin{aligned}
G_{ij}(\mathbf{p},t)&=G_{ij}(\mathbf{p},0)\exp(-iE_{rp}t-\Gamma_p t),\quad \text{for}\quad t>0,\\
&=G_{ij}(\mathbf{p},0)\exp(-iE_{rp}'|t|-\Gamma_p'|t|),\\
&\qquad\qquad \text{for}\quad t<0.
\end{aligned}\quad (2.18)$$

Since a quasi-particle with momentum $\mathbf{p}$ and spin component $\frac{1}{2}$ consists of a certain mixture of the excitations $c_{p\frac{1}{2}}^\dagger(t)|\Psi_0\rangle$ and $a^\dagger(t)c_{-p-\frac{1}{2}}(t)|\Psi_0\rangle$, the probability amplitude of finding the quasi-particle $(\mathbf{p},\frac{1}{2})$ at time $t>0$ when it was present at time 0 is seen, from the definition in Eq. (2.3), to be a linear combination of all four components of the propagator $G_{ij}(\mathbf{p},t)$ for $t>0$. We conclude, therefore, from Eqs. (2.17) and (2.18) for the case $t>0$, that the energy and damping of a quasi-particle $(\mathbf{p},\frac{1}{2})$ are determined by the nearest pole to the positive real axis of the analytic continuation of $G(\mathbf{p},p_0)$ into the lower right plane. Similarly for the $t<0$ case we recognize from Eqs. (2.3), (2.17), and (2.18) that the energy and the lifetime of a quasi-particle $(-\mathbf{p},-\frac{1}{2})$ are determined by the nearest pole to the negative real axis of the analytic continuation of $G(\mathbf{p},p_0)$ into the upper left plane. Since the quasi-particle excitations $(\mathbf{p},\frac{1}{2})$ and $(-\mathbf{p},-\frac{1}{2})$ are formed here by adding an electron to the system or by subtracting it from it, respectively, their lifetimes $1/\Gamma_p$ and $1/\Gamma_p'$ are in principle different.

Nambu's ansatz for $G(p)$ [Eq. (2.11)] presupposes

[7] L. F. Kadanoff and P. C. Martin, Phys. Rev. **124**, 670 (1961).
[8] V. M. Galitskii and A. B. Migdal, Soviet Phys.—JETP **7**, 96 (1958).

first the existence of delta-function like peaks in all four components of the spectral weight function around the same energies E_{rp} and $-E_{rp}'$ with the same half widths Γ_p and Γ_p', respectively. Second it presumes that $E_{rp}'=E_{rp}$ and $\Gamma_p'=\Gamma_p$ since it assumes that $\phi(p)$ is an even function in p_0. Third this ansatz presupposes the reality of the off-diagonal spectral weight functions since it does not contain a term with τ_2.[9] The pole of the Nambu propagator $G(p)$ [Eq. (2.11)] below the positive real axis, for instance, is obtainable from the equation

$$E_{rp}-i\Gamma_p=(\tilde{E}(p)/\tilde{Z}(p))_{p_0=E_{rp}-i\Gamma_p}, \tag{2.19}$$

where $\tilde{E}(p)/\tilde{Z}(p)$ is the analytic continuation of $E(p)/Z(p)$ for $p_0>0$ into the lower right p_0 plane. For $\Gamma_p/E_{rp}\ll 1$, we have approximately

$$\begin{aligned} E_{rp}&=\mathrm{Re}[E(p)/Z(p)]_{p_0=E_{rp}};\\ \Gamma_p&=-A_p^{-1}\,\mathrm{Im}[E(p)/Z(p)]_{p_0=E_{rp}},\end{aligned} \tag{2.20}$$

where the quantity A_p is given in Eqs. (2.23). If we expand the square root $E(p)$ in terms of $\mathrm{Im}[\chi(p)/E_{rp}]$ and $\mathrm{Im}[\phi(p)/E_{rp}]$, we obtain, by neglecting terms of order $(\Gamma_p/E_{rp})^2$, the following expressions:

$$E_{rp}=E_p/Z_p, \tag{2.21}$$

$$\Gamma_p=-(Z_pA_p)^{-1}\times\mathrm{Im}\left[p_0\zeta(p)+\frac{\bar{\epsilon}_p}{E_p}\chi(p)+\frac{\phi_p}{E_p}\phi(p)\right]_{p_0=E_{rp}}, \tag{2.22}$$

where we have used the notation

$$E_p=(\bar{\epsilon}_p^2+\phi_p^2)^{\frac{1}{2}};\qquad \bar{\epsilon}_p=\epsilon_p+\mathrm{Re}[\chi(p)]_{p_0=E_{rp}};$$
$$\phi_p=\mathrm{Re}[\phi(p)]_{p_0=E_{rp}};\quad Z_p=1-\mathrm{Re}[\zeta(p)]_{p_0=E_{rp}}; \tag{2.23}$$
$$A_p=1-\mathrm{Re}\left(\frac{d}{dp_0}\frac{E(p)}{Z(p)}\right)_{p_0=E_{rp}}.$$

We shall see later how the special combination of imaginary parts of $p_0\zeta(p)$, $\chi(p)$, and $\phi(p)$ in the expression for the inverse decay time Γ_p [Eq. (2.22)] gives rise to the important coherence factors in the transition probabilities, which make the striking difference between the behavior of the superconducting and normal state excitations. We want to emphasize that all three imaginary parts have to be taken into account in order to get the right answer. The term with $p_0\zeta(p)$ has been considered by Eliashberg[10] in his derivation of damping caused by single phonon emission (see Sec. III).

The quantities $\zeta(p)$, $\chi(p)$, and $\phi(p)$ appearing in Eqs. (2.21) and (2.22) are obtainable from the self-consistency condition, Eq. (2.13). This matrix condition splits into three coupled integral equations if we equate the coefficient of 1, τ_3 and τ_1 on both sides. An additional term proportional to τ_2 arises from the second term on the right-hand side of the equation; however, this can be shown to vanish identically. We write these integral equations in the following form:

$$\phi(p)=-i\int\frac{d^4q_1}{(2\pi)^4}\frac{\phi_1D_1}{N_1}+\iint\frac{d^4q_1d^4q_2}{(2\pi)^8}\frac{D_1D_2}{N_1N_2N_3}\times[\phi_1(\alpha_2\alpha_3+\bar{\epsilon}_2\bar{\epsilon}_3)+\phi_2(\alpha_1\alpha_3+\bar{\epsilon}_1\bar{\epsilon}_3)-\phi_3(\alpha_1\alpha_2-\bar{\epsilon}_1\bar{\epsilon}_2)-\phi_1\phi_2\phi_3], \tag{2.24}$$

$$p_0\zeta(p)\pm\chi(p)=i\int\frac{d^4q_1}{(2\pi)^4}\frac{D_1}{N_1}(\alpha_1\pm\bar{\epsilon}_1)-\iint\frac{d^4q_1d^4q_2}{(2\pi)^8}\frac{D_1D_2}{N_1N_2N_3}\times[(\alpha_1\pm\bar{\epsilon}_1)(\alpha_2\pm\bar{\epsilon}_2)(\alpha_3\pm\bar{\epsilon}_3)+\phi_1\phi_2(\alpha_3\mp\bar{\epsilon}_3)-\phi_1\phi_3(\alpha_2\pm\bar{\epsilon}_2)-\phi_2\phi_3(\alpha_1\pm\bar{\epsilon}_1)], \tag{2.25}$$

where

$$\alpha(p)=p_0Z(p),\quad N(p)=p_0^2[Z(p)]^2-[E(p)]^2,$$

and where subscripts 1 and 2 on D denote arguments q_1 and q_2, respectively, and subscripts 1, 2, and 3 on α, $\bar{\epsilon}$, ϕ, and N denote arguments $(p-q_1)$, $(p-q_2)$, and $(p-q_1-q_2)$, respectively.

