Some Experimental Consequences of Triangle Singularities in Production Processes*

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(Received March 9, 1962)

The infinite anomalous threshold singularities in the amplitude for a production reaction may, in certain circumstances, lie close to the physical region. The possibility then arises that they can be "detected" through characteristic peaking effects which they produce, as sometimes happens for the more familiar pole-type singularities. Examples based on triangle graphs are discussed.

I. INTRODUCTION

ONE knows, on the evidence of perturbation theory, that the transition amplitudes for production reactions are in general characterized by the occurrence of various anomalous threshold singularities, both real and complex.[1,2] The recent work of Polkinghorne and of Stapp[3] suggests that this should obtain more generally for any unitary theory. It is our object here to consider whether these singularities can ever be "noticed" experimentally. The hope, of course, is that, under favorable kinematic circumstances, they may appreciably influence the shape of a cross-section curve in some characteristic manner, in much the same sense as the familiar pole-type singularities sometimes do. In fact our singularities are not poles, but provided that they are infinities rather than simple branch points, they may be expected to produce noticeable effects when they approach close to the physical region. According to the rules given by Landau and by Polkinghorne and Screaton,[4] only the simplest graphs produce singularities that are infinite. We shall confine ourselves here to a discussion of simple triangle graphs, which in fact are the only graphs with three external vertices that do produce infinities.

Even with this restriction the general situation is still very complicated and not very easily surveyed. For a reaction involving n particles ($n\geqslant 5$) there are many ways of disposing the n external momenta at the three vertices, and there are $N=3n-10\geqslant 5$ independent variables to be considered, aside from spins. The singularities for a given triangle graph lie on a part of a $(2N-2)$ manifold in the $2N$-dimensional space of N complex variables. One might expect a singularity effect to manifest itself most directly in the physical amplitude for those points in the physical region which lie closest to the singular part of the manifold. For example, a plot of the amplitude as a function of physical variables lying on a one-parameter curve which passes through a "close" point might be expected to show a peak in the neighborhood of this point. As in corresponding discussions of pole-type singularities, however, we of course cannot know in advance how close is close enough.

II. TRIANGLE SINGULARITIES

Rather than cope with the complicated generalities of the structure of the physical manifold and of the various singularity manifolds for different kinds of graphs, let us turn directly to what is a particularly simple class of graphs. Consider the production reaction

$$k_1+k_2\rightarrow q+(K_1+K_2+\cdots),$$

where each letter represents a particle and also its 4-momentum. In particular we shall denote by m the mass of the particle q: $q^2=m^2$. We now restrict ourselves to triangle graphs (see Fig. 1) in which the colliding particles k_1 and k_2 join at one vertex, the single particle q emerges from a second vertex, and the remaining particles K_1, K_2, $\cdots$ all join at the third vertex. Aside from spins, the amplitude for such a graph depends on only two independent variables, which we take to be

$$W^2=(k_1+k_2)^2;\quad s=(K_1+K_2+\cdots)^2.$$

We study the dependence of the amplitude on the variable s, for fixed W^2.

Let M denote the sum of the masses of the particles K_1, K_2, $\cdots$. The physical range of the variable s, for fixed W, is given by

$$M^2<s<(W-m)^2\equiv s_1. \tag{1}$$

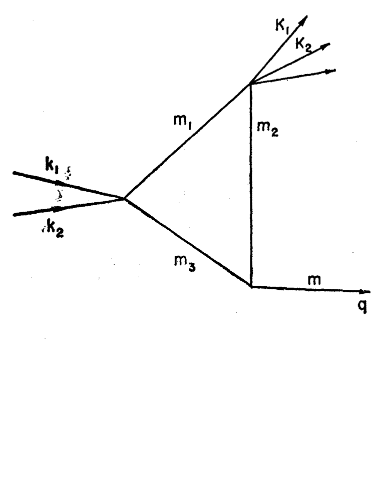

FIG. 1. The basic triangle graph under consideration.

