Matrix Elements of the n-p Interaction and the Odd-Group Model for a Heavy Nucleus

MARTIN G. REDLICH
*704 Keeler Avenue, Berkeley, California**

(Received December 15, 1961)

In the double-closed-shell-core model, the interaction between each pair of outer nucleons is similar to that between two free nucleons. It appears possible that the wave functions for low states of odd-A *heavy* nuclei obtained with this model are generally approximately those of the odd-group model. Energy matrices for a zero-range, charge-independent two-nucleon potential with triplet/singlet strength ratios $\rho=1$ and 1.5 were calculated for all states with $J=7/2$ and $9/2$ of the configuration $(1h_{9/2}p)^2J'(2g_{9/2}n)$ of two protons (p) and one neutron (n), for Po^{211}. In an expansion of the wave function of the lowest state, with $J=9/2$, the odd-group model wave function, characterized by $J'=0$, has amplitude 0.97 for $\rho=1$, and 0.92 for $\rho=1.5$. There are four higher states with $J=9/2$ and four with $J=7/2$. Their wave functions have large components of two or more states with $J'>0$, but small or zero components of the $J'=0$ state. A more general result is valid for certain special potentials, including the pure tensor potential.

I. INTRODUCTION

IN a usual, simple form of the j-j coupling shell model[1] of an odd-A nucleus, the angular momenta of the even nucleons couple to $J'=0$, and those of the odd nucleons couple to J, the total angular momentum of the nucleus. This form is often called the *odd-group model*. Early calculations by Mayer and Racah showed that if a zero-range interaction is assumed between each pair of β odd nucleons in a configuration j^β, then the lowest state has $J=j$. Furthermore, the addition of one nucleon to $\beta-1$ even nucleons does not add to the interaction energy of the lowest state.[1,2] One sometimes speaks of a *single-particle model*, neglecting both the even nucleons and $\beta-1$ nucleons of the odd type.

In the more detailed *double-closed-shell-core model*, the set of outer-nucleon states contains all states of one main proton shell and all states of one main neutron shell. For $A=19$, calculations have been made on the basis of this model,[3,4] assuming a potential $V(1,2)$ between each pair of the three outer nucleons. These calculations have led to wave functions far removed from those of the odd-group model.

It has long seemed puzzling, however, that the odd-group model appears to have empirical validity for heavy nuclei.[5] Apparently striking examples can be seen in the proton shell below Pb^{208}. The set of predicted single-hole proton states is:

$$(3s_{1/2})^{-1},\quad (2d_{3/2})^{-1},\quad (2d_{5/2})^{-1},\quad (1h_{11/2})^{-1},\quad (1g_{7/2})^{-1}.$$

Two examples are: (1) Seven odd Tl isotopes[6] with $A=195$ to 207, $N=114$ to 126, and $Z=81$. The data for each of six isotopes are consistent with the assignments

$$1/2^+,\quad 3/2^+,\quad 5/2^+,$$

for the lowest three observed states. In the seventh isotope, Tl^{207}, only two states have so far been found; their assignments are $(1/2^+)$ and $(3/2^+)$. Also (2): A similar situation appears for three isotopes of Au with $A=193$, 195, 197, $N=114$, 116, 118, and $Z=79$. All data[6] are consistent with the assignments

$$3/2^+,\quad 1/2^+,\quad 3/2^+,\quad 5/2^+,\quad 11/2^-,$$

for the lowest five observed states. Four of these states have the single-particle spin and parity predicted by the theory. In both examples, the energies of corresponding levels do not change much with N. It should be noted, however, that an attempt has recently been made to interpret excited states of some of the Tl isotopes on the basis of an excited-core+single-nucleon model.[7]

The experimental results suggest that for heavy nuclei some matrix elements of the n-p interaction may be small in comparison with those of the n-n or p-p interaction, and that the nuclear wave functions for many low states may be quite accurately those of the odd-group model. It has been questioned whether this is consistent with a charge-independent two-nucleon interaction.[8] The possibility that the n-p interaction

* Part of this investigation was made in the Department of Physics and Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts, on a National Science Foundation post-graduate fellowship, and in the Department of Physics, Washington University, St. Louis, Missouri. IBM computer calculations were sponsored by the Department of Physics, University of California, Berkeley, California.

[1] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley & Sons, Inc., New York, 1955); E. Feenberg, *Shell Theory of the Nucleus* (Princeton University Press, Princeton, New Jersey, 1955); earlier references are given in these two books.

[2] G. Racah and I. Talmi, Physica **18**, 1097 (1952).

[3] J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) **A229**, 536 (1955).

[4] M. G. Redlich, Phys. Rev. **99**, 1427 (1955).

[5] L. Eisenbud and E. P. Wigner, *Nuclear Structure* (Princeton University Press, Princeton, New Jersey, 1958); or *Handbook of Physics*, edited by E. U. Condon and H. Odishaw (McGraw-Hill Book Company, Inc., New York, 1958), Part 9, Chap. 1.

[6] K. Way, N. B. Gove, F. Everling, C. L. McGinnis, G. H. Fuller, and R. Nakasima, *Nuclear Data Sheets*, National Academy of Sciences, National Research Council (U. S. Government Printing Office, Washington, D. C.), including 1960, Set 1; K. Way, N. B. Gove, C. L. McGinnis, and R. Nakasima in the section for $A=21$ to 212 of H. H. Landolt and R. Börnstein, *Zahlenwerte und Funktionen aus Naturwissenschaften und Technik* (Springer-Verlag, Berlin, 1961), New Series, Vol. I/1.

[7] A. de-Shalit, Phys. Rev. **122**, 1530 (1961).

