PHYSICAL REVIEW VOLUME 127, NUMBER 3 AUGUST 1, 1962

Generators of Coordinate Transformations in the Penrose Formalism of General Relativity

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(Received March 28, 1962)

A new covariant expression for the generator of coordinate transformations is developed in the general theory of relativity. It is hoped that this expression may facilitate the determination of the commutation relations between the independent dynamical variables of Penrose. Since the Penrose function is defined on a characteristic surface, a program for quantization is discussed which can be applied to dynamical variables defined on such null surfaces. In addition, the relationship of the new conservation law to the usual physical constants of Lorentz-covariant theories is indicated, the relationship being effected when there exists a Killing bivector in the Riemannian manifold.

I. INTRODUCTION

IN a recent series of papers, Penrose[1] has shown how to construct a complete, independent set of quantities which uniquely characterize in a systematic fashion the solutions of all the physical theories of interest, including among others the Dirac, Maxwell, linearized gravitation, and Einstein gravitation theories. The unusual thing about Penrose's procedure is that in each of the above-mentioned cases the solutions are completely characterized by the specification of a single complex function on a null cone. For the Maxwell theory the appropriate function is a suitably defined projection of the Maxwell tensor into the null cone. For the linearized gravitation theory, and for the Einstein general theory of relativity, the appropriate function is a similarly defined projection of the curvature tensor into the null cone. Specification of (what we hereafter refer to as) the Penrose function on the light cone uniquely determines the solution (via the appropriate field equations) in a surprising fashion. The portion of the Penrose function defined on the forward (backward) light cone determines the solution uniquely in the interior of the forward (backward) light cone. For space-like distances the properties of the solution are more difficult to analyze and the stability of the determination is questionable.

In view of the fact that the Penrose function provides a complete, nonredundant characterization of solutions it would appear to be ideal for use in constructing the quantization of the corresponding classical theory. For this purpose it is necessary to determine the classical Poisson brackets of the Penrose function at different world points on the null cone.

It is usual to define the Poisson brackets as a particular set of differential operations upon pairs of functions of canonical variables which are defined on a space-like hypersurface, generally taken to be of constant time. Instead of proceeding blindly to attempt a construction of "canonical variables" and then impose canonical commutation relations upon them, if one makes an effort to understand what the point of the whole procedure is, it is readily seen that there is no need to confine one's considerations to space-like hypersurfaces. Dirac has shown,[2] for Lorentz -covariant theories, how one can construct commutation relations between quantities defined on arbitrary surfaces, whether space-like or null. For particle theories, the only requirement placed upon the surface is that each particle trajectory can be uniquely labeled by a set of properties determined at the point of intersection of the trajectory with the surface. Thus, in particular, each particle trajectory must intersect the surface. The independent set of quantities which specify the trajectory we will call the independent dynamical variables of the theory. For the particular case where the surface is chosen to have the equation $t=\text{const}$, the independent dynamical variables may be taken to be the three position coordinates and three momentum coordinates of the particle at that instant of time, in other words, the usual canonical variables. Given the independent dynamic variables, Dirac demands that, for Lorentz-covariant theories, it is possible to construct from them a particular set of generators of canonical transformations on the classical phase space which provide a representation of the Lorentz group. That is, the Poisson brackets of the particular set of generators constructed should satisfy the commutation relations of the Lorentz group. The procedure for quantization is then to maintain the algebraic relationship between the dynamical variables and the generators as closely as possible, but to regard the dynamical variables and the generators as linear operators on a Hilbert space, such that the generators now provide a representation of the Lorentz group on the Hilbert space. That is, the representation space for physical states is altered from the classical phase space to Hilbert space, but the algebraic relations between the dynamic variables and the generators, and the group being represented (and, consequently, the Poisson brackets) are preserved.

For the case of the general theory of relativity the relevant group is not the Lorentz group, but the group

* Supported in part by National Science Foundation.

[1] R. Penrose, Ann. Phys. (New York) **10**, 171 (1960); also "Null Hypersurface Initial Data for Classical Fields of Arbitrary Spin and for General Relativity" (to be published); see the Appendix for a brief summary.

