

Analyticity in the Complex Angular Momentum Plane of the Coulomb Scattering Amplitude*

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The complex angular momentum J -plane analytic properties of the scattering amplitude for Coulomb potential have been studied. It is found that the only singularities are poles (Regge poles) in the J plane. These poles are discussed in detail, with their application to relativistic S -matrix theory kept in view.

I. INTRODUCTION

RECENTLY, Regge has introduced, for the non-relativistic potential problem, the notion of simultaneous analyticity in the complex angular momentum J and energy E .¹ The importance of this notion for the relativistic S matrix is also now well recognized.² At the present stage, however, the practical utilization of analyticity in J is hampered by lack of a detailed knowledge of the singularities in the complex J plane and their behavior with energy. The present note considers a solvable case—that of the Coulomb potential—in order to illuminate a number of pertinent questions.

Regge, in his paper on potential scattering, considered analyticity in only the right-hand half of the J plane ($\text{Re}J > -1/2$). It was shown that, for real E , singularities in this part of the J plane consist at most of a finite number of poles (Regge poles) in the first quadrant ($\text{Re}J > -1/2$, $\text{Im}J \geq 0$). A question naturally arises at this point about the nature of the line $\text{Re}J = -1/2$. It may be recognized that $J = -1/2$ is precisely the point for which the regular and irregular solutions of the Schrödinger equation change their roles. One is not sure, then, whether the Schrödinger equation together with its boundary conditions can be analytically continued into the left-hand half of the J plane. In view of this circumstance, a feeling was prevalent that the line $\text{Re}J = -1/2$ may be the natural boundary of the domain of analyticity in J . Thus, a continuation to the left of this line may not be possible. If such a boundary exists in the nonrelativistic (NR) potential scattering, a similar one may exist also for the relativistic S matrix at, say, $\text{Re}J = J_0$.³ In that case, systems with $J_0 > J$ would not have the same significance as systems with $J > J_0$. The former might possibly be associated with “elementary” particles. Thus the question whether such a natural boundary exists or not is of considerable interest. What is significant, in general, is the continua-

tion in the J plane of S -matrix elements, not the Schrödinger equation or the wave function. The answer for the case we consider is that there is nothing special about the line $\text{Re}J = -1/2$ as far as S -matrix elements are concerned. As a matter of fact, we find for NR Coulomb scattering that the S matrix is analytic in the entire J plane except for an infinite number of Regge poles (i.e., a meromorphic function). It should be emphasized, for the relevance of this result to relativistic S -matrix theory, that the Coulomb scattering amplitude satisfies the Mandelstam representation.

As is well known, the Regge poles move as the energy is varied. The scattering system has bound states or resonances, depending on whether the energy is below or above the threshold, when the real part of the Regge pole (a) assumes a physical J value; (b) is an increasing function of energy while passing through this J value. It is good to see an explicit example of this behavior. Of course for our problem there are no resonances, but one sees that the bound states are correctly given by Regge poles. This study of this solvable case also clarifies one further perplexing point, which is as follows: Recall that the double spectral functions (dsf) in the Mandelstam representation are nonvanishing only above two-particle thresholds in both energy and momentum transfer. In particular for NR potential scattering, the dsf (there is only one) is nonvanishing only when the energy is positive. Now, bound-state poles in the energy variable occur for negative energies, so naively one would think that the dsf could not possibly give rise to poles below threshold. However, in the example herein one sees explicitly that this naive feeling is not correct. In fact, the same Regge poles simultaneously control the oscillations in the dsf and determine the bound states.

We discuss in the next two sections the Coulomb scattering of nonrelativistic and Klein-Gordon (and Dirac) particles, respectively. The last section is a summary and gives some further results.

II. NONRELATIVISTIC COULOMB SCATTERING

The Schrödinger equation for the attractive Coulomb potential, which we consider first, is⁴

$$\nabla^2\psi + (k^2 + e^2/r)\psi = 0, \quad (2.1)$$

⁴ In this section $\hbar = 1 = 2m$, where m is the mass of the scattered particle.

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¹ T. Regge, *Nuovo cimento* **14**, 951 (1959); **18**, 947 (1960); T. Regge, A. M. Longoni, and A. Bottino (to be published).