[8] Reference 5, Sec. 7.5, pp. 55–56.

may be weaker than the p-p or n-n interaction is also mentioned in a recent paper on the odd-odd nuclei.[9]

In the present paper, the results of a calculation for Po^{211} are given. The calculation is based upon a restricted double-closed-shell-core model, with only one proton state and one neutron state. The core is Pb^{208} and the configuration is $(1h_{9/2}p)^2J',(2g_{9/2}n)J$. In this notation p refers to a proton and n to a neutron. All matrix elements of two charge-independent zero-range two-nucleon interactions are calculated for this configuration with $J=9/2$ and $7/2$. The lowest state has $J=9/2$, and is predominantly the odd-group model state, with $J'=0$. The purpose of the calculation is to give some theoretical orientation about the validity of the odd-group model. It appeared best to start with the heaviest nuclei, in the trans-Pb region, even though fewer data are available in this region. No attempt will be made to account in detail for the data on Tl and Au, most of which require calculations with more complex configurations.

One result, however, is applicable to a configuration $(jp)^\alpha(j'n)^\beta$ with any even α, and odd β (Sec. V). There are certain potentials which, alone, differ appreciably from the two-nucleon potential, but which, within one configuration, lead to precisely the odd-group model wave function. Among these is the tensor interaction.

II. FRAMEWORK OF THE CALCULATION

Part of the discussion of this section is based upon that of the recent paper by Newby and Konopinski.[10,11] The nuclei Pb^{209} and Bi^{209} are single-particle nuclei even for the double-closed-shell-core model. The 83rd proton state is taken to be $1h_{9/2}$, and the 127th neutron state is assumed to be $2g_{9/2}$, consistent with the data[6] for the ground states of these single-particle nuclei.

In the present paper, harmonic oscillator wave functions with exponentials $\exp[-\frac{1}{2}(r/a)^2]$ are used. The parameter $\nu=a^{-2}$ is determined by

$$R(1h_{9/2})=\langle 1h_{9/2}|r^2|1h_{9/2}\rangle^{\frac{1}{2}}=1.2\times 210^{\frac{1}{3}}=7.13\text{ F}. \quad (1)$$

This leads to $\nu=0.12776\text{ F}^{-2}$. The maximum of the distribution $|r\psi(1h_{9/2})|^2$ vs r is at $r=6.85=1.152\times 210^{\frac{1}{3}}$ F. The probability/unit volume $|\psi(1h_{9/2})|^2$ is equal to 1/4 its maximum value[12] at $r=8.71=1.465\times 210^{\frac{1}{3}}$ F. The classical turning point for a $1h_{9/2}$ orbit is at $\sqrt{2}R(1h_{9/2})=10.09=1.697\times 210^{\frac{1}{3}}$ F. The same ν is used for the $2g_{9/2}$ wave function, and leads to a somewhat larger neutron radius: $R(2g_{5/2})=7.66$ F.

Range of the interaction. Next we consider a Gaussian potential,

$$V(r)=V_0\exp[-(r/b)^2]^{\frac{1}{2}}. \quad (2)$$

In the calculations for $A=19$ of reference 4, a Serber potential $V(r)(1+P_M)$ was used. The parameter ratio was $b/a=0.75$. The same b with a determined by (1) would lead to $b/a=0.54$. Calculations for Pb^{206}, however, have been made[13] with a longer-range singlet interaction and $b/a=0.79$.

To obtain some idea of the changes in matrix elements with b/a, we consider Figs. 1 and 2 of a paper by Flowers.[14] There, the levels of $(1g_{9/2}p)^\alpha$ for $\alpha=2$ and 3 are given as a function of b/a for the pure space-exchange (M) interaction $V(r)P_M$. The figures show that for the entire range $0\leq b/a\leq 0.9$, there is a group of levels of $(1g_{9/2}p)^2$ with seniority number $v=2$ bunched together and far above the $J=0$, $v=0$ level. Similarly, for $(1g_{9/2}p)^3$, there is a group of $v=3$ levels far above the J$=9/2$, $v=1$ ground state. For $b/a=\infty$, the levels have energies proportional to $J(J+1)$; the level diagram looks considerably different. Probably the energies of the levels of $(1h_{9/2}p)^2$, which are of interest here, are similar functions of b/a as those of $(1g_{9/2}p)^2$. If this is also true for the levels of $(1h_{9/2}p)(2g_{9/2}n)$, then the results to be given in Sec. III for a zero-range interaction are similar to the ones which would be obtained with an interaction of appropriate longer range.

The zero-range interaction. The parameter F^0. The two-nucleon potential is

$$V_\rho(1,2)=(D_s\Pi_s+D_t\Pi_t)r_1^{-2}\delta(r_1-r_2)\cdot\delta(\Omega_1-\Omega_2), \quad (3)$$

where Ω_1, Ω_2 are angular co-ordinates, D=potential strength, Π=a projection operator, and the subscripts are s=singlet, t=triplet, and $\rho=D_t/D_s$. The strength D_s is kept fixed throughout this paper. The definition of the Slater integral F^0 given by Edmonds,[15] p. 115, with obvious abbreviations for n_1l_1, n_2l_2, n_3l_3, n_4l_4, is used here. We assume that for $V_1(1,2)$, with $\rho=1$,

$$F^0(1h,2g)=-0.1598\text{ MeV (exactly)}. \quad (4a)$$

Then, using $1h$ and $2g$ harmonic-oscillator wave functions with the same ν, one obtains[16]

$$\begin{aligned}F^0(2g^2)&=-0.27202\text{ MeV},\\ F^0(1h^2)&=-0.31217\text{ MeV}.\end{aligned} \quad (4b)$$

We define a singlet potential $V_0(r)=\Pi_s^{-1}V_0(1,2)$, where $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, and find for the volume integral, over all space:

$$\int V_0(r)d\mathbf{r}=D_s.$$

Using (3), (4b), the algebraic expression for $F^0(1h^2)$, and the value of ν determined by (1), we obtain $D_s=-866$ MeV F^3. An increase of 4.34% in $R(1h_{9/2})$