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* Supported in part by the U. S. Air Force Office of Scientific Research, Air Research and Development Command.

† On leave of absence from St. John's College, Cambridge, England.

[1] P. V. Landshoff and S. B. Treiman, Nuovo cimento 19, 1249 (1961).

[2] L. F. Cook and J. Tarski, J. Math. Phys. 3, 1 (1962).

[3] J. C. Polkinghorne (to be published); H. P. Stapp (to be published).

[4] L. D. Landau, Nuclear Phys. 13, 181 (1959); J. C. Polkinghorne and G. R. Screaton, Nuovo cimento 15, 925 (1960).

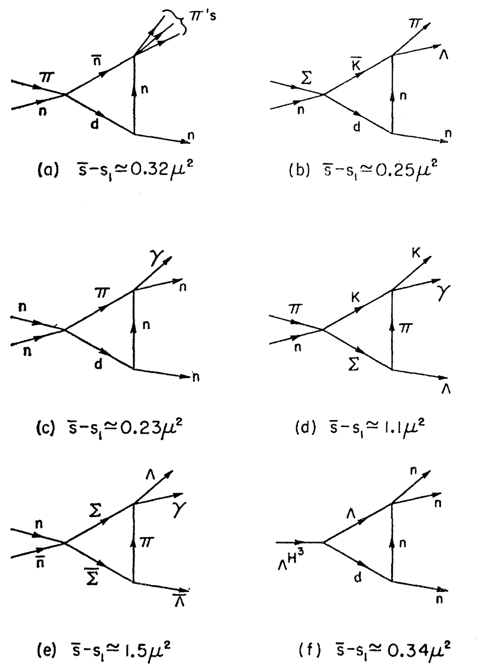

FIG. 2. Examples of graphs for which the anomalous threshold singularity at $\bar{s}$ lies close to the upper end s_1 of the physical region, when the center-of-mass energy is $W=m_1+m_3$. The separation $\bar{s}-s_1$ is listed for each example; μ denotes the pion mass.

Now the triangle graph in question produces a normal threshold singularity at

$$s_n=(m_1+m_2)^2. \tag{2}$$

If W is large enough, and if the internal particles are stable, as we assume, then the graph also produces an anomalous singularity.[1] For $W>m_1+m_3$ this occurs at the complex point $\bar{s}$ given by

$$\bar{s}=m_1^2+m_2^2-2m_1m_2Z, \tag{3}$$

where

$$Z=zz'+i[(z^2-1)(1-z'^2)]^{\frac{1}{2}}, \tag{4}$$

$$z=(m_1^2+m_3^2-W^2)/2m_1m_3, \tag{5}$$

$$z'=(m_2^2+m_3^2-m^2)/2m_2m_3. \tag{6}$$

For $W=m_1+m_3$, $\bar{s}$ is real and we have

$$\bar{s}-s_1=[1+(m_1/m_3)][m_2^2-(m_3-m)^2], \tag{7}$$

$$s_n-\bar{s}=(m_1/m_3)[m^2-(m_2-m_3)^2]. \tag{8}$$

As W decreases below m_1+m_3, $\bar{s}$ moves away from the physical region, toward the normal threshold. This is of course just what we do not want to happen, so we shall not consider the case $W<m_1+m_3$ any further.

However, we also note that the physical amplitude is obtained by allowing the variable s to approach the real axis from above, whereas for $W>m_1+m_3$ the singularity $\bar{s}$ lies in the lower half s plane. Thus the case where Re $\bar{s}\gg s_n$ is also uninteresting, since here a normal cut intervenes between s and $\bar{s}$ even when they are "close" to one another. Inspection of Eqs. (3) to (7) shows that for $W\gg m_1+m_3$ the singularity $\bar{s}$ moves far away from the physical region, except when $z'\approx 1$; but in the latter case Re $\bar{s}\gg s_n$. All of these considerations therefore suggest that the most hopeful prospects for "detecting" the influence of the anomalous singularity corresponds to choosing W in a more or less narrow region near m_1+m_3.