[2] P. A. M. Dirac, Revs. Modern Phys. **21**, 392 (1949).

of general coordinate transformations. It would appear from the above discussion of Dirac's work that the appropriate procedure to use for the quantization of the Einstein theory is to construct the generators of the group of general coordinate transformations and express them as functions of an independent set of dynamical variables. However, due to the fact that the group of general coordinate transformations is a function group rather than a Lie group, the generators of coordinate transformations vanish modulo the field equations. The independent set of dynamical variables of general relativity ("observables") are precisely those quantities which are invariant under coordinate transformations. The program for quantization requires that we determine the canonical transformations generated by the observables and thereby determine the commutation relations between them. Then, apart from questions of factor ordering, the observables are to be regarded as Hermitian operators on a linear vector space and the infinitesimal unitary transformations which they generate are to have the same commutation relations as in the classical theory.

An important step in the determination of true observables and the canonical transformations which they generate is the recognition of those expressions formed from the field variables which generate purely coordinate transformations. In the usual notation of general relativity the expressions for the generators of coordinate transformations are well known.[3] An essential difficulty arises, however, when we seek to employ these known expressions in the Penrose formalism. For, in Penrose's construction, the Einstein field equations are satisfied identically by the very way the symbols of Penrose are defined. The "field equations" in the Penrose formalism correspond to the Bianchi identities of the usual theory. The usual expression for the generating density of a coordinate transformation[3]

$$C^\rho(\xi)=-2\xi^\mu G_\mu{}^\rho \tag{1}$$

(where $G_\mu{}^\rho$ are the Einstein equations, and ξ^μ is the descriptor of the infinitesimal coordinate transformation $\bar{X}^\mu=X^\mu+\xi^\mu$) is then identically zero in this formalism. (We shall work throughout in the Lagrangian formalism of Bergmann and Schiller[4] since this enables us to treat canonical transformations in a manifestly covariant fashion.)

In Sec. II of this paper we are therefore concerned with the construction of an alternative expression for the generator of coordinate transformations which can be expressed readily and simply in the Penrose notation. In Sec. III we digress somewhat to discuss some of the properties of the new covariant conservation laws which we have obtained, in particular how they might be related to the preferred conserved quantities of Lorentz covariant theories. In an appendix we briefly sketch some of the relevant aspects of the Penrose formalism and show how the generator of coordinate transformations derived in Sec. II is expressed in Penrose's notation. The fact that the Penrose function is defined on a light cone appears to present no essential obstacle to the quantization program particularly in view of the perspective we have gained from the works of Dirac.[2]

II. GENERATORS OF COORDINATE TRANSFORMATIONS

Consider now the expression

$$\begin{aligned}C^\rho(r)&=r_{\alpha\beta;\gamma}[C^{\alpha\beta\gamma\rho}+g^{\rho\alpha}R^{\beta\gamma}+\tfrac{1}{6}Rg^{\alpha\gamma}g^{\beta\rho}]\\&\equiv r_{\alpha\beta;\gamma}\bar{C}^{\alpha\beta\gamma\rho},\end{aligned} \tag{2}$$

where $r_{\alpha\beta}$ is an arbitrary antisymmetric tensor and $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, which is related to the Riemann tensor $R_{\alpha\beta\gamma\delta}$ via

$$\begin{aligned}C_{\alpha\beta\gamma\delta}=R_{\alpha\beta\gamma\delta}&\\-\tfrac{1}{2}(g_{\alpha\gamma}R_{\beta\delta}+g_{\beta\delta}R_{\alpha\gamma}-g_{\delta\gamma}R_{\alpha\delta}-g_{\delta\alpha}R_{\beta\gamma})&\\+\tfrac{1}{6}R(g_{\alpha\gamma}g_{\beta\delta}-g_{\beta\gamma}g_{\alpha\delta}).&\end{aligned} \tag{3}$$

When the Einstein field equations are satisfied $C^\rho(r)$ does not vanish, but is equal to $r_{\alpha\beta;\gamma}C^{\alpha\beta\gamma\rho}$. Taking the divergence of Eq. (2), we find after a bit of computation that

$$C^\rho(r)_{;\rho}=\tfrac{1}{2}(r^\rho{}_{\beta;\rho\gamma}+r^\rho{}_{\gamma;\rho\beta})G^{\beta\gamma}. \tag{4}$$

Thus, we can conclude from the Bergmann-Schiller formalism that

$$C(r)\equiv\int_S C^\rho(r)dS_\rho \tag{5}$$

generates a canonical transformation such that

$$\delta g_{\mu\nu}=-\tfrac{1}{2}(r_\mu{}^\rho{}_{;\rho\nu}+r_\nu{}^\rho{}_{;\rho\mu}). \tag{6}$$