[9] A. de-Shalit and J. D. Walecka, Nuclear Phys. **22**, 184 (1961).
[10] N. Newby, Jr., and E. J. Konopinski, Phys. Rev. **115**, 434 (1959).
[11] The ν, b, and D_s which will be given here equal 2ν, $\beta^{-\frac{1}{2}}$, and $-pD(\pi/\beta)^{\frac{1}{2}}$, respectively, in the notation of Newby and Konopinski.
[12] W. J. Swiatecki, Proc. Roy. Soc. (London) **A205**, 238 (1951).
[13] W. W. True and K. W. Ford, Phys. Rev. **109**, 1675 (1958).
[14] B. H. Flowers, Proc. Roy. Soc. (London) **A215**, 398 (1952).
[15] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).
[16] N. Zeldes, Nuclear Phys. **2**, 1 (1956/7).

TABLE I. Comparison of theoretical and experimental interaction energies. The neutrons are in state $2g_{9/2}$, the protons in $1h_{9/2}$. All energies are in MeV, and have been multiplied by -1.

Nucleus	J^π	$\rho=D_t/D_s$	Theoretical interaction energy Nuclear	Coulomb	Total	Interaction energy from experiment[a]
Pb^{210}	0^+	b	1.36		1.36	1.36
Po^{210}	0^+	b	1.56	−0.24	1.32	1.33
Bi^{210}	0^-	1	0.80		0.80	∼0.77
		1.5	1.20		1.20	
	1^-	1	0.41		0.41	∼0.82
		1.5	0.61		0.61	
Po^{211}	$9/2^+$	1	1.94	−0.24	1.70	2.00
		1.5	2.14	−0.24	1.90	

[a] From the data of reference 21.
[b] Only the singlet interaction appears here.

would increase $|D_s|$ to 984 MeV F³, the value of the volume integral for a Gaussian potential between two free nucleons in the singlet state, using the appropriate scattering length and effective range.[17]

Some consequences of this model for $A=210$. The first calculations with the present model for this mass number were made by Pryce.[18] Hoff and Hollander[19] showed that approximate agreement between theory and experiment can be obtained for the energies of the 0^+, 2^+, 4^+, and 6^+ states of Po^{210}. Recently, detailed calculations were made by Newby and Konopinski,[10] and Banerjee and Zeh.[20] Some results of the present, less detailed model will, however, be given for orientation purposes. We assume $(1h_{9/2}p)^2$ and $(2g_{9/2}n)^2$ for Po^{210} and Pb^{210}. The calculated nuclear interaction energies for the 0^+ states are $5F^0(1h^2)$ and $5F^0(2g^2)$. They are given in Table I, together with the Coulomb energy of $(1h_{9/2}p)^2$, $J=0$ and experimental results. The Slater integrals (4b) lead to good agreement with experiment for these two states. The interaction energy from experiment was calculated from the neutron and proton binding energies which were obtained from various data.[21]

The Coulomb energy was calculated by the usual expansion in terms of Slater integrals. The result, adjusted for the smaller radius used by Newby and Konopinski,[10] checks the value for $J=0$ of their Table V. The results for $J>0$ of their table were used to calculate the Coulomb energies of the two states of Po^{211} given in Table I. The quantum-mechanical result for $J=0$ is close to that of a classical calculation based on the assumption that the two protons are uniformly distributed over a thin shell with radius $R(1h_{9/2})$; this leads to $e^2/R(1h_{9/2})=0.20$ MeV.

The triplet interaction enters the calculation for Bi^{210}, with $(1h_{9/2}p)(2g_{9/2}n)$. Energies of the low 0^- and 1^- states are compared with experimental results in Table I. The calculated energy for the 1^- state is certainly too small in magnitude. However, Newby and Konopinski[10] showed that with configuration interaction, this state may be predominantly $(1h_{9/2}p)(1i_{11/2}n)$. They also found that a longer-range scalar+tensor interaction may lead to the observed energies of the 0^- and 1^- states. There is still uncertainty about the magnitude and detailed form of the triplet interaction. Nevertheless, it seems possible that the magnitude of the *matrix elements* of the n-p interaction for $(1h_{9/2}p)^2(2g_{9/2}n)$ which will be given in Sec. III is correct to a factor of ≤ 2, as it may be for Bi^{210}.

III. CALCULATIONS FOR $(1h_{9/2}p)^2(2g_{9/2}n)$, WITH J=9/2 AND 7/2

Altogether there are 35 states of this configuration. A complete set of states may be described as follows: The angular momenta of the two protons couple to J', which equals 0, 2, 4, 6, or 8 because of the Pauli principle; the neutron's angular momentum $j'=9/2$ is then added to J' to give J. Thus, five among the 35 states have $J=9/2$, and four have $J=7/2$. Only one state has a $J'=0$ parent.

Method of calculation. Calculations were made for the interaction $V_1(1,2)$ given by (3):

$$V_1(1,2)=D_s\frac{\delta(r_1-r_2)}{r_1^2}\delta(\Omega_1-\Omega_2)=D_s\frac{\delta(r_1-r_2)}{4\pi r_1^2}\sum_{k=0}^{\infty}(2k+1)\mathbf{C}^{(k)}(\Omega_1)\cdot\mathbf{C}^{(k)}(\Omega_2), \tag{5}$$

where $\mathbf{C}^{(k)}$ is an irreducible tensor operator with components defined in Edmonds, reference 15, Eq. (2.5.31). Cal-

[17] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 109 (1950).
[18] M. H. L. Pryce, Proc. Phys. Soc. (London) **A65**, 773 (1952).
[19] R. W. Hoff and J. M. Hollander, Phys. Rev. **109**, 447 (1958).
[20] P. Banerjee and H.-D. Zeh, Z. Physik **159**, 170 (1960).
[21] B. M. Foreman, Jr., and G. T. Seaborg, J. Inorg. & Nuclear Chem. **7**, 305 (1958).