We can therefore most easily survey the possibilities by taking the simple case $W=m_1+m_3$ and looking for diagrams for which $\bar{s}-s_1$ is small enough to be interesting, e.g., $\bar{s}-s_1\lesssim$ (pion mass)^2.

Some examples are shown in Fig. 2 and it is easy enough, by inspection of the equations, to generate others. All are somewhat impracticable in that they involve emission of photons or require very high (and narrowly defined) incident-particle energies or are concerned with processes that are as yet not very common. The sixth, a hyperfragment decay, involves a weak interaction. Notice that, in the notation of Fig. 1, the number and nature of the particles emitted at the (m_1,m_2) vertex is irrelevant in determining the location of s_1 and $\bar{s}$, provided the choice is consistent with selection rules and, for given W, with energy-momentum conservation. Thus in Fig. 2(a) the particles emitted at the $(n,\bar{n})$ vertex could consist of any number of pions and, say, $K\bar{K}$ pairs, up to the limit set by available energy. Similarly, the nature of the incident particles in each graph has no effect on the location of the singularity. Thus, again in Fig. 2(a), the incident (πn) could be replaced by $(\bar{n}d)$. This might seem ideal for our purpose, since with our chosen value for W the antinucleon would be at rest, as is very convenient for experimental study. However, our triangle singularity $\bar{s}$ then coincides with the nucleon-exchange pole in the diagram obtained by "dissolving" the $(\bar{n}d)$ interaction, so that our effect, although expected to be there, would be swamped by another.[5] The same remark applies to the hypertriton-decay diagram of Fig. 2(f) if it is valid to regard the hypernucleus as a (Λd) scattering state. One might, however, say that this model should be modified by inclusion of the wave function for the Λ in the hypernucleus and that this could have the effect of changing the pole singularity precisely into one of the triangle type.

III. AN EXAMPLE

We illustrate the way in which the anomalous singularity influences the transition amplitude by considering in detail one example, that of Fig. 2(a). We compute the Feynman amplitude for this graph with neglect of structure effects at the vertices, since we are only concerned with the variation of the amplitude over a narrow range near the singularity. For the same reason we also neglect spin effects, treating all particles as spinless. A closed form for the Feynman integral has been ob-

[5] This has been called to our attention by Professor R. Blankenbecler.

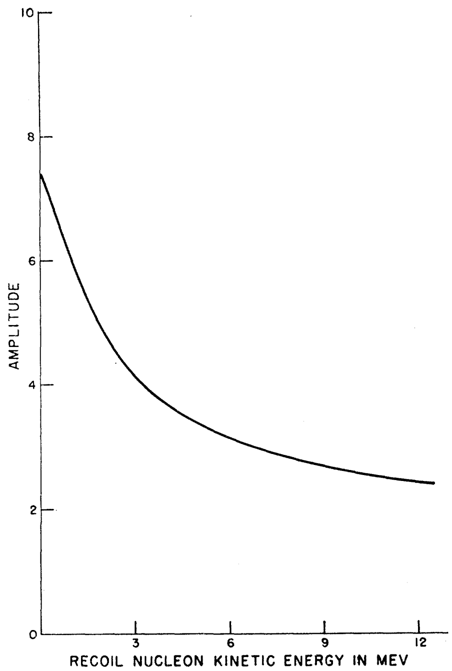

FIG. 3. Amplitude (on arbitrary scale) for the graph corresponding to FIG. 2(a), as a function of center-of-mass kinetic energy of outgoing nucleon. The total center-of-mass energy is fixed at a value corresponding to the sum of masses of nucleon and deuteron.

tained by Wu.[6] This contains a large number of Spence functions, however, and is unattractive for computation purposes. We prefer instead to write down a dispersion relation in the variable s. The weight function is easily obtained in explicit form, and the dispersion integral can readily be evaluated numerically. Details are set out in the Appendix.