² G. F. Chew and S. C. Frautschi, *Phys. Rev. Letters* **7**, 394 (1961); **8**, 11 (1962); G. F. Chew, S. C. Frautschi, and S. Mandelstam, *Phys. Rev.* **126**, 1204 (1962); M. L. Goldberger and R. Blankenbecler, Talk at International Conference on the Theory of Weak and Strong Interactions, La Jolla, 1960 (unpublished).

³ M. Froissart has shown $J_0 \leq 1$; see *Phys. Rev.* **123**, 1053 (1961).

where $k^2 = E$ is the energy, e^2 = the coupling constant of the Coulomb field. This, as usual, yields for the J th partial wave the equation

$$\psi_J''(r) + [k^2 + (e^2/r) - J(J+1)/r^2] \psi_J(r) = 0. \quad (2.2)$$

As is well known, both these equations can be solved exactly. The S -matrix element for energy E and angular momentum J is given by

$$S(E, J) = \Gamma(J+1 - ie^2/2\sqrt{E}) / \Gamma(J+1 + ie^2/2\sqrt{E}). \quad (2.3)$$

The unitarity of the S matrix, which reads as

$$\bar{S}(E, J) S(E, J) = 1, \quad (2.4)$$

where the bar over S denotes the complex conjugation of the functional form only, is obviously satisfied by (2.3).

It is obvious that the only J -plane singularities of $S(E, J)$, as given by (2.3), are poles of the gamma function in the numerator, since $\Gamma(z)$ is a nonvanishing meromorphic function in the entire z plane with simple poles at $z=0, -1, -2, \dots$.

Thus, the position of the n th Regge pole $J = \alpha_n(E)$ is given by

$$J+1 - ie^2/2\sqrt{E} = \alpha_n(E) + 1 - ie^2/2\sqrt{E} = -n, \quad \text{i.e.,} \quad J = \alpha_n(E) = -n + \alpha_0(E) \quad (\text{for } n=0, 1, 2, \dots), \quad (2.5)$$

where

$$\alpha_0(E) = -1 + ie^2/2\sqrt{E}. \quad (2.6)$$

We can now trace the trajectory of the Regge poles in the J plane as we vary the energy along the real axis. For very large negative values of energies, the n th Regge pole approaches $J = -n$. As one varies the energy from $E = -\infty$ toward $E = 0 - \epsilon$, they all remain on the real J axis and move in step with each other toward the point $\text{Re } J = +\infty, \text{Im } J = 0$. The energy at which the n th Regge pole crosses a physical J value is given by

$$E_{n,J} = -e^4/4(n+J+1)^2 \quad (\text{for } J=0, 1, 2, \dots). \quad (2.7)$$

These are just the energies of the bound states in an attractive Coulomb potential. For positive energies the n th Regge pole jumps to the straight line $\text{Re } J = -n$ and approaches $J = -n$ as $E \rightarrow +\infty$ (Fig. 1).

The residue of the n th Regge pole in $S(E, J)$ is given by

$$(-)^n/n! \Gamma(n+2+2\alpha_n(E)), \quad (2.8)$$

and vanishes at a number of points, $J_{n,m} = (-m-n-2)/2$, on the negative real axis in the J plane. It is interesting to consider these points in reference to the following identity:

$$S(E, J) - S(E, -J-1) = (2/\pi) \Gamma(J+1 - ie^2/2\sqrt{E}) \times \Gamma(-J - ie^2/2\sqrt{E}) \cos \pi(J+1) \sin(\pi ie^2/2\sqrt{E}), \quad (2.9)$$

which leads to

$$S(E, J) = S(E, -J-1)$$

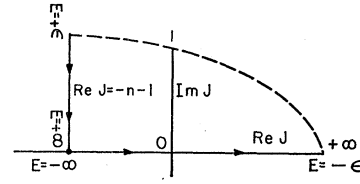


FIG. 1. The trajectory with energy (real) of the n th Regge pole for nonrelativistic attractive Coulomb potential.

for

$$(i) \quad \cos \pi(J+1) = 0, \quad \text{i.e., for any } E, \text{ if } J = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots; \quad (2.10a)$$

$$(ii) \quad \sin(\pi ie^2/2\sqrt{E}) = 0, \quad \text{i.e., for any } J, \text{ if } ie^2/2\sqrt{E} = 0, \pm 1, \pm 2, \dots \quad (2.10b)$$