culations were also made for

$$\mathcal{V}(1,2)=\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 V_1(1,2)=D_s\frac{\delta(r_1-r_2)}{4\pi r_1^2}\sum_{K=0}^{\infty}\mathbf{T}^{(K)}(\Omega_1,\boldsymbol{\sigma}_1)\cdot\mathbf{T}^{(K)}(\Omega_2,\boldsymbol{\sigma}_2), \tag{6}$$

with

$$\mathbf{T}^{(K)}(\Omega_1,\boldsymbol{\sigma}_1)=\sum_{k=K-1}^{K+1}(2k+1)[\boldsymbol{\sigma}_1^{(1)}\times\mathbf{C}^{(k)}(\Omega_1)]^{(K)},$$

where $\boldsymbol{\sigma}^{(1)}=2\mathbf{s}^{(1)}$, and $\mathbf{s}^{(1)}$ is the spin operator in the usual irreducible tensor form. For the two protons, the zero-range interaction is $\neq 0$ only in the singlet state, and $V_1(1,2)$ and $-\frac{1}{3}\mathcal{V}(1,2)$ have the same matrix elements.

The p-p interaction matrix elements for $(1h_{9/2}p)^2$ were calculated directly.[22] The radial integral is given in (4b). Matrix elements of the n-p potential (5) can be expressed as a sum $\sum A_k$ of at most five terms, with $k=0$, 2, 4, 6, and 8. Each term is given by

$$A_k=2(j^2J',\ j'J\,|\,\mathbf{C}^{(k)}(\Omega_2)\cdot\mathbf{C}^{(k)}(\Omega_3)\,|\,j^2J'',\ j'J)\cdot(2k+1)F^0(1h,2g). \tag{7}$$
$$\text{Coordinates: } 1,2\quad 3\qquad 2\qquad 3\quad 1,2\quad 3$$

A simple expression for (7) is obtained by application first of Edmonds (7.1.6) and then (7.1.8). The expression $(j\|\mathbf{C}^{(k)}\|j)$, which appears in the application of these formulas, is given by Edmonds (7.1.9). The calculation of the matrix elements of (6) is entirely analogous. The expression $(j\|\mathbf{T}^{(K)}\|j)$ is evaluated by Edmonds (7.1.5).

Matrices of the interactions V_1 and $\mathcal{V}$. For $J=9/2$, the matrices are: A, the matrix of the p-p interaction $V_1(1,2)$; B, the matrix of the total n-p interaction $V_1(1,3)+V_1(2,3)$; and C, the matrix of the total n-p interaction $\mathcal{V}(1,3)+\mathcal{V}(2,3)$. These matrices follow; each matrix element is given in MeV:

MATRIX $-A$

$J'=$	0	2	4	6	8
0	1.5608	0	0	0	0
2	0	.3784	0	0	0
4	0	0	.1965	0	0
6	0	0	0	.1164	0
8	0	0	0	0	.0629

MATRIX $-B$

$J'=$	0	2	4	6	8
0	.3196	.1732	.1207	.0860	.0531
2	.1732	.4850	.1692	.0632	.0550
4	.1207	.1692	.3702	.2017	.0878
6	.0860	.0632	.2017	.4154	.1857
8	.0531	.0550	.0878	.1857	.5622

MATRIX $-C$

$J'=$	0	2	4	6	8
0	0	0	0	0	0
2	0	.0675	.1329	.1427	.0702
4	0	.1329	.2772	.1336	.1175
6	0	.1427	.1336	.3300	.1834
8	0	.0702	.1175	.1834	.4828

Here, the exact results, which contained only integers and square roots of integers times F^0 have been approximated to four decimal places.

For $J=7/2$, the analogous matrices, A', B', and C' are:

MATRIX $-A'$

$J'=$	2	4	6	8
2	.3784	0	0	0
4	0	.1965	0	0
6	0	0	.1164	0
8	0	0	0	.0629

MATRIX $-B'$

$J'=$	2	4	6	8
2	.2186	−.0004	.0215	.0421
4	−.0004	.3426	.0825	.0469
6	.0215	.0825	.3814	.0852
8	.0421	.0469	.0852	.4377

MATRIX $-C'$

$J'=$	2	4	6	8
2	.1272	.1037	.0331	−.0043
4	.1037	.1297	.0866	.0400
6	.0331	.0866	.2423	.0917
8	−.0043	.0400	.0917	.3753

Check of the calculation. Matrix elements of A were checked by expansion of the $(1h_{9/2}p)^2$ wave functions in terms of those for L-S coupling, and separate calculation of the matrix elements of $(1h\ p)^2$, ^1L. Matrix elements of B, C, B', and C' were obtained independently by expansion of all three-particle wave functions in terms of fractional parentage coefficients[23] and use of the energies of all states of $(1h_{9/2}p)^2$ and $(1h_{9/2}p)(2g_{9/2}n)$. Approximations to four or five significant figures were made throughout this calculation, and it is therefore less accurate. Agreement was obtained to ≤ 0.0002 MeV for all matrix elements.

Most of the six-j-symbols needed for both calculations are given in one table.[24] The remaining ones, involving an argument 9, were calculated by a recursion formula, or by direct algebraic formulas.