[The hypersurface S over which the integration in Eq. (5) is to be taken is customarily understood to be everywhere space-like. However, if the arbitrary bivector field $r_{\alpha\beta}$ is taken to have a compact support, the surface integration may readily be deformed into either the future or past light cone if we so desire.] Equation (6) is precisely the change induced in the metric tensor by means of an infinitesimal coordinate transformation whose descriptor is

$$\xi^\mu=\tfrac{1}{2}r^{\mu\rho}{}_{;\rho}. \tag{7}$$

We find that the constant of motion $C(r)$ defined by Eqs. (2) and (5) generates a subgroup of the group of general coordinate transformations described by an arbitrary bivector field $r_{\mu\nu}$. From Eq. (7), we see that

$$\xi^\mu{}_{;\mu}=-\tfrac{1}{2}r^{\mu\rho}{}_{;\mu\rho}=0, \tag{8}$$

and therefore this is precisely the subgroup whose Jacobian is unity. Although this is not an invariant subgroup (since the group of general coordinate transformations is simple), it is probably sufficiently large

[3] P. G. Bergmann, Phys. Rev. 112, 287 (1958).
[4] P. G. Bergmann and R. Schiller, Phys. Rev. 89, 4 (1953).

for the purpose of deducing the required commutation relations of the Penrose function.

If ξ^ρ and η^ρ are two arbitrary vector fields which may be taken to be the descriptors of two infinitesimal coordinate transformations, the commutation relations for the group of general coordinate transformations may be characterized by defining the vector field λ^ρ,

$$\lambda^\rho \equiv \xi^\mu \eta^\rho{}_{;\mu} - \eta^\mu \xi^\rho{}_{;\mu}, \tag{9}$$

to be the descriptor of the infinitesimal coordinate transformation which is the commutator of the infinitesimal coordinate transformations described by ξ^ρ and η^ρ. In particular, if ξ^ρ is derived from a bivector field $r^{\rho\sigma}$ via Eq. (7), and if η^ρ is analogously determined by a bivector field $s^{\rho\sigma}$, that is,

$$\eta^\rho = \tfrac{1}{2} s^{\rho\sigma}{}_{;\sigma}, \tag{10}$$

then it is readily seen that λ^ρ as defined by Eq. (9) is also derivable from a bivector $t^{\rho\sigma}$:

$$\lambda^\rho = \tfrac{1}{2} t^{\rho\sigma}{}_{;\sigma}, \tag{11}$$

where

$$t^{\rho\sigma} = \tfrac{1}{2}(s^{\rho\alpha}{}_{;\alpha} r^{\sigma\beta}{}_{;\beta} - r^{\rho\alpha}{}_{;\alpha} s^{\sigma\beta}{}_{;\beta}). \tag{12}$$

We are now in a position to state the commutation relations which we must demand of the constants of motion which generate the above coordinate transformations. If we denote by $C(s)$ and $C(t)$ the constants of motion which are determined by the bivectors $s^{\rho\sigma}$ and $t^{\rho\sigma}$, respectively, in the identical fashion that $C(r)$ of Eq. (5) is determined by the bivector $r^{\rho\sigma}$, and if in addition the relation among the three bivectors $r^{\rho\sigma}$, $s^{\rho\sigma}$, and $t^{\rho\sigma}$ is defined by Eq. (12), then what we require is

$$[C(r), C(s)] = C(t), \tag{13a}$$

or more explicitly,

$$\left[\int r_{\alpha\beta;\gamma} \bar{C}^{\alpha\beta\gamma\rho} dS_\rho, \int s_{\alpha\beta} \bar{C}^{\alpha\beta\gamma\rho} dS_\rho\right] = \frac{1}{2}\int (s_{\alpha\sigma}{}^{;\sigma} r_{\beta\mu}{}^{;\mu} - s_{\beta\sigma}{}^{;\sigma} r_{\alpha\mu}{}^{;\mu})_{;\gamma} \bar{C}^{\alpha\beta\gamma\rho} dS_\rho. \tag{13b}$$

Since the bivectors $r_{\alpha\beta}$ and $s_{\alpha\beta}$ are completely arbitrary, they may be regarded as test functions in the above integrals.

It should be noted that the generating density found in this section [Eq. (2)] is weakly equal to a pure curl field. The constant of motion given by Eq. (5) may, therefore, be converted into a 2-surface integral. Thus, when the field equations are satisfied we have

$$C(r) = \oint_{\partial S} r_{\alpha\beta} C^{\alpha\beta\mu\nu} dS_{\mu\nu}. \tag{14}$$

The value of the generator $C(r)$ depends only on the boundary value of the bivector $r_{\alpha\beta}$, as must be the case since it generates purely coordinate transformations.