Energy and damping of a dressed phonon are obtainable from the nearest pole below the positive real axis, for instance, of the phonon propagator given by the first term in Eq. (2.7). If we make the ion plasma approximation, where ω_q is set equal to ω_{pl}, we derive [with the help of Eq. (2.9)] for $\Pi(q)$) a pole $(\Omega_q-i\Theta_q)$, which in the case $\Theta_q\ll\Omega_q$ is given approximately by

$$\Omega_q=\frac{\omega_{\mathrm{pl}}}{|K(q,\Omega_q)|}[\mathrm{Re}K(q,\Omega_q)]^{\frac{1}{2}} \tag{2.26}$$

and

$$\Theta_q=\frac{\omega_{\mathrm{pl}}^2}{2\Omega_q|K(q,\Omega_q)|^2B_q}\,\mathrm{Im}K(q,\Omega_q), \tag{2.27}$$

where B_q is

$$B_q=1-\frac{\omega_{\mathrm{pl}}^2}{2\Omega_q}\,\mathrm{Re}\left(\frac{d}{dq_0}\frac{1}{K(q)}\right)_{q_0=\Omega_q}. \tag{2.28}$$

For the dielectric constant $K(q)$ we obtain from Eqs. (2.8), (2.14), and (2.11) the general expression

$$K(q)=1+2iv_q\int\frac{d^4p}{(2\pi)^4}\frac{(p_0+q_0)p_0Z(p+q)Z(p)+\bar{\epsilon}(p+q)\bar{\epsilon}(p)-\phi(p+q)\phi(p)}{N(p+q)N(p)}. \tag{2.29}$$

[9] See Nambu's discussion of this point, reference 3.
[10] G. M. Eliashberg, Soviet Phys.—JETP 11, 696 (1960).

III. PARTICLE AND PHONON DAMPING BY SINGLE PROCESSES

In this section only those contributions to particle or phonon decay are considered which arise from the lowest order self-energy or polarization diagram, respectively, shown in Fig. 1. In general the processes participating in a certain contribution to decay are described by the right half of the self-energy or the polarization diagram (if time runs from right to left) when the diagram is cut through one of its intermediate states. Let us first determine the damping of a particle caused by emission of single phonons Γ_p^{ph}. This quantity is obtained from Eq. (2.22) and from the first order terms of Eqs. (2.24) and (2.25) by including only the phonon part of $D(q)$ [see first term of Eq. (2.7), and Eq. (2.9) with $\omega_q=\omega_{\mathrm{pl}}$]. The expression for Γ_p^{ph} is written as

$$\Gamma_p^{\mathrm{ph}}=\int\frac{d^3q}{(2\pi)^3}\frac{v_q\omega_{\mathrm{pl}}^2}{Z_pA_p}\times\mathrm{Im}\int_{-\infty}^{+\infty}\frac{dz}{(2\pi i)}H(z)L(z-E_{rp}),\quad(3.1)$$

where

$$H(z)=\frac{E_pzZ_{p-q}(z)+\bar{\epsilon}_p\bar{\epsilon}_{p-q}(z)-\phi_p\phi_{p-q}(z)}{E_p[z^2(Z_{p-q}(z))^2-(E_{p-q}(z))^2]}\quad(3.1\mathrm{a})$$

and

$$L(z)=\left\{[K(q,z)]^2\left[z^2-\frac{\omega_{\mathrm{pl}}^2}{K(q,z)}\right]\right\}^{-1}.\quad(3.1\mathrm{b})$$

In deriving Eq. (3.1) we have used the fact that $K(q,q_0)=K(q,-q_0)$ for real q_0, and therefore $L(q_0)=L(-q_0)$. The general analytic properties of the functions $H(z)$ and $L(z)$ in the complex z plane are known from the spectral representations of the particle and the phonon propagator, respectively. Thus $H(z)$ has a branch cut running from $-\infty$ to 0 infinitesimally above and from 0 to ∞ infinitesimally below the real axis. $L(z-E_{rp})$ has a branch cut running from $-\infty$ to E_{rp} infinitesimally above and from E_{rp} to ∞ infinitesimally below the real axis. Therefore the z-integration contour along the real axis in Eq. (3.1) can be deformed in a contour C running from $-\infty$ to E_{rp}, surrounding the end point $E_{rp}+i\eta$ of the one branch cut, and running back to $-\infty$ just above both parts of the branch cuts of $H(z)$ and $L(z-E_{rp})$ which lie above the real axis. This is shown in Fig. 2. Making use of the fact that H and L go over into their complex conjugate values if we cross their branch cuts, we encounter two different contributions to the z integral. The first is given by $1/\pi$ times an integral from $-\infty$ to 0 along the real axis with the integrand $\mathrm{Im}[H(x)L(x-E_{rp})]$, and thus it is real. The second contribution is equal $1/\pi$ times an integral from 0 to E_{rp} along the real axis with integrand $H(x)\,\mathrm{Im}L(x-E_{rp})$. We can assume, in the limits of the desired accuracy, that $-\mathrm{Im}[E_{p-q}(x)/Z_{p-q}(x)]=\eta(x)$ and $-\mathrm{Im}[\omega_{\mathrm{pl}}^2/K(q,x)]=\epsilon(x)$ are (positive) infinitesimals. Then we get, approximately,

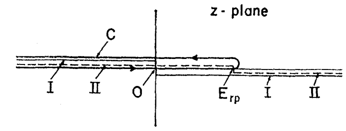

FIG. 2. The contour C of integration in the complex z plane. I (solid line) and II (dashed line) represent schematically the branch cuts corresponding to the particle and phonon propagator, respectively. E_{rp} is the quasi-particle energy.