Wave functions. Calculations were made for $\rho=D_t/D_s=1$ and 1.5. The potential $V_{1.5}(1,2)$ is given by

$$V_{1.5}(1,2)=\tfrac{1}{8}(11+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)V_1(1,2)=\tfrac{1}{4}(5+P_\sigma)V_1(1,2), \tag{8}$$

where $P_\sigma=$ the spin-exchange operator, and $V_1(1,2)$ is given by (5). The ratio $\rho=1.5$ leads to approximately the triplet-singlet state splitting in the deuteron. It is the ratio which was used in extensive calculations of magnetic and quadrupole moments, taking into account configuration interaction.[25] The matrices $\|V_\rho\|$ for the total interaction

$$V_\rho=V_\rho(1,2)+V_\rho(1,3)+V_\rho(2,3)$$

[22] A. de-Shalit, Phys. Rev. **91**, 1483 (1953); see also N. Newby and E. J. Konopinski, reference 10, p. 438.
[23] See, for example, reference 4, p. 1429; the modifications needed because T is not used here can be made directly.
[24] M. Rotenberg, R. Bivins, N. Metropolis, and J. K. Whooten, Jr., *The 3-j and 6-j Symbols* (Technology Press, Cambridge, Massachusetts, 1959).
[25] H. Noya, A. Arima, and H. Horie, Suppl. Progr. Theoret. Phys. (Kyoto), No. 8, 33 (1958). Earlier references are given there.

are, for $J=9/2$,

$$\|V_1\|=A+B, \qquad \|V_{1.5}\|=\frac{11}{8}\left(\frac{8}{11}A+B+\frac{1}{11}C\right).$$

For $J=7/2$, it is necessary only to replace A, B, C by A', B', C'.

Diagonalization of $\|V_1\|$ and $\|V_{1.5}\|$ for $J=9/2$ leads to five states denoted a, b, c, d, e, in order of increasing energy. The wave functions $\Psi(9/2\ a)$ and the energies for the lowest state are:

$$\rho=1, \quad E=-1.94 \text{ MeV}, \quad \Psi(9/2\ a)=0.968\psi(0)+0.184\psi(2)+0.126\psi(4)+0.094\psi(6)+0.068\psi(8);$$
$$\rho=1.5, \quad E=-2.14 \text{ MeV}, \quad \Psi(9/2\ a)=0.916\psi(0)+0.274\psi(2)+0.205\psi(4)+0.166\psi(6)+0.132\psi(8).$$

Here $\psi(J')$ is the wave function of $(1h_{9/2}p)^2J'(2g_{9/2}n)$, with $J=9/2$. The state with $J'=0$, which is the state of the odd-group model, is dominant. The total probability of all states with $J\neq 0$ is 6.3% for $\rho=1$, and 16.2% for $\rho=1.5$. (It may be noted that first-order perturbation theory leads to the following values of these quantities: 4.2% and 8.6%.)

The corresponding results for the lowest state with $J=7/2$ are:

$$\rho=1, \quad E=-0.67 \text{ MeV}, \quad \Psi(7/2\ a)=0.432\psi(2)+0.510\psi(4)+0.541\psi(6)+0.511\psi(8);$$
$$\rho=1.5, \quad E=-0.92 \text{ MeV}, \quad \Psi(7/2\ a)=0.263\psi(2)+0.491\psi(4)+0.583\psi(6)+0.592\psi(8).$$

The energies due to nuclear interactions of all the states with $J=9/2$ and of all those with $J=7/2$ are given in Fig. 1 for the interactions V_1 and $V_{1.5}$. The levels of $(1h_{9/2}p)^2$ are also drawn there. Probably, the remaining 26 states of $(1h_{9/2}p)^2(2g_{9/2}n)$, which have $J\neq 7/2$ or $9/2$, lie approximately in the already densely filled region of the states $9/2\ c$ to $9/2\ e$. It is not surprising that the state $9/2\ a$ lies almost 1 Mev below all other states with $J=9/2$, because they do not have large $J'=0$ components, and the $(1h_{9/2}p)^2$ $J'=0$ state is by far the lowest for the two protons. The $J=7/2$ states have no $J'=0$ component. In Table I, the energies E for the state $9/2\ a$ are compared with the interaction energy obtained for the ground state from experimental data.

The calculated eigenvectors and energies of the second state with J$=9/2$ are:

$$\rho=1, \quad E=-0.99 \text{ MeV}, \quad \Psi(9/2\ b)=-0.249\psi(0)+0.649\psi(2)+0.460\psi(4)+0.401\psi(6)+0.379\psi(8);$$
$$\rho=1.5, \quad E=-1.30 \text{ MeV}, \quad \Psi(9/2\ b)=-0.386\psi(0)+0.440\psi(2)+0.444\psi(4)+0.467\psi(6)+0.491\psi(8).$$

For either value of ρ, this eigenvector and also the eigenvectors for $9/2\ c$, d, e, and $7/2\ a$, b, c, d show large amplitudes for two or more of the $J'=2$, 4, 6, and 8 states. It is apparent that the wave functions of the higher states of this configuration are considerably more complicated than that of the ground state, if small amplitudes are neglected. Furthermore, configuration interaction may be greater for the higher states. This may be true also for larger configurations; the single-particle-type states of the Tl and Au isotopes, which were described in Sec. I, may be the ground states of different configurations.

Among the three outer nucleons, there are two $n-p$ bonds, but there is only one $p-p$ bond; that is, the neutron interacts with two protons, but each proton interacts with only one proton. Nevertheless, the total n-p interaction energy for the state of $(1h_{9/2}p)^2(2g_{9/2}n)$ with $J'=0$ is small compared with the p-p interaction energy, as can be seen from the matrices A, B, and C. This is partly due to the small overlapping of $1h$ and $2g$ radial wave functions, as shown by a comparison of (4a) with (4b). Another important factor is: The probability that a proton and a neutron be in a state of $(1h_{9/2}p)(2g_{9/2}n)J''$ which has low energy is quite small, especially when compared with the 100% probability that the two protons be in the very low $J'=0$ state. For the other states, this situation is generally reversed: The diagonal n-p interaction matrix elements are larger than those of the p-p interaction.