This, for example, implies that for any given E , if there is a Regge pole at $J = n - \frac{1}{2}$, then at $J = -n - \frac{1}{2}$ there must also be a Regge pole. All the points $J_{n,m}$ can be understood in terms of either Eq. (2.10a) if $J_{n,m}$ is a negative half odd integer, or by (2.10b) if $J_{n,m}$ is a negative integer. The reflection property (2.10a) involves neither energy nor the parameters of the potential, and was conjectured by Mandelstam to be true for a general potential.⁵

The function $\alpha_n(E)$ is real analytic in E with a branch point at $E=0$ and cut along the positive real axis. It satisfies the dispersion relation

$$\text{Re} \alpha_n(E) = \alpha_n(-\infty) + \frac{P}{\pi} \int_0^\infty \frac{\text{Im} \alpha_n(E') dE'}{(E' - E)} \quad (2.11)$$

with

$$\alpha_n(-\infty) = -n-1 \quad (2.12)$$

and

$$\text{Im} \alpha_n(E) = e^2/2\sqrt{E}. \quad (2.13)$$

We would now like to point out two respects in which we expect the behavior of Regge poles for our problem—infinite-range Coulomb potential—to differ from that of the more interesting short-range potentials. First, one knows that one cannot make bound states of arbitrarily high angular momentum with short-range potentials. Thus we expect the trajectory of the Regge pole to leave the real J axis and move into the first quadrant from some finite value of J at $E=0$, and not go up to infinity. Second, it may be observed from Fig. 1 that the real part of the Regge pole has a very violent jump at $E=0$. This is inevitable in the Coulomb case, since at $E = -\epsilon$ the real part of the Regge pole is $\text{Re } J = +\infty$, and in order to have a finite real part i.e., large momentum-transfer behavior being given by a finite power for physical energy region for scattering), the Regge pole has to make this violent jump. This is reflected in having a \sqrt{E} in the denominator for $\text{Im} \alpha_n(E)$

⁵ S. Mandelstam (private communication to Professor G. F. Chew).

in Eq. (2.13). However, there is no reason for the real part of the Regge pole to be discontinuous at $E=0$ for the short-range potentials. For them we expect the factor \sqrt{E} in $\text{Im}\alpha_n(E)$ to be present in the numerator rather than in the denominator.

For the Coulomb problem, we can also write in closed form the scattering amplitude $A(E, t)$, whose partial-wave expansion is given by

$$A(E, t) = (1/2i\sqrt{E}) \sum_{J=0}^{\infty} (2J+1)[S(E, J)-1]P_J(1+t/2E), \quad (2.14)$$

where t is the negative of the momentum-transfer square. The closed form is

$$A(E, t) = [\Gamma(1 - ie^2/2\sqrt{E})/\Gamma(ie^2/2\sqrt{E})] (i/2\sqrt{E}) (-t/4E)^{-1+ie^2/2\sqrt{E}} \quad (2.15)$$

$$= [\Gamma(-\alpha_0(E))/\Gamma(1+\alpha_0(E))](i/2\sqrt{E}) \times (-t/4E)^{\alpha_0(E)}. \quad (2.16)$$

The energy and momentum-transfer singularities of $A(E, t)$ as given by Eq. (2.15) are as follows: The gamma function in the numerator gives the bound-state poles in the energy E , with the correct degeneracy. There is a branch point in E at $E=0$, which gives the physical branch cut. In the t variable, there is a branch cut starting at $t=0$, which we take to run over the real t axis from 0 to $+\infty$.

The discontinuity $A_t(E, t)$ of $A(E, t)$, in crossing the branch cut in t with E fixed is given by

$$A_t(E, t) = (i\pi/2\sqrt{E}) [(t/4E)^{\alpha_0(E)}\theta(t)/[\Gamma(1+\alpha_0(E))]^2]. \quad (2.17)$$

It should be observed that $A_t(E, t)$ has no bound-state poles in E , even though $A(E, t)$ does. One may also write down the expression for the dsf $\rho(E, t)$, i.e., the discontinuity in $A_t(E, t)$ for fixed t over the cut in E . This is

$$\rho(E, t) = \frac{i\pi}{4\sqrt{E}} \left[\frac{(t/4E)^{\alpha_0(E)}}{[\Gamma(1+\alpha_0(E))]^2} + \frac{(t/4E)^{\alpha_0^*(E)}}{[\Gamma(1+\alpha_0^*(E))]^2} \right] \times \theta(E)\theta(t). \quad (2.18)$$

It is seen that the dsf has oscillations in the momentum transfer whose magnitude and period are controlled, respectively, by the real and imaginary parts of $\alpha_0(E)$.