III. COVARIANT CONSERVATION LAWS

In this section we want to digress from the discussion of commutation relations and elaborate some of the properties of the covariant conservation law [Eq. (4)] found in the preceding section. The conserved quantity, $C(r)$ of Eq. (5) is defined by means of an arbitrary auxiliary geometric quantity, $r_{\alpha\beta}$. Although this is precisely what is required for the discussion of the commutation relations which we carried out in the preceding section, it is undesirable if we wish to obtain truly intrinsic geometric conservation theorems which may be related to physical quantities such an energy, momentum, and angular momentum. Since $r_{\alpha\beta}$ is a completely arbitrary bivector field it is in fact rather easy to construct intrinsic, geometric conserved quantities by employing geometrically determined bivectors. For example, there are the six eigen-bivectors of the Weyl tensor, or the four Ruse-Debever-Penrose null bivectors determined by the Weyl tensor. However, there is as yet no convincing argument which can relate the usual physical constants to the conserved quantities obtained by employing any of these geometric bivectors as a particular $r_{\alpha\beta}$ in Eq. (2).

The usual constants of motion of Lorentz-covariant theories are closely related to the existence of ten Killing vector fields in Minkowski space.[5,6] In the present formalism preferred conservation laws of a similar character can be obtained when there exist Killing bivector fields. We shall call a bivector field, $r_{\alpha\beta}$, "Killing" if its covariant derivative $r_{\alpha\beta;\gamma}$ is a completely antisymmetric tensor. That is, we may write

$$r_{\alpha\beta;\gamma} = |g|^{\frac{1}{2}} r^\sigma \epsilon_{\sigma\alpha\beta\gamma}, \tag{15}$$

where the vector field r^σ is the dual of the tensor $r_{\alpha\beta;\gamma}$. Provided the Ricci tensor of the space vanishes, after a somewhat lengthy calculation one can confirm from Eq. (15) that r^σ is necessarily a Killing vector field. We see, therefore, that the existence of a Killing bivector field places a heavy restriction on the Riemannian manifold.

Let $r_{\alpha\beta}$ now be a Killing bivector field. If we define the bivector $C^{\alpha\beta}(r)$ thus

$$C^{\alpha\beta}(r) = r_{\mu\nu} C^{\mu\nu\alpha\beta}, \tag{16}$$

then

$$C^{\alpha\beta}(r)_{;\beta} = r_{\mu\nu;\beta} C^{\mu\nu\alpha\beta} + r_{\mu\nu} C^{\mu\nu\alpha\beta}{}_{;\beta} = 0, \tag{17}$$

the first term on the right vanishing as a consequence o Eq. (15) and the cycle identity of the Weyl tensor, the second term on the right vanishing as a consequence of the Einstein field equations and the Bianchi identity. We obtain in this fashion a conservation law striking similar to the Gaussian flux integral for the total charge in the Maxwell theory. For it follows from Eq. (17)

[5] A. Trautman, Lectures on Relativity, Kings College, London, 1958 (unpublished).
[6] A. Komar, Phys. Rev. (to be published).

that the constant of the motion

$$C(r)\equiv\int_S C^{\alpha\beta}(r)_{;\beta}dS_\alpha=\oint_{\partial S}C^{\alpha\beta}(r)dS_{\alpha\beta} \tag{18}$$

is surface independent, provided the integral on the right is performed over any two-surface that includes all the singularities and sources of the gravitational field. The surface-independent constant constructed in Eq. (18) is more nearly analogous to the preferred physical constants of Lorentz-covariant theories, and the construction here presented is essentially equivalent to the more familiar construction based on the explicit use of the Killing vector fields.[6] The principal virtue of the present treatment is that it can more readily be translated into the Penrose formalism.

IV. CONCLUSION

We have found a new conservation theorem, Eq. (4), which appears to be particularly useful for the program of determining the commutation relations for the known independent dynamical variables of the gravitational field. We also show in Sec. III, how the preferred conservation laws of Lorentz-covariant physical theories may be related to our conserved quantities. Whether the relationship established here will play a role in the further elaboration, and in particular in the program for the quantization of gravitation theory, remains to be seen.

In future papers we hope to carry out the program indicated above of quantizing the Einstein theory in a formalism which employs the Penrose function. This will entail among other things, (a) determining the commutation relations of the Penrose function, and (b) applying a similar program to the quantization of the simpler and more familiar Maxwell and linearized gravitation theories. Investigation