The results are most conveniently expressed, not in terms of s, but rather in terms of a variable T which denotes the kinetic energy, in the over-all center-of-mass system, of the outgoing nucleon. This is related to the variables W and s by

$$T=[(W-m)^2-s]/2W=(s_1-s)/2W. \qquad (9)$$

For our simple choice $W=n+d$ (≈ 3 BeV) the transition amplitude is real and its dependence on T is shown in Fig. 3. The sharp increase in the amplitude towards small T will be noted: the amplitude is roughly doubled in a width $\Delta T\approx 3$ MeV. In the region of rapid variation the squared amplitude varies as T^{-1}, an effect which should outweigh the phase-volume effect, which goes as $T^{\frac{1}{2}}\,dT$. That is to say, insofar as the graph in question dominates all others at small T, the experimental nucleon-recoil spectrum should show a noticeable, narrow bump at small T. We may also remark here that, for fixed-recoil nucleon energy T, our amplitude is independent of the angle of emission of the nucleon. This could be tested if one had reason to suspect that the graph in question were in fact dominating the reaction. Similar remarks hold for all the graphs under discussion.

The sharp effect indicated in Fig. 3 corresponds to the precise choice $W=n+d$. As W is increased, $\bar{s}$ becomes complex, but since z' is very nearly unity, ($z'=1-\epsilon$, $\epsilon\approx 1/800$), Im $\bar{s}$ remains small for a long while. However, Re $\bar{s}$ very quickly moves past the normal threshold at $4n^2$. By the t me that W has increased by 50 MeV above the value $n+d$, all noticeable peak effects have disappeared, even though Re $\bar{s}$ and s_1 are very close to one another. As we have already said earlier, when the cut intervenes between $\bar{s}$ and a point s "near" to $\bar{s}$, the distance between the points must really be measured along a path that goes around the normal threshold, without crossing any cuts; and this distance increases rapidly here with increasing W.

We also learn in this example that there is no reason to suppose that only singularities that occur on what is usually taken to be the physical sheet should be effective. Singularities which are uncovered when the normal cuts are distorted may well be important if they lie near the physical region ("nearness" again being somehow defined in terms of a distance along a path that crosses no cuts). In fact this observation is relevant for the results of Fig. 3. The singularity at $\bar{s}$ is only logarithmic, so that one might have expected that no noticeable effect persist at s_1. What comes into play here, however, is another singularity at s_1. This one lies on the "unphysical" sheet reached through the cut attached to $\bar{s}$. This singularity, first noticed in triangle graphs by Cutkosky,[7] is an example of a second-type singularity,[8] whose position depends only on the masses of external particles in the graph. Here it behaves as $1/(s_1-s)^{\frac{1}{2}}$ near s_1 and it combines with the logarithmic singularity at $\bar{s}$ in a subtle way to produce the sharp peaking effects noted here. How this comes about can be seen in more detail from the discussion given in the Appendix.

We note finally that the general character of the effects discussed above for the graph of Fig. 2(a) holds also for the other graphs under consideration: a sharp peaking in the spectrum of the recoil particle at low energies, for incident energy W in a narrow interval near m_1+m_3. The effect rapidly disappears with increasing energy W.

The present investigation was prompted in part by urgings of Professor M. L. Goldberger, who has long regarded the actuality of anomalous singularities as a critical test of present-day notions. He bears no responsibility for the results discussed here, however. We also call attention to a discussion of other aspects of these singularities in production reactions, given by R. F. Sawyer.[9]

[6] A. C. T. Wu (private communication).

[7] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).

[8] D. B. Fairlie, J. Nuttall, P. V. Landshoff, and J. C. Polkinghorne (to be published).

[9] R. F. Sawyer, Phys. Rev. Letters 7, 213 (1961).