We shall now discuss the case of a repulsive Coulomb potential. Formally we obtain all the results for this case from those above, if we replace e^2 by $-e^2$. Therefore we now have

$$S(E, J) = \Gamma(J+1+ie^2/2\sqrt{E})/\Gamma(J+1-ie^2/2\sqrt{E}). \quad (2.19)$$

Again, we see that $S(E, J)$ is a meromorphic function of J in the entire J plane. The n th Regge pole $\alpha_n(E)$ is now given by

$$\alpha_n(E) = -n-1-ie^2/2\sqrt{E}. \quad (2.20)$$

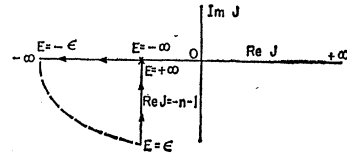


FIG. 2. The trajectory of the n th Regge pole with energy (real) for NR repulsive Coulomb potential.

The trajectory of the n th Regge pole, as the energy is varied along the real axis, is shown in Fig. 2. The important point to note is that even in the case of pure repulsive potential, we have an infinite number of Regge poles. Since for this case there can be neither bound states nor resonances, all the Regge poles are confined to the left-hand half of the J plane.

III. RELATIVISTIC COULOMB SCATTERING

The treatment of the Coulomb scattering of Klein-Gordon particles is quite analogous to that of the preceding section. Hence, we shall just state the results.

We have, for positive integers J and real E ,⁶

$$\phi_J'' + [(E \pm e^2/r)^2 - m^2 - J(J+1)/r^2]\phi_J = 0. \quad (3.1)$$

We obtain, for complex J and E ,

$$S(E, J) = \Gamma(J' - J_0(E))/\Gamma(J' + 2 + J_0(E)), \quad (3.2)$$

$$A(E, t) = [\Gamma(-J_0(E))/\Gamma(1+J_0(E))][1/2(m^2 - E^2)^{\frac{1}{2}}] [t/4(m^2 - E^2)]^{J_0(E)}, \quad (3.3)$$

where

$$J' = [(J+1/2)^2 - e^4]^{\frac{1}{2}} - 1/2$$

and

$$J_0(E) = -1 \pm e^2 E / (m^2 - E^2)^{\frac{1}{2}}. \quad (3.4)$$

The branch cuts arising from the factor $[m^2 - E^2]^{\frac{1}{2}}$ shall be taken to run on the real axis from $E = -\infty$ to $E = -m$ and from $E = +m$ to $E = +\infty$ in the E plane; that of the factor $[(J+1/2)^2 - e^4]^{\frac{1}{2}}$ from $J = -(1/2) - e^2$ to $J = -(1/2) + e^2$ on the real axis in the J plane. It may also be pointed out that the significance of E here is the same as that of the invariant energy square variable in the relativistic problem, since here we consider the scattering by a fixed center of force, i.e., from an infinite-mass particle.

Thus we see that the singularities in the J plane of $S(E, J)$ are not only the Regge poles, but there is also a branch cut due to the factor $[(J+1/2)^2 - e^4]^{\frac{1}{2}}$. That this branch cut really is there and is not an apparent one may be seen from noting that near $J'=0$ we have

$$S(E, J) = [\Gamma(-J_0(E))/\Gamma(1+J_0(E))] \times \left\{ 1 + J' \left(\frac{1}{J_0(E)} - \pi \cot \pi J_0(E) \right) + \dots \right\}. \quad (3.5)$$

⁶ In this section $\hbar=c=1$. The upper and lower signs before e^2 refer respectively, to attractive and repulsive Coulomb potential, in this section.

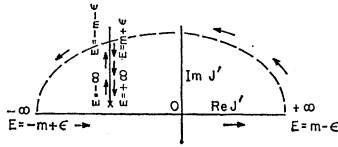


FIG. 3. The trajectory of the n th Regge potential in the J' plane for Klein-Gordon particle scattering by attractive Coulomb potential.