$$\mathrm{Im}\int_{-\infty}^{\infty}\frac{dz}{2\pi i}H(z)L(z-E_{rp})\approx\pi\int_0^{E_{rp}}dx\frac{\mathrm{Re}w(x)}{|K(q,x-E_{rp})|^2[\mathrm{Re}Z_{p-q}(x)]^2E_p}\times\delta\left[x^2-\left(\mathrm{Re}\frac{E_{p-q}(x)}{Z_{p-q}(x)}\right)^2\right]\delta\left[(x-E_{rp})^2-\mathrm{Re}\frac{\omega_{\mathrm{pl}}^2}{K(q,x-E_{rp})}\right].$$

$\delta(x)$ is the delta function, and $w(x)$ is an abbreviation for the numerator of $H(x)$ in Eq. (3.1a). Since the zero of the function $\{x-\mathrm{Re}[E_{p-q}(x)/Z_{p-q}(x)]\}$ is $x=E_{rp-q}$, and that of the function $\{y+[\omega_{\mathrm{pl}}(\mathrm{Re}K(q,y))^{\frac{1}{2}}/|K(q,y)|]\}$ is $y=-\Omega_q$, we derive from Eqs. (3.1) and (3.1a) and from the last expression the result

$$2\Gamma_p^{\mathrm{ph}}=\int\frac{d^3q}{(2\pi)^3}\frac{\pi v_q\omega_{\mathrm{pl}}^2}{\Omega_q|K(q,\Omega_q)|^2}\frac{1}{2}\left(1+\frac{\bar{\epsilon}_p\bar{\epsilon}_{p-q}-\phi_p\phi_{p-q}}{E_pE_{p-q}}\right)\frac{\delta(E_{rp}-E_{rp-q}-\Omega_q)}{Z_pZ_{p-q}A_pA_{p-q}B_q}.\quad(3.2)$$

The notation is given by Eqs. (2.23). Evidently the transition probability from the particle state $\mathbf{p}$ into the particle state $(\mathbf{p}-\mathbf{q})$ plus the phonon state $\mathbf{q}$ is equal to $(2\pi)^3$ times the integrand of the right-hand side of Eq. (3.2). The term in brackets multiplied by $\frac{1}{2}$ is the well-known coherence factor valid for this process. The correction terms Z, A, and B in the denominator, which are given by Eqs. (2.23) and (2.28), are result of the use of Nambu's formulation of the theory.

Γ_p^{ph} is evaluated from the exact expression in Eq. (3.2) to the lowest order approximation, by which we mean that ζ_p and χ_p are set equal to zero and therefore Z_p, A_p equal to one and $\bar{\epsilon}_p$ equal to ϵ_p. Further ϕ_p is taken to be the constant BCS energy-gap value, ϵ_0, and B_q is set equal to one. The integration variable

$\vartheta = \sphericalangle(\mathbf{p},\mathbf{q})$ is changed into $E'=E_{p-q}$, and the variable $q=|\mathbf{q}|$ into Ω_q. One can show from the argument of the delta function, that $\Omega_q \leq (E_{rp}-\epsilon_0)$ is a necessary condition and $\Omega_q > 2(c/v_F)(E_{rp}-\epsilon_0)$ is a sufficient condition for the integral over E' to be different from zero. v_F is the velocity of the electrons at the Fermi surface, and c is the velocity of sound. We shall neglect the latter condition since c/v_F is much smaller than one, and furthermore, since the low frequencies Ω_q are screened out anyway by the dielectric constant. The dE' integration can be carried out with the help of the delta function. The term with $\bar{\epsilon}_{p-q}$ in the coherence factor drops out, because both signs of $\bar{\epsilon}_{p-q}$ are encountered. Approximating q by Ω_q/c, we obtain

$$2\Gamma_p^{\mathrm{ph}} = \frac{e^2 m \omega_{\mathrm{pl}}^2}{p} \int_0^{\min(\Omega_m;\, E_{rp}-\epsilon_0)} \frac{d\Omega}{\Omega^2 |K(\Omega/c,\Omega)|^2} \frac{(E_{rp}-\Omega)}{[(E_{rp}-\Omega)^2-\epsilon_0^2]^{\frac{1}{2}}} \left(1-\frac{\epsilon_0^2}{E_{rp}(E_{rp}-\Omega)}\right). \tag{3.3}$$

Ω_m is the maximum phonon frequency. The absolute square of the dielectric constant at $q_0=\Omega_q$ is, apart from terms of order $(c/v_F)^2$, equal to the square of the real part at $q_0=0$. We will approximate $\mathrm{Re}K(q,0)$ by the Thomas-Fermi value, that is $\mathrm{Re}K(q,0)=[1+(\omega_{\mathrm{pl}}/cq)^2]$, which leads to $\Omega_q=cq$ for small values of q according to Eq. (2.26). Further, we can set $\mathrm{Re}K(q,0)\approx(\omega_{\mathrm{pl}}/cq)^2$, if we consider values of E_{rp} close to ϵ_0 and neglect terms of order $(E_{rp}/\omega_{\mathrm{pl}})^2$ in comparison to one. Under this restriction the $d\Omega$ integration in Eq. (3.3) is carried out easily to give

$$2\Gamma_p^{\mathrm{ph}} = (\alpha r_s)\left(\frac{\epsilon_0}{\omega_{\mathrm{pl}}}\right)^2 \epsilon_0 f\left(\frac{E_{rp}}{\epsilon_0}\right), \tag{3.4}$$

where

$$f(x) = \left(\tfrac{1}{3}x^2+\frac{13}{6}\right)(x^2-1)^{\frac{1}{2}} - \left(2x+\frac{1}{2x}\right)\ln[x+(x^2-1)^{\frac{1}{2}}], \tag{3.5}$$

and where $(e^2m/p_F)=\alpha r_s$, $\alpha=(4/9\pi)^{\frac{1}{3}}$, and r_s is the Wigner-Seitz radius in Bohr units. For $x\gg1$ the function $f(x)$ tends to the normal state value $\frac{1}{3}x^3$. If we set $x=(1+y^2)^{\frac{1}{2}}$, and if $f[(1+y^2)^{\frac{1}{2}}]$ is expanded in terms of y for $y<1$, one gets

$$f[(1+y^2)^{\frac{1}{2}}] = \frac{8}{105}y^7 - \frac{224}{2205}y^9 + \cdots. \tag{3.6}$$

This differs from the result derived by Eliashberg,[10] the reason being, that the coherence factor does not appear in his formulas. In fact by neglecting the term with ϵ_0^2 in the brackets of the integrand in Eq. (3.3) we obtain a function $f[(1+y^2)^{\frac{1}{2}}]$ which has an expansion for $y=1$ starting with $(2/15)y^5$, and this gives the Eliashberg result. We conclude from Eqs. (3.4) and (3.6) that Γ_p^{ph} tends to zero as $(E_{rp}-\epsilon_0)^{7/2}$ when E_{rp} approaches ϵ_0. It seems natural to compare the decay of an elementary excitation in the superconducting state to that in the normal state by considering both decay rates as a function of the energy available for phonon creation. This energy, which we measure in units of the gap energy, is given by $w=(E_{rp}-\epsilon_0)/\epsilon_0$ for the superconducting state, and by $w=\bar{\epsilon}_p/\epsilon_0$ for the normal state. In terms of this available energy w the ratio of the decay rates in the superconducting and normal states is given by $f(1+w)/\frac{1}{3}w^3$. This ratio is plotted as a function of w in Fig. 3. It is interesting that a maximum occurs at w approximately 1.4; this maximum is about 1.42.