APPENDIX

The weight functions in the representation for the triangle graph may be evaluated by the standard method of Cutkosky.[7] The result, when the singularity at $\bar{s}$ appears on the physical sheet, is proportional to

$$\int_{\bar{s}}^{s_n} \frac{2\pi}{[K(s')]^{\frac{1}{2}}} \frac{ds'}{s'-s} + \int_{s_n}^{s_2} \frac{2}{[K(s')]^{\frac{1}{2}}} \times \tan^{-1}\left(\frac{[K(s')L(s')]^{\frac{1}{2}}}{a(s')}\right) \frac{ds'}{s'-s-i\epsilon} + \int_{s_2}^{\infty} \frac{1}{[-K(s')]^{\frac{1}{2}}} \times \ln\left(\frac{a(s')+[-K(s')L(s')]^{\frac{1}{2}}}{a(s')-[-K(s')L(s')]^{\frac{1}{2}}}\right)\frac{ds'}{s'-s}, \quad \text{(A1)}$$

where

$$\begin{aligned} s_2 &= (W+m)^2, \\ K(s') &= (s_2-s')(s'-s_1), \\ L(s') &= [s'-(m_1+m_2)^2][s'-(m_1-m_2)^2], \\ a(s') &= s'^2+s'(2m_2^2-m_1^2-m_3^2-W^2-m^2) \\ &\quad +(W^2-m^2)(m_1^2-m_2^2), \end{aligned} \quad \text{(A2)}$$

and s_1, s_n, $\bar{s}$ are defined, respectively, in (1), (2), (3).

We have written (A1) in the form directly applicable to our computation discussed in Sec. III, where we took $W=m_1+m_3$ so that $\bar{s}$ was real and $s_1<s_n$. In this case the square roots in (A1) are to be taken to have positive values and the logarithm is real. The arctangent takes the value π at $s'=s_n$ and 0 at $s'=s_2$, there being a zero of $a(s')$ in between these points. The first integral in (A1) may be evaluated in closed form and the result is

$$\frac{4\pi}{[-K(s)]^{\frac{1}{2}}}\left\{\tan^{-1}\left(\left[\frac{(s_n-s_1)(s_2-s)}{(s_2-s_n)(s_1-s)}\right]^{\frac{1}{2}}\right) - \tan^{-1}\left(\left[\frac{(\bar{s}-s_1)(s_2-s)}{(s_2-\bar{s})(s_1-s)}\right]^{\frac{1}{2}}\right)\right\}, \quad \text{(A3)}$$

where again the radicals are positive and the arctangents lie between 0 and π. The remaining two integrals in (A1) must be estimated numerically.

When $s_1>s_n$ and when $\bar{s}$ is complex the equations (A1) and (A3) are still formally correct. The sheets of the logarithm and arctangents must be determined by continuation from the previous case.

We may notice some qualitative features of our function for the case discussed in Sec. III. The value of (A3) at $s=s_1$ is

$$\frac{4\pi}{s_2-s_1}\left[\left(\frac{s_2-\bar{s}}{\bar{s}-s_1}\right)^{\frac{1}{2}} - \left(\frac{s_2-s_n}{s_n-s_1}\right)^{\frac{1}{2}}\right]^0. \quad \text{(A4)}$$

When, as in our case, $\bar{s}$ and s_1 are close together this is large. The second integral in (A1) produces a contribution of the same order of magnitude.

As s is taken below s_1 the contribution (A3) falls off rapidly, both because $-K(s)$ increases and because the arctangents become nearly equal. The second integral, however, falls off more slowly. $a(s')$ has a zero at a point s^* close to s_n and a substantial part of the contribution from the second integral arises from the range s^* to about $2s^*$. A crude idea of the behaviour may be obtained by approximating the arctangent as $\pi/2$ in this range and then the result of the integration is as in (A3), with s_n replaced by $2s^*$ and $\bar{s}$ by s_n. As s varies over a small range below s_1, the arctangents are sensibly constant and so we see that the behaviour is as $1/(s_1-s)^{\frac{1}{2}}$. This is in accord with our assertions in Sec. III concerning the interplay of the singularity at $\bar{s}$ with the second-type singularity at s_1. The effect of the nearness of the $\bar{s}$ singularity is mainly felt indirectly in that it causes the zero s^* of $a(s)$ to be close to s_n.