It is to be noted, however, that this branch cut does not move with energy. It is a fixed branch cut. The trajectory of the n th Regge pole, which is given by

$$J' + 1 \mp e^2 E / (m^2 - E^2)^{1/2} = -n, \quad (\text{for } n = 0, 1, 2, \dots), \quad (3.6)$$

is shown in the J' plane in Fig. 3. Here, one may make a remark which has some relevance from the point of view of a relativistic S matrix, i.e., that the Regge pole is an analytic function of E in the neighborhood of $E = 0$. The Regge poles also give correct bound-state energies in the case of attractive potential. The amplitude is seen to have the following Mandelstam representation

$$A(E, t) = \frac{1}{\pi^2} \int_0^\infty \frac{dt'}{t' - t} \int_m^\infty dE' \left\{ \frac{\rho_1(E', t')}{E' - E} + \frac{\rho_2(E', t')}{E' + E} \right\}. \quad (3.7)$$

For the case of Coulomb scattering of Dirac particles, we have entirely similar results. For every given J value there are two S -matrix elements having orbital angular momentum $L = J \pm 1/2$. We have

$$S(E; J, L = J - 1/2) = \frac{[J + (1/2) + i\rho'] \Gamma(\beta_{J+1/2} - i\rho)}{\Gamma(\beta_{J+1/2} + 1 + i\rho) \exp\{i\pi[J + (1/2) - \beta_{J+1/2}]\}},$$

$$S(E; J, L = J + 1/2) = \frac{[J + (1/2) - i\rho'] \Gamma(\beta_{J+1/2} - i\rho)}{\Gamma(\beta_{J+1/2} + 1 + i\rho) \exp\{i\pi[J + (1/2) - \beta_{J+1/2}]\}},$$

where

$$\rho = \rho' E, \quad \rho' = \pm e^2 / [E^2 - m^2]^{1/2}, \quad \beta_J = [J^2 - e^4]^{1/2}.$$

The point to note is that the singularities in the J plane of the S -matrix elements having the same J are the same, and these are the same as those for Klein-Gordon particles, apart from the replacement $J' - 1/2$ for J' due to spin $1/2$ of Dirac particles. On the contrary, the singularities of the two amplitudes having the same L but different J , i.e., $J = L \pm 1/2$, are not the same in the L plane. They are displaced with respect to each other.

IV. SUMMARY AND INTERPRETATION OF THE RESULTS

We have, therefore, the following results and further remarks for nonrelativistic potential scattering:

(a) For potential scattering, in general, the existence of an *a priori* natural boundary of analytic continuation in angular momentum at any place has been ruled out. Such a boundary may exist for some potentials, but it seems unlikely.

(b) The only singularities anywhere in the J plane for Coulomb scattering are Regge poles. The n th Regge pole starts from and comes back to $J = -n$ as the energy (taken real) is increased from $E = -\infty$ to $E = +\infty$. This may in general be true even for a superposition of Yukawa potentials. The feeling is based on the following circumstance. We know that for large E , the scattering amplitude converges uniformly to the Born approximation. Thus for the J th partial wave for the potential

$$V(r) = \int_M^M \sigma(\mu) \frac{e^{-\mu r}}{r} d\mu \quad (4.1)$$

we have

$$A(E, J) \equiv \frac{1}{2i} [S(E, J) - 1]$$

$$\underset{E \rightarrow \infty}{\approx} e^{i\phi} \frac{1}{2E} \int_m^M \sigma(\mu) d\mu Q_J \left(1 + \frac{\mu^2}{2E} \right), \quad (\pi > \phi > 0) \quad (4.2)$$

and expression that is analytic, except for poles at $J = -1, -2, \dots$. This circumstance also makes it plausible to the author that there may be nothing but Regge poles, since the other singularities do not seem to show up for $E \rightarrow \infty e^{i\phi} / (\pi > \phi > 0)$. However, the reader is warned to take these remarks with a skeptical attitude, as they do not constitute proofs.