IV. DISCUSSION OF THE RESULTS AND THEIR POSSIBLE EXTENSION

1. Configuration interaction. Neutron and proton radii. A more detailed calculation for Po^{211} or Bi^{211} would include all states of the main neutron shell and proton shell above Pb^{208}. Probably, the largest admixtures to the state $(1h_{9/2}p)^2(2g_{9/2}n)$ $J'=0$, $J=9/2$ are from the states of $(2f_{7/2}p)^2(2g_{9/2}n)$ and $(1i_{13/2}p)^2(2g_{9/2}n)$ with $J'=0$, because (a) these odd-group model states will lie especially low, and (b) the off-diagonal matrix elements between any two of the three states mentioned, each of which has $J'=0$, are especially large. Therefore, it seems possible that the total weight (quantum-mechanical probability) of all odd-group model states may still be very high even for a more detailed calculation.

We saw in Sec. II that the root-mean-square radius of the outer neutron, $R(2g_{9/2})$, exceeds that of the outer proton, $R(1h_{9/2})$, by 0.529 F. It has been suggested that the neutron radius of a heavy nucleus may be 1 F greater than the proton radius.[26] If we increase $R(2g_{9/2})$

[26] M. H. Johnson and E. Teller, Phys. Rev. **93**, 357 (1954).

TABLE II. The weight $\mathfrak{W}$ of the odd-group model state for the ground states of several configurations.

Configuration(s)	$(1d_{5/2}p)^2(1d_{5/2}n)$		$\left\{\begin{matrix}(1d_{5/2}p)^2(2s_{1/2}n)\\(1d_{5/2}p)(2s_{1/2}p)(1d_{5/2}n)\end{matrix}\right\}$		$(1h_{9/2}p)^2(2g_{9/2}n)$
J	5/2		1/2		9/2
b/a	0	0.75	0	0.75	0
$\mathfrak{W}$	0.75	0.72	0.61	0.60	0.94

so that it equals $R(1h_{9/2})+1$ F, thus introducing a $\nu'=\nu(2g_{9/2})$ in addition to $\nu=\nu(1h_{9/2})$, then the ratio $F^0(1h,2g)/F^0(1h^2)$ is decreased by 6.4%. This should lead to still slightly higher probabilities for the $J'=0$ state than were obtained in Sec. III.

2. Non-validity of the model for light nuclei. Contrast with Ne^{19}. *Isotopic spin.* The matrices of the ordinary (W) interaction (2) between each pair of outer nucleons were calculated for configurations of Ne^{19}, using harmonic-oscillator wave functions. The states were specified by $(jp)(j'p)J',(j''n)J$; in this scheme, the wave function is antisymmetric with respect to interchange of the two protons. $\mathfrak{W}$ is defined as the weight of the odd-group model state, with the angular momenta of the two outer proton states coupling to $J'=0$. In Table II, $\mathfrak{W}$ is given for the ground state of the specified configuration(s) for Ne^{19}, together with a result for Po^{211}. The range/radius ratio b/a is defined in Sec. II. These results can also be obtained in an isotopic-spin scheme, with totally antisymmetric A-particle wave functions. (In the present notation, the $1s^41p^{12}$ closed shells are omitted.) For $(1d_{5/2})^3$ and $(1d_{5/2})^2(2s_{1/2})$, the energy due to interactions between outer-nucleon states, and $\mathfrak{W}$ have been obtained by direct calculation in both schemes. For $(1d_{5/2})^2(2s_{1/2})$, formation of antisymmetric wave functions with $T=\frac{1}{2}$ leads to linear combinations of wave functions of two different configurations:

$$(1d_{5/2}p)^2(2s_{1/2}n) \quad \text{and} \quad (1d_{5/2}p)(2s_{1/2}p)(1d_{5/2}n).$$

Therefore, these two configurations were grouped together in the above table.

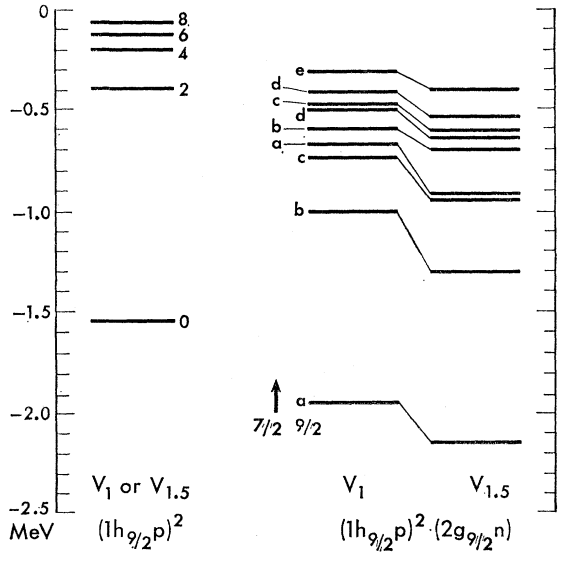

FIG. 1. The nuclear interaction energies for all states of $(1h_{9/2}p)^2$ and all states with $J=9/2$ and 7/2 of $(1h_{9/2}p)^2(2g_{9/2}n)$ for charge-independent, zero-range interactions V_1 and $V_{1.5}$.

For the wave functions of Ne^{19} which had been obtained by diagonalization of matrices of the Serber interaction for all states of all configurations of the $1d$, $2s$ shell,[4] the weight $\mathfrak{W}_t$ of all odd-group model states, each with $J'=0$, can be calculated. The results are:

(T,J)	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{5}{2})$	$(\frac{1}{2},\frac{3}{2})$
$\mathfrak{W}_t$	0.51	0.51	0.26

An analogous quantity $\mathfrak{W}_t'$ can be defined for L-S coupling. The odd-group model state is defined by $(lp)^2$, ^1S, $(l'n)$. The values of $\mathfrak{W}_t'$ are lower than those of $\mathfrak{W}_t$ by 0.02 to 0.04 for the three states of the above table.