Now we shall determine the contribution to the decay rate of a particle due to excitation of the medium via Coulomb interaction. The lowest order contribution, denoted by Γ_p^{Coul}, is obtained from the lowest order self-energy terms of Eqs. (2.24), (2.25) by inserting only the Coulomb interaction part of $D(q)$, i.e., $v_q/K(q)$. The general expression for Γ_p^{Coul} is identical to that for Γ_p^{ph}, Eq. (3.1), except that the function $L(z-E_{rp})$ is replaced now by $(1/\omega_{\mathrm{pl}}^2)[K(q,z-E_{rp})]^{-1}$. Since the branch cut of this new function is identical to that of the previous one, the z integration can be carried out in complete analogy to the case of Γ_p^{ph} by deforming the contour. One notices, however, that in the case of the term containing $zZ_{p-q}(z)$ in the numerator of $H(z)$ [Eq. (3.1a)] the contour integral over the large semicircle in the upper half of the z plane does not vanish as it should. This difficulty is overcome by subtracting the term $(1/\omega_{\mathrm{pl}}^2)$ from $L(z-E_{rp})$ providing thus an additional factor $v_q\Lambda(q, z-E_{rp})$ in the numerator. This procedure does not alter the value of the original z integral since $zZ_{p-q}(z)$ is an odd function of z. The

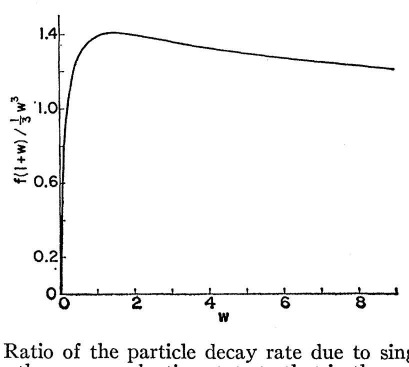

FIG. 3. Ratio of the particle decay rate due to single phonon emission in the superconducting state to that in the normal state, as a function of available energy for phonon creation. This ratio is given by $f(1+w)/\frac{1}{3}w^3$, where $w=(E_{rp}-\epsilon_0)/\epsilon_0$ for the numerator function and $w=\bar{\epsilon}_p/\epsilon_0$ for the denominator function. E_{rp} is the quasi-particle energy, ϵ_0 is the gap energy, and ϵ_p is the normal state energy of an excitation relative to the ground state.

general result for $2\Gamma_p^{\text{Coul}}$ turns out to be

$$2\Gamma_p^{\text{Coul}}=-\int\frac{d^3q}{(2\pi)^3}\frac{2v_q^2\,\text{Im}[\Lambda(q,E_{rp}-E_{rp-q})]\Theta(E_{rp}-E_{rp-q})}{|K(q,E_{rp}-E_{rp-q})|^2Z_pZ_{p-q}A_pA_{p-q}}\frac{1}{2}\left(1+\frac{\bar{\epsilon}_p\bar{\epsilon}_{p-q}-\phi_p\phi_{p-q}}{E_pE_{p-q}}\right). \tag{3.7}$$

$\Theta(x)$ is the step function, which equals one for $x>0$ and zero for $x<0$. The general result for $\text{Im}\Lambda(q,q_0)$ in the pair approximation will be derived later [see Eqs. (2.27) and (3.23)]. In lowest-order approximation ($Z_p=1$, $A_p=1$, $\bar{\epsilon}_p=\epsilon_p$) and under the assumption $\phi_p=\epsilon_0$ this quantity is determined by

$$\text{Im}\Lambda(q,q_0)=-\pi\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\left(1-\frac{\epsilon_{p+q}\epsilon_p-\epsilon_0^2}{E_{p+q}E_p}\right)\delta(E_{p+q}+E_p-q_0). \tag{3.8}$$

We are interested here in values of q_0, which exceed the threshold for pair production, $2\epsilon_0$, by energies of the order ϵ_0 only. Under this and the further restriction that $q/p_F>2(q_0-\epsilon_0)/E_F$, it is easy to evaluate the integral Eq. (3.8). We merely quote the result:

$$\text{Im}\Lambda(q,q_0)=-\frac{m^2}{2\pi q}\left[(q_0+2\epsilon_0)E(k)-\frac{4\epsilon_0q_0}{q_0+2\epsilon_0}K(k)\right] \tag{3.9}$$

for $q_0>2\epsilon_0$ and zero for $q_0<2\epsilon_0$, with

$$k=(q_0-2\epsilon_0)/(q_0+2\epsilon_0). \tag{3.9a}$$

$K(k)$ ane $E(k)$ are Legendre's complete elliptic integrals of the first and second kind. Expanding these elliptic integrals in power series in k^2 and retaining the zero- and first-order terms only we have

$$\text{Im}\Lambda(q,q_0)=-\frac{m^2}{4q}\left(\frac{q_0^2+4\epsilon_0^2}{q_0+2\epsilon_0}\right)+\frac{m^2}{16q}\left(\frac{q_0-2\epsilon_0}{q_0+2\epsilon_0}\right)^2\left(q_0+2\epsilon_0+\frac{4\epsilon_0q_0}{q_0+2\epsilon_0}\right)+\cdots. \tag{3.10}$$

Now the integral in Eq. (3.7) is evaluated to lowest-order approximation and by setting $\phi_p=\epsilon_0$. The polar angle integration is transformed into a dE' integration. For wave numbers q with $q/p_F>2(E_{rp}-\epsilon_0)/E_F$ (E_F is the Fermi energy) the limits of this integration become ϵ_0 and $(E_{rp}-2\epsilon_0)$, and the term with $\bar{\epsilon}_{p-q}$ in the coherence factor drops out because both signs of this quantity appear. The above restriction put on the wave number q also allows us to use the expression in Eq. (3.9) for $\text{Im}\Lambda(q,q_0)$, and furthermore, to replace the absolute square of the dielectric constant at $q_0=(E_{rp}-E_{rp-q})$ by the square of the real value at $q_0=0$. We can neglect the restriction placed on q to a good degree of accuracy since we shall consider values of E_{rp} exceeding the threshold $3\epsilon_0$ for particle damping by pair production by energies of order ϵ_0 only, while the upper limit of the dq integration is $2p_F$. If E_{rp} is so small that $(E_{rp}-3\epsilon_0)^2/(E_{rp}+3\epsilon_0)^2\ll 1$, then $\text{Im}\Lambda(q,q_0)$ can be approximated by the zeroth order term of its expansion in Eq. (3.10), and the dE' integration can be carried out. The dq integration from 0 to $2p_F$ provides simply a factor, independent of E_{rp}, which is estimated by setting $\text{Re}K(q,0)=[1+(2\alpha r_s/\pi)(p_F/q)^2]$. An additional factor $\frac{1}{2}$ in the second term of this expression is introduced in comparison to the corresponding Thomas-Fermi expression. This factor accounts, in a rough way, for the much lower values of the actual dielectric constant in comparison to those given by the Thomas-Fermi approximation for wave numbers q close to the upper limit $2p_F$ of the dq integration. The final result is