(c) We found that the position of a Regge pole satisfies a dispersion relation of the following form for the Coulomb scattering:

$$\text{Re} \alpha_n(E) = \alpha_n(-\infty) + \frac{P}{\pi} \int_0^\infty dE' \frac{\text{Im} \alpha_n(E')}{(E' - E)}. \quad (4.3)$$

We assume this result is true in general. Now Regge tells us that one has

$$\text{Im} \alpha_n(E') \geq 0, \quad \text{for } E' \geq 0 \text{ if } \text{Re} \alpha_n(0) > -1/2. \quad (4.4)$$

Thus all the derivatives of $\text{Re} \alpha_n(E)$ are positive for $0 \geq E$, and $\alpha_n(E)$ is a monotonically increasing real-valued function of E below threshold.

(d) The Regge poles correctly give the bound states.

(e) The Mandelstam conjecture,⁷

$$S(E, J) = S(E, -J - 1), \quad J = \pm 1/2, \pm 3/2, \dots \quad (4.5)$$

holds true for the Coulomb case and may be true in general.

To these above remarks may be added the following results on scattering of relativistic particles by Coulomb potential of a fixed source.

⁷ This has been conjectured by S. Mandelstam also; see reference 5.

(f) There are fixed branch cuts in the J plane which do not move with energy. These arise because the centrifugal term $J(J+1)/r^2$ has been modified to $[J(J+1)-e^4]/r^2$. Thus we may not expect these fixed branch cuts for nonrelativistic potential cases as long as the centrifugal term does not get modified.

(g) The Regge poles considered as a function of energy E are analytic at $E=0$ and have two branch cuts running over the real axis, one from $E=+m$ to $E=+\infty$ and another from $E=-\infty$ to $E=-m$. The significance and implications of the latter are not clear.

In conclusion, we wish to mention that some aspects of the Coulomb scattering problem have also been considered by Goldberger and Blankenbecler.⁸

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⁸ R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).

Negative Meson Absorption in Liquid Hydrogen*†

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A theoretical study has been made of the atomic processes involved in the absorption of π^- and K^- mesons in liquid hydrogen, with the main purpose of setting an upper limit on the fraction of K mesons that react with the proton from the P state. The principal mode of de-excitation of the mesonic atom, Auger ionization of neighboring hydrogen atoms, was calculated in Born approximation for all the appropriate initial and final values of n . The Stark mixing process discussed by Day, Snow, and Sucher, which allows S -state reaction from high n orbitals, was calculated quantitatively using an impact parameter method and including the effect of the S -state energy shift. In agreement with Day *et al.*, it was found that practically all of the K^- mesons react from the S state; the actual fraction of P -state reactions is less than 1%. In addition, the calculated cascade time for π^- in liquid hydrogen is compatible with the experimental value.

1. INTRODUCTION

OF importance to the analysis of low-energy \bar{K} -nucleon scattering data is the answer to the question: From what orbital angular momentum states do K^- mesons absorbed in liquid hydrogen react with the protons? In this paper we attempt to place onto a more quantitative footing the conclusion of Day, Snow, and Sucher¹ that the reaction is overwhelmingly from the S state.

Until recently, the final word on the subject of negative meson absorption in hydrogen was that of Wightman.² Wightman was particularly concerned with the π^- , and with demonstrating that π^- decay could not

compete with the moderation, atomic capture, and de-excitation into low atomic orbitals. There is no question of nuclear reaction from the P state simply because the rate for capture is demonstrably much smaller than the nP to $1S$ radiation rate.³ For K^- , in contrast, this is not the case; in fact, the rather meager data indicate that these two rates might be comparable.⁴

The development that has stimulated interest in this problem is the observation of Day *et al.*¹ that the electric fields of neighboring hydrogen atoms cause the K -mesonic atom to make transitions among the degenerate states of principal quantum number n . Since the mesonic atom is neutral, rather small, and has no electrons of its own, it can pass freely into the hydrogen-atom electron cloud where it feels the field from the proton. The resulting "Stark effect" makes the strong S -state absorption effective even for orbital angular momenta $l > 0$. Day *et al.* concluded that this effect suffices to ensure K^- absorption (i.e., inelastic reaction with the proton) at high n via the S state, before the atom can de-excite to low n where P -state capture might be important. Further, Russell and Shaw⁵ pointed out

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⁴ L. W. Alvarez, University of California Radiation Laboratory Report UCRL-9354, 1960 (unpublished).

⁵ J. E. Russell and G. L. Shaw, Phys. Rev. Letters **4**, 369 (1960).