The situation is different for heavy nuclei, with neutron and proton states belonging to different main harmonic-oscillator shells. If we consider those states of a configuration which have the maximum T_ζ possible for this configuration, we can form wave functions with only one value of the total isotopic spin,[27] namely, $T=T_\zeta$. An example is the 211-particle configuration consisting of the Pb^{208} double-closed-shell states $+(1h_{9/2}p)^2(2g_{9/2}n)$, with $T_\zeta=43/2$. States with higher T_ζ, such as $(1h_{9/2}n)^2(2g_{9/2}n)$, do not exist, because the $1h_{9/2}n$ shell is already filled. Similarly, states of $(1h_{9/2}p)(2g_{9/2}p)(1h_{9/2}n)$ do not exist, in contrast to the $(1d_{5/2})^2(2s_{1/2})$ example. The states with high T have high symmetry in isotopic spin and low symmetry in space and ordinary spin for the neutrons and protons taken together. The results of Sec. III, however, suggest the possibility that low-lying states with rather high space symmetry for the neutrons and protons *separately* may generally exist for heavy nuclei, and that this separate symmetry is considerably lower for light nuclei.

3. Single-particle and many-particle states. Experimental data. We can expect that calculations for each configuration $(1h_{9/2}p)^2(j'n)$ will lead to a lowest state with $J=j'$, and that the wave function of this state is predominantly that of the odd-group model. We expect therefore to find several low-lying single-particle-type levels of Po^{211} with spins and parities predicted by the shell model. In addition, we expect to find a

[27] L. Eisenbud and E. P. Wigner, reference 5, Sec. 7.4, pp. 53–54; M. G. Mayer and J. H. D. Jensen, reference 1, p. 250.

second state with $J=9/2$, labeled b in Fig. 1, approximately within the group of single-particle states. The weight of the $J'=0$ state in the state $9/2\ b$ is only 6.2% for V_1 and 14.9% for $V_{1.5}$. Possibly, a similar situation exists for the three Au isotopes discussed in Sec. I. The data are consistent with an assignment of $3/2^+$ to the second excited state and also to the ground state of each isotope.

From experiment, the only definitely known[6] excited state of Po^{211} is at 1.29 MeV. It has a half-life of 25 sec, and $J\geq 19/2$. The highest J which could arise from an odd-group model state in this shell is 15/2, from the single-particle state $1j_{15/2}$. The 1.29-MeV state is therefore more complicated. The present theory leads to a level scheme very similar to that of Fig. 1 for the configuration $(1h_{9/2}p)(2g_{9/2}n)^2$ of Bi^{211}. A level scheme for this nucleus with four excited states having energies up to 829 keV has been deduced from experiments[6]; however, the spins of these states are not known. All or some of these states may be of single-particle type, because they lie in the interval below the group of most of the complex higher states.

V. EXACT VALIDITY OF THE ODD-GROUP MODEL WITHIN ONE CONFIGURATION FOR CERTAIN INTERACTIONS. LARGER CONFIGURATIONS

We consider here any interaction $V(1,2)$ which can be expanded as a sum of scalar products of irreducible tensor operator $\mathbf{U}^{(K)}$. Examples are the expansions (5) and (6) for $V_1(1,2)$ and $\mathcal{V}(1,2)$. Racah and Talmi[2] showed that within a configuration with only one type of particles, such as $(jp)^\alpha$, the pairing property holds for $V(1,2)$ provided that only the terms with odd K have non-zero diagonal or off-diagonal matrix elements between states of this configuration.

This result can be extended directly, for example, to a configuration $(jp)^\alpha J'(j'n)^\beta$, with α even and β odd. If only the terms with odd K in the expansion of $V(1,2)$ have non-zero matrix elements for this configuration, then: (a) The pairing property holds separately for the total n-n and the total p-p interaction. (b) The total n-p interaction is diagonal with respect to neutron and proton seniority. (c) For the states with $J'=0$, the diagonal matrix element of the n-p interaction equals 0.

Statement (b) implies that the odd-group model is precisely valid for one state of this configuration: There is no admixture to the only state with minimum neutron and proton seniority, which has $J'=0$, and $J=j'$. Statements (b) and (c) follow directly from Racah's theorem that an odd tensor operator is diagonal with respect to the seniority and from the algebra of irreducible tensor operators. These results had already been obtained by French[28] for the configuration $(jp)^\alpha(jn)^\beta$ (with $j=j'$), in an investigation of the states corresponding to the symplectic group.

[28] J. B. French, Nuclear Phys. **15**, 393 (1960).

An example of an interaction of this type is

$$\mathcal{V}'(1,2)=\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 V'(\mathbf{r}_1,\mathbf{r}_2)=(1-2P_\sigma)\cdot V'(\mathbf{r}_1,\mathbf{r}_2),$$

where $V'(\mathbf{r}_1,\mathbf{r}_2)$ is any central potential. In Sec. III, the matrices C and C' of the n-p interaction for this potential with $V'(\mathbf{r}_1,\mathbf{r}_2)=V_1(1,2)$ are given. The properties (b) and (c) are illustrated by the zeros of C. It must be noted, however, that $\mathcal{V}'(1,2)$ does not resemble closely the interaction between two free nucleons.

The tensor interaction. This interaction, $S_{12}V_T(\mathbf{r}_1,\mathbf{r}_2)$, can be expanded in terms of scalar products of irreducible tensor operators $\mathbf{U}^{(K)}$, and only the matrix elements with odd K are $\neq 0$, as was already pointed out by Racah and Talmi.[2,29] We assume now that the two-nucleon interaction can be represented as[30]

$$V_C(\mathbf{r}_1,\mathbf{r}_2)+S_{12}V_T(\mathbf{r}_1,\mathbf{r}_2). \tag{9}$$

The zero-range approximation $V_1(1,2)$, as used in this paper probably gives satisfactory values for the matrix elements of $V_C(\mathbf{r}_1,\mathbf{r}_2)$. To represent an increase in triplet strength, the potential (8) was introduced. It can be written as

$$V_{1.5}(1,2)=V_1(1,2)+\tfrac{1}{8}(3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)V_1(1,2). \tag{10}$$

Instead, a longer-range, tensor interaction could be used to give the increased triplet interaction, and might lead approximately to the 93.7% weight of the $J'=0$ state which was obtained for $\rho=1$, rather than the 9.9% lower weight obtained with (10).