$$2\Gamma_p^{\text{Coul}}=\frac{\pi^2}{4}\left[\left(\frac{\alpha r_s}{2\pi}\right)^{\frac{1}{2}}\tan^{-1}\left(\frac{2\pi}{\alpha r_s}\right)^{\frac{1}{2}}+\frac{\alpha r_s}{2\pi+\alpha r_s}\right]\times\frac{\epsilon_0^2}{E_F}g\left(\frac{E_{rp}}{\epsilon_0}\right). \tag{3.11}$$

The function $g(x)$ is different from zero only for $x>3$, and then it is given by

$$g(x)=\left(\frac{x}{2}-1+\frac{1}{x}\right)[(x-2)^2-1]^{\frac{1}{2}}-\left(\frac{19}{2}-\frac{2}{x}\right)\times\ln\{x-2+[(x-2)^2-1]^{\frac{1}{2}}\}+\frac{8(x+2-x^{-1})}{[(x+2)^2-1]^{\frac{1}{2}}}\times\ln\tfrac{1}{4}\{x^2-5+[(x+2)^2-1]^{\frac{1}{2}}[(x-2)^2-1]^{\frac{1}{2}}\}. \tag{3.12}$$

For $x\gg 3$ the function $g(x)$ tends to the normal state value $\frac{1}{2}x^2$, which agrees with the result of DuBois.[4]

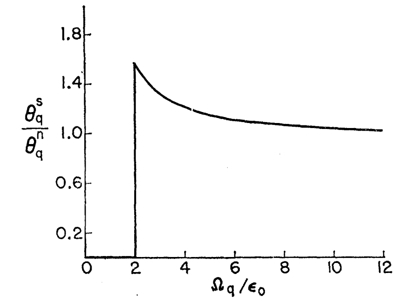

FIG. 4. Ratio of the phonon decay rate due to decay into single pairs in the superconducting state to that in the normal state, as a function of phonon frequency in units of the energy gap, Ω_q/ϵ_0. The difference of the phonon frequencies in the superconducting and normal states is neglected. The ratio is denoted by Θ_q^s/Θ_q^n.

The comparison of the particle damping caused by pair production with that due to emission of phonons will be made in section IV after the second order correction to Γ_p^{Coul} has been calculated (see the results in Fig. 5).

We determine also the level width of a phonon, $2\Theta_q$, caused by decay into single pairs of particles. Θ_q is given by Eqs. (2.27) and (2.29) with $q_0=\Omega_q$. The dp_0 integration in the expression Eq. (2.29) for the dielectric constant can be done in a way analogous to the procedure employed in the case of Γ_p^{ph} and Γ_p^{Coul}. Here we recognize that the integrand is a sum of products of two functions, where the first one has a branch cut running from $-\infty+i\eta$ to $-q_0+i\eta$, and then from $-q_0-i\eta$ to $\infty-i\eta$, and where the second one has a branch cut running from $-\infty+i\delta$ to $0+i\delta$ and from $0-i\delta$ to $\infty-i\delta$ (η and δ are positive infinitesimals). Therefore the contour over the real axis can be deformed in a contour enclosing both parts of the two branch cuts which lie above the real axis, for instance. Thus the imaginary part of the dp_0 integral is given by $1/\pi$ times an integral from $-q_0$ to 0 along the real axis, the integrand being the product of the imaginary parts of the two functions. The general result for Θ_q turns out to be

$$2\Theta_q=\frac{\pi v_q\omega_{\mathrm{pl}}^2}{|K(q,\Omega_q)|^2\Omega_q B_q}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\left(1-\frac{\bar{\epsilon}_{p+q}\bar{\epsilon}_p-\phi_{p+q}\phi_p}{E_{p+q}E_p}\right)\times\frac{\delta(E_{rp+q}+E_{rp}-\Omega_q)}{Z_{p+q}Z_pA_{p+q}A_p}. \quad (3.13)$$

The transition probability from the phonon state $\mathbf{q}$ into the pair of particles state $(\mathbf{p}+\mathbf{q})$ and $-\mathbf{p}$ is given by $(2\pi)^3$ times the integrand of Eq. (3.13). One recognizes the appearance of the coherence factor valid for this process, and the correction terms B, Z, and A in the denominator, which also were found in the case of particle damping. The integral in Eq. (3.13) has been evaluated earlier to the lowest-order approximation, the result being given by Eqs. (3.8) and (3.9) with $q_0=\Omega_q$. If we neglect the difference in phonon frequencies and dielectric constants in the superconducting and normal states, the ratio of the level widths of a phonon in the two states, Θ_q^s/Θ_q^n, as a function of $x=\Omega_q/\epsilon_0$, is found to be

$$\frac{\Theta_q^s}{\Theta_q^n}=\left(\frac{x+2}{x}\right)E\left(\frac{x-2}{x+2}\right)-\frac{4}{(x+2)}K\left(\frac{x-2}{x+2}\right) \quad (3.14)$$

for $x\geq 2$, and equal to zero for $x<2$. This ratio is plotted vs x in Fig. 4. It is interesting to notice that the level width of a phonon in the superconducting state rises like a step function to the value $(\pi/2)\Theta_q^n$ once the phonon energy reaches the threshold energy, $2\epsilon_0$, for pair production. This can be seen by taking the limit $q_0\rightarrow 2\epsilon_0$ in the expansion of $\mathrm{Im}\Lambda(q)$ in Eq. (3.10).

IV. SECOND-ORDER EFFECTS ON PARTICLE DAMPING

The contributions to the particle decay rate, which arise from the second order self-energy diagram in Fig. 1 are obtainable from the general expression for Γ_p in Eq. (2.22) and from the second-order terms of the self-consistency conditions in Eqs. (2.24) and (2.25). The imaginary part of the double integral over the fourth components of the intermediate state momentum vectors q_1, q_2 can be written as a sum of expressions having the following form

$$\mathrm{Im}\int_{-\infty}^{+\infty}dy\exp(-iE_{rp}y)\times\prod_{j=1}^{2}\int_{-\infty}^{+\infty}\frac{dz_j}{(2\pi i)}\exp(iz_jy)H_j(z_j)L_j(z_j-E_{rp})\times\int_{-\infty}^{+\infty}\frac{dz_3}{(2\pi)}\exp(iz_3y)H_3(z_3). \quad (4.1)$$

The dy integration and the exponential functions come from the introduction of the well-known integral representation for the delta function $\delta(z_1+z_2+z_3-E_{rp})$. The function $H_j(z_j)$ $(j=1, 2, 3)$ has a branch cut in the complex z_j plane, which is identical to that of the particle propagator $G(\mathbf{p}-\mathbf{q}_j\, z_j)$ for $j=1, 2$ or $G(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2, z_3)$ for $j=3$. The function $L_i(z_i-E_{rp})$ $(i=1, 2)$ has a branch cut which is identical to that of the interaction propagator $D(\mathbf{q}_i, z_i-E_{rp})$. The dz_j contour along the real axis in Eq. (4.1) can be deformed for fixed y so that it encloses those parts of the branch cuts of functions H_j and L_j (or of H_3 alone for $j=3$) which lie above the real axis in case $y>0$ or below the real axis in case $y<0$. Using the fact that the functions H_j and L_i have discontinuities at their branch cuts equal to $2i\,\mathrm{Im}H_j$ or $2i\,\mathrm{Im}L_i$, respectively, one obtains then

from Eq. (4.1);

$$\operatorname{Im}\Big[\frac{i}{\pi^3}\int_0^\infty dy\,\exp(-iE_{rp}y)\prod_{j=1}^{2}\Big\{\int_{-\infty}^{0}dx_j\,\exp(ix_jy)\,\operatorname{Im}[L_j(x_j-E_{rp})H_j(x_j)]$$

$$+\int_0^{E_{rp}}dx_j\,\exp(ix_jy)H_j(x_j)\,\operatorname{Im}L_j(x_j-E_{rp})\Big\}\int_{-\infty}^{0}dx_3\,\exp(ix_3y)\,\operatorname{Im}H_3(x_3)$$

$$+\frac{i}{\pi^3}\int_{-\infty}^{0}dy\,\exp(-iE_{rp}y)\prod_{j=1}^{2}\Big\{\int_0^{E_{rp}}dx_j\,\exp(ix_jy)L_j(x_j-E_{rp})\,\operatorname{Im}H_j(x_j)$$

$$+\int_{E_{rp}}^{\infty}dx_j\,\exp(ix_jy)\,\operatorname{Im}[L_j(x_j-E_{rp})H_j(x_j)]\Big\}\int_0^\infty dx_3\,\exp(ix_3y)\,\operatorname{Im}H_3(x_3)\Big]. \quad (4.2)$$

Let us consider first the contributions which arise from the terms $\operatorname{Im}(H_j)\operatorname{Re}(L_j)$ $(j=1, 2)$, and let us use the fact that $\operatorname{Im}H_j(x_j)$ $(j=1, 2, 3)$ gives rise to a delta function $\delta(x_j^2-E_{rj}^2)$. Then we obtain a result, which contains the delta function $\delta(E_{rp-q1}+E_{rp-q2}+E_{rp-q1-q2}-E_{rp})$ as a factor. Obviously this delta function corresponds to a time ordering of the second-order self-energy diagram where a symmetrical cut is possible and leads to a final state containing a particle pair with momenta $(\mathbf{p}-\mathbf{q}_2)$, $(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2)$ and the original particle with momentum $(\mathbf{p}-\mathbf{q}_1)$. Thus, we obtain a correction to Γ_p^{Coul}, which we will denote by $\Gamma_p^{(2x)}$. If we include solely the Coulomb interaction part of $D(q)$ into $\operatorname{Re}(L_j)$, we obtain the following contribution to the decay rate:

$$2\Gamma_p^{(2x)\text{Coul}}=-\frac{\pi}{4}\int\int\frac{d^3q_1d^3q_2}{(2\pi)^6}\frac{v_{q_1}v_{q_2}\delta(E_{r1}+E_{r2}+E_{r3}-E_{rp})}{\operatorname{Re}K(q_1, E_{rp}-E_{r1})\operatorname{Re}K(q_2, E_{rp}-E_{r2})Z_pZ_1Z_2Z_3A_pA_1A_2A_3}$$

$$\times\Big\{\Big[1-\frac{\bar\epsilon_1\bar\epsilon_3-\phi_1\phi_3}{E_1E_3}-\frac{\bar\epsilon_2\bar\epsilon_3-\phi_2\phi_3}{E_2E_3}+\frac{\bar\epsilon_1\bar\epsilon_2+\phi_1\phi_2}{E_1E_2}\Big]+\frac{\bar\epsilon_p}{E_p}\Big[\frac{\bar\epsilon_1}{E_1}+\frac{\bar\epsilon_2}{E_2}-\frac{\bar\epsilon_3}{E_3}-\frac{\bar\epsilon_1\bar\epsilon_2\bar\epsilon_3-\bar\epsilon_1\phi_2\phi_3-\bar\epsilon_2\phi_1\phi_3-\bar\epsilon_3\phi_1\phi_2}{E_1E_2E_3}\Big]$$

$$-\frac{\phi_p}{E_p}\Big[\frac{\phi_1}{E_1}+\frac{\phi_2}{E_2}+\frac{\phi_3}{E_3}-\frac{\bar\epsilon_1\bar\epsilon_3\phi_2+\bar\epsilon_1\bar\epsilon_2\phi_3+\bar\epsilon_2\bar\epsilon_3\phi_1+\phi_1\phi_2\phi_3}{E_1E_2E_3}\Big]\Big\}. \quad (4.3)$$

Subscripts 1, 2, and 3 denote vectors $(\mathbf{p}-\mathbf{q}_1)$, $(\mathbf{p}-\mathbf{q}_2)$, and $(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2)$, respectively, and otherwise the notation is that of Eqs. (2.21) and (2.23). We evaluate this expression by setting $Z_p=1$, $A_p=1$, $\bar\epsilon_p=\epsilon_p$, and $\phi_p=\epsilon_0$. Integrations over the polar angles ϑ_1, ϑ_2 and the azimuth difference $\varphi=\varphi_1-\varphi_2$ of vectors $\mathbf{q}_1$, $\mathbf{q}_2$ with respect to $\mathbf{p}$ as polar axis are transformed into dE_1, dE_2, and dE_3 integrations, respectively. From the argument of the delta function we see that the dE_1, dE_2 integrations are restricted by the step function $\Theta(E_{rp}-E_1-E_2-\epsilon_0)$. First the dE_3 integration is carried out with the help of the delta function. One can show by using the condition $E_1<E_{rp}$, $E_2<E_{rp}$, and the expression

$$\epsilon_3=\epsilon_1+\epsilon_2-\epsilon_p+\frac{q_1q_2}{m}(\cos\vartheta_1\cos\vartheta_2+\sin\vartheta_1\sin\vartheta_2\cos\varphi), \quad (4.4)$$

that the dE_3 integration always gives a result different from zero, and that it provides both signs of ϵ_3, if the wave numbers q_i $(i=1, 2)$ lie inside the interval from $2p_F(E_{rp}/E_F)^{\frac{1}{2}}$ to $\sqrt{2}p_F$. Since we shall consider values of the particle energy E_{rp} which exceed the threshold for pair production, $3\epsilon_0$, by an energy of order ϵ_0 only, these conditions on q_i are fulfilled for the overwhelming parts of the dq_1, dq_2 integration intervals extending from 0 to $2p_F$. Also we recognize that under these restrictions on q_i the inequalities $E_{|p|\mp|q_i|}>E_{rp}$ hold. Therefore the lower limits of the dE_1, dE_2 integrations become ϵ_0, the upper limits are determined by the step function $\Theta(E_{rp}-E_1-E_2-\epsilon_0)$, and both signs of ϵ_1 and ϵ_2 occur symmetrically in these integrals. The number of dE_1, dE_2 integrations is reduced considerably by the fact

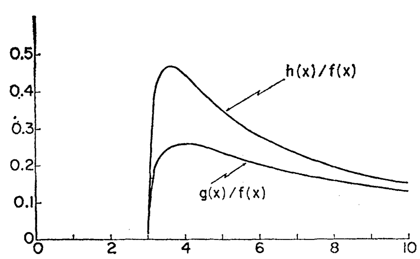

FIG. 5. Ratios $g(x)/f(x)$ and $h(x)/f(x)$ versus $x=E_{rp}/\epsilon_0$, where E_{rp} is the quasi-particle energy and ϵ_0 the gap energy. These ratios determine the energy dependence of the ratios $\Gamma_p^{\text{Coul}}/\Gamma_p^{\text{ph}}$ and $\Gamma_p^{(2x)\text{Coul}}/\Gamma_p^{\text{ph}}$, respectively. Γ_p^{Coul} and $\Gamma_p^{(2x)\text{Coul}}$ are the first and second order contributions to the particle decay rate which arise from emission of single pairs via Coulomb interaction, and Γ_p^{ph} is the decay rate due to single phonon emission. All contributions refer to the superconducting state.

that all terms linear in $\bar{\epsilon}_1$, $\bar{\epsilon}_2$, and $\bar{\epsilon}_3$ in the integrand of Eq. (4.3) drop out. The dq_1dq_2 integral provides a factor, independent of E_{rp}, which is estimated by neglecting the restrictions on the wave numbers q_i under which the above considerations were made, and further, by using the same approximation for the dielectric constant as that used in the case of $\Gamma_p{}^{\mathrm{Coul}}$. Under these approximations we get the final result

$$2\Gamma_p{}^{(2x)\,\mathrm{Coul}}=-\tfrac{1}{4}(\alpha r_s)\left[\tan^{-1}\left(\frac{2\pi}{\alpha r_s}\right)^{\frac{1}{2}}\right]^2\frac{\epsilon_0{}^2}{E_F}h\left(\frac{E_{rp}}{\epsilon_0}\right). \quad (4.5)$$

The function $h(x)$ is different from zero only if $x>3$, and then it equals

$$h(x)=\left(\frac{x}{2}-1-\frac{3}{x}\right)[(x-2)^2-1]^{\frac{1}{2}}-\left(\frac{11}{2}+\frac{6}{x}\right)$$
$$\times\ln\{x-2+[(x-2)^2-1]^{\frac{1}{2}}\}+\frac{4(2x+4+3x^{-1})}{[(x+2)^2-1]^{\frac{1}{2}}}$$
$$\times\ln\tfrac{1}{4}\{x^2-5+[(x+2)^2-1]^{\frac{1}{2}}[(x-2)^2-1]^{\frac{1}{2}}\}. \quad (4.6)$$

If we include solely the phonon propagator in $\mathrm{Re}(L_i)$, we obtain a contribution $\Gamma_p{}^{(2x)\mathrm{ph}}$. The essential difference between the expressions for $\Gamma_p{}^{(2x)\,\mathrm{Coul}}$ [see Eq. (4.3)] and $\Gamma_p{}^{(2x)\mathrm{ph}}$ is that the former contains the propagators for intermediate virtual strings of pairs and the latter contains the propagators for intermediate virtual phonons. Evaluation of $\Gamma_p{}^{(2x)\mathrm{ph}}$ gives once more a dependence on the excitation energy which is governed by the function $h(E_{rp}/\epsilon_0)$ and a factor which is about 0.3 of that occurring in Eq. (4.5) if $(\alpha r_s)\approx 2$.

We are now in a position where we can compare the over-all decay rate of a particle due to emission of single pairs with that due to emission of single phonons. From the Eqs. (3.11), (4.5), and (3.4) we obtain ratios in the form

$$\begin{aligned}\Gamma_p{}^{\mathrm{Coul}}/\Gamma_p{}^{\mathrm{ph}}&=(\omega_{\mathrm{pl}}{}^2/E_F\epsilon_0)A_1(\alpha r_s)[g(x)/f(x)],\\ \Gamma_p{}^{(2x)\,\mathrm{Coul}}/\Gamma_p{}^{\mathrm{ph}}&=-(\omega_{\mathrm{pl}}{}^2/E_F\epsilon_0)A_2(\alpha r_s)[h(x)/f(x)],\end{aligned} \quad (4.7)$$

where $x=E_{rp}/\epsilon_0$. The values of the factors $A_1(\alpha r_s)$ and $A_2(\alpha r_s)$ are about 1 and $\frac{1}{4}$ if (αr_s) is set equal to 2, for instance. The factor $(\omega_{\mathrm{pl}}{}^2/E_F\epsilon_0)$ is found to be maximal for superconductors like tin and tantalum, for which the values are about 0.1. We see from Fig. 5 that the functions $g(x)/f(x)$ and $h(x)/f(x)$ have maxima at about $x=4.1$ and $x=3.5$, respectively; these maxima are approximately 0.26 and 0.48, respectively. From these results we conclude that particle damping due to single pair emission is at most only a few per cent of that due to single phonon emission.

We have estimated also the effect of successive emission of two phonons on the decay rate of a particle. The terms responsible for this correction are $\mathrm{Re}(H_i)\,\mathrm{Im}(L_i)$ $(i=1,2)$ appearing in Eq. (4.2), where in L_i only the phonon part is included. The $d^3q_1d^3q_2$ integral which determines this contribution, $\Gamma_p{}^{2\mathrm{ph}}$, to the decay rate contains essentially a delta function $\delta(E_{rp}-\Omega_{q1}-\Omega_{q2}-E_{rp-q1-q2})$, which ensures energy conservation for the processes in question, and denominators, which come from the propagators for virtual particles with momenta $(\mathbf{p}-\mathbf{q}_i)$. Recall that in evaluating $\Gamma_p{}^{(2x)\,\mathrm{Coul}}$ we made use of the fact that the wave numbers q_i were allowed to extend to relatively high values, namely, $2p_F$, while the energies E_{rp-qi} were restricted to values smaller than $(E_{rp}-\epsilon_0)$. Now the situation is reversed, because according to the argument of the delta function the phonon frequencies Ω_{qi} must be smaller than $(E_{rp}-\epsilon_0)$, while no restrictions are placed on the E_{rp-qi}. By using this in the expression for ϵ_3 in Eq. (4.4), one recognizes that for values of $(E_{rp}-\epsilon_0)$ of order ϵ_0, one can make the approximations $\epsilon_3=\epsilon_1+\epsilon_2-\epsilon_p$ and $\epsilon_i=\epsilon_p-v_Fq_i\cos\vartheta_i$. [Recall our notation: subscripts 1, 2, 3 for vectors $(\mathbf{p}-\mathbf{q}_1)$, $(\mathbf{p}-\mathbf{q}_2)$, $(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2)$, respectively; $\vartheta_i=\measuredangle(\mathbf{p},\mathbf{q}_i)$; $\varphi=$azimuth difference of vectors $\mathbf{q}_1$, $\mathbf{q}_2$.] Thus the $d\varphi$ integration becomes trivial. Next the $d\vartheta_2$ integration, for instance, is transformed in a dE_3 integration and is carried out with the help of the delta function. One can show that the result is always different from zero and provides both signs of ϵ_3, if the condition $\Omega_{q1}<\Omega_{q2}$ is fulfilled. For this reason we multiply the original integrand by the step function $2\Theta(\Omega_{q2}-\Omega_{q1})$, which leaves the integral unchanged because the integrand is symmetric in Ω_{q1} and Ω_{q2}. Now the $d\vartheta_1$ integration is transformed into a $d\epsilon_1$ integration, and the integration is performed. The result is expanded in a power series in c/v_F, and finally the $d\Omega_{q1}$ and $d\Omega_{q2}$ integrations are carried out for the lowest order term. Then we find that the ratio $\Gamma_p{}^{2\mathrm{ph}}/\Gamma_p{}^{\mathrm{ph}}$ has an order of magnitude given by $(\epsilon_0/\omega_D)^2(c/v_F)$. Thus we can say that two-phonon emission has a completely negligible effect on particle decay in comparison to single phonon emission if the excess excitation energy of the particle over the gap, $(E_{rp}-\epsilon_0)$, is of the order of the gap energy.

V. CONCLUDING REMARKS

In the preceding sections we have seen that single phonon emission yields the overwhelming contribution to the decay rate of a quasi-particle. The comparison of this dominant contribution to decay in the superconducting state, $\Gamma_p{}^{\mathrm{ph},s}$, to that in the normal state, $\Gamma_p{}^{\mathrm{ph},n}$, has been made by considering both contributions as a function of the excess energy which is available for creation of phonons. We find that the ratio $\Gamma_p{}^{\mathrm{ph},s}/\Gamma_p{}^{\mathrm{ph},n}$ is always greater than one except for excess energies which are smaller than about 0.5 gap energies, and that it exhibits a maximum with a value of approximately 1.4 at an excess energy of about 1.4 gap energies (see Fig. 3).

The values obtained for $\Gamma_p{}^{\mathrm{ph},s}$ have been determined from an exact integral formulation [see Eq. (3.2)] by making the following approximations. First, we have

neglected those self-energy terms which are also present in the normal state, i.e., ζ_p and χ_p. Second, we have employed the normal state dielectric constant. Third, the energy gap, i.e., ϕ_p, has been approximated by the constant BCS value ϵ_0. The errors in the ratio $\Gamma_p^{\mathrm{ph},s}/\Gamma_p^{\mathrm{ph},n}$ which have been introduced by making the first two of these approximations are insignificant because they turn out to be of order $(\epsilon_0/\omega_D)^2$. However, a consideration of the dependence of the gap on the energy and the resulting modification in the density of states might give rise to a significant change in this ratio.

It also would be of great interest, especially with regard to the theory of thermal conductivity, to see how the ratio $\Gamma_p^{\mathrm{ph},s}/\Gamma_p^{\mathrm{ph},n}$ is altered when we go to finite temperatures. This calculation will be presented as part of a forthcoming investigation which is based on an extension of Nambu's self-consistency conditions to the case of finite temperatures.

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Critical Fields of Thin Superconducting Films. I. Thickness Effects

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A theoretical model is presented with which the critical magnetic fields of thin superconducting films can be calculated from any theory of superconductivity for which the kernel of the current-vector potential relationship is known. The model is worked out in detail for the nonlocal theory of Pippard with specular boundary conditions, and the critical field is shown to be a function of film thickness and the nonlocal parameters ξ and $\xi_0\lambda_L^2$. The results are compared to critical-field data for pure indium films and are found to predict very well the observed thickness dependence of critical field. On the basis of reasonable assumptions, ξ_0 and $\lambda_L(0)$ are calculated from the indium critical field data to be 2600 ± 400 Å and 350 ± 30 Å, respectively.

INTRODUCTION

To interpret critical magnetic field measurements on superconducting films, a theory is needed which includes both strong-field effects and nonlocal effects—strong-field, to describe phenomena occurring at the critical field; nonlocal to adequately describe thickness and mean-free-path effects. Such a theory does not exist at present. It is the purpose of this paper to show how the critical fields of superconducting films can be related to the nonlocal microscopic parameters by the use of the Ginzburg-Landau theory[1] together with the nonlocal theories. The general scheme is as follows: Using the Ginzburg-Landau results, the critical field of a film is related to its susceptibility in a weak magnetic field. Using the nonlocal calculations of Schrieffer,[2] the weak-field susceptibility is related to the nonlocal parameters. Combining the theoretical expressions, the film critical field can be expressed directly in terms of the nonlocal parameters. The resulting model is compared to critical field data for pure indium films and is shown to be in good agreement. Because of the purity of these films, mean-free-path effects are unimportant and the detailed discussion of the theoretical model is limited to thickness effects. In a subsequent paper, mean-free-path effects will be discussed in detail and the results will be compared to critical-field data for alloy films.

THEORETICAL MODEL

For films thin enough so that the order parameter ψ_0 can be considered constant over the thickness of the film, Eqs. (61) and (62) of Ginzburg-Landau[1] give the following expressions for the film critical field:

$$(h_c/H_c)^2=\psi_0^2(2-\psi_0^2)/[1-(1/\eta)\tanh\eta], \tag{1}$$

and

$$(h_c/H_c)^2=[4\psi_0^2(\psi_0^2-1)\cosh^2\eta]/[1-(1/2\eta)\sinh 2\eta], \tag{2}$$

where

$$\eta\equiv\psi_0 a/\delta_0. \tag{3}$$

The quantity a is the film half-thickness, h_c is the film critical field, H_c is the bulk critical field, and δ_0 is the weak-field penetration parameter. For $h_c/H_c>1$, Eqs. (1) and (2) can be solved numerically to obtain a

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