A larger configuration. The statements (a), (b), and (c) apply to all possible (α,β). The extension of the calculations of Sec. III for $(1h_{9/2}p)^2(2g_{7/2}n)$, however, is more difficult. For example, in $(1h_{9/2}p)^2(2g_{7/2}n)^3$, there are already 35 states with $J=9/2$. One of these has neutron seniority $v_n=1$, and $v_p=0$; this is the odd-group model state. Perhaps the total weight of the 34 other states, with $v_n+v_p>1$, in the lowest state is as small as the corresponding total weight of the four other states in $(1h_{9/2}p)^2(2g_{9/2}n)$. At present, however, it appears that a detailed calculation is necessary to ascertain this magnitude.

I am grateful to Professor E. P. Wigner and Professor E. Feenberg for their advice and interest.

Note added in proof. Silverberg[31] has just published calculations for Tl^{205}. They include a calculation based upon the interaction (2) with a Serber mixture and triplet/singlet strength ratio 1.485, and harmonic oscillator wave functions. The range/radius ratio is $b/a=0.79$. The calculation was made for three values

[29] The explicit expansion is given in the Appendix of the paper by Walecka and de-Shalit, reference 9, and in Eq. (15) of H. Horie and K. Sasaki, Prog. Theoret. Phys. (Kyoto) **25**, 475 (1961).

[30] This possibility is discussed on p. 103 of J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

[31] L. Silverberg, Arkiv Fysik **20**, 355 (1961).

of J. All states of all three-hole configurations with $(3s_{1/2}p)^{-1}$, $(2d_{3/2}p)^{-1}$, $(3p_{1/2}n)^{-1}$, $(2f_{5/2}n)^{-1}$, and $(3p_{3/2}n)^{-1}$ are included. The probabilities of the following table were calculated from the amplitudes of Silverberg's Table 5. Here $\mathcal{W}_t(J')$ is the total weight of all states of type $(jn)^{-1}(j'n)^{-1}J'(j''p)^{-1}J$. The total number of states is n; that is, $n\times n$ matrices were diagonalized.

Calculated energy (keV)	J	n	$\mathcal{W}_t(0)$	$\mathcal{W}_t(1)$	$\mathcal{W}_t(2)$	$\mathcal{W}_t(3)$	$\mathcal{W}_t(4)$
880	5/2	18		0.001	0.942	0.004	0.053
318	3/2	19	0.476	0.001	0.523	0.000	
0	1/2	12	0.890	0.003	0.107		

Here, the odd-group model is a very good approximation for the ground state, with $J=1/2$. The probability for $J'=0$ is larger than the 83.8% probability obtained with triplet/singlet ratio $\rho=1.5$ in the calculation of the present paper for $(1h_{9/2}p)^2(2g_{9/2}n)$. For the $J=3/2$ state, however, there is a very large probability $\mathcal{W}_t(2)$. The main reason for this result is the existence of low-lying states of $(3p_{1/2}n)^{-1}(3p_{3/2}n)^{-1}$ and $(3p_{1/2}n)^{-1}(2f_{5/2}n)^{-1}$ with $J'=2$ for Pb^{206}. The energy of the $(2d_{5/2}p)^{-1}$ level in Tl^{207} has not yet been measured, and the weight $\mathcal{W}_t(0)$ for the $J=5/2$ state of Tl^{205} was not calculated at all.

Recent papers[32,33] include additional calculations for Bi^{210}, based upon a detailed model. A two-nucleon Gaussian potential with $b=2$ F is used, and several configurations are included. The work has just been extended[34] for Po^{212} to several four-nucleon configurations.

A new calculation[35] for the ground state with $J=9/2$ of Po^{211} has been based upon δ-function interaction matrix elements, adjusted in an empirical way to approximate matrix elements for a longer-range potential. The calculation included $(1h_{9/2}p)^2(2g_{9/2}n)$ and also higher configurations. The resulting wave function has a considerably lower amplitude for the $J'=0$ state of $(1h_{9/2}p)^2(2g_{9/2}n)$ than that of the exact calculation of the present paper. This is still true if the $(1h_{9/2}p)^2(2g_{9/2}n)$ part of the wave function is separately normalized. The work of Flowers[14] and the discussion in Sec. II, however, suggest the possibility that an exact calculation for an appropriate non-zero b/a would lead to $J'=0$ amplitudes close to those of the present calculation. We saw in Sec. IV (Table II) that exact calculation for two states in the $1d$, $2s$ shell leads to nearly equal values of $\mathcal{W}$ for $b/a=0$ and 0.75.

[32] L. A. Sliv, G. A. Sogomonova, and Yu. I. Kharitonov, J. Exptl. Theoret. Phys. (U. S. S. R.) **40**, 946 (1961) [translation: Soviet Phys.—JETP **13**, 661 (1961)].

[33] Yu. I. Kharitonov, L. A. Sliv, and G. A. Sogomonova, Nuclear Phys. **28**, 210 (1961).

[34] I. M. Band, L. A. Sliv, and Yu. I. Kharitonov, J. Exptl. Theoret. Phys. (U. S. S. R.) **41**, 1274 (1961) [translation: Soviet Phys.—JETP **14**, 908 (1962)].

[35] H.-D. Zeh and H. J. Mang, Nuclear Phys. **29**, 529 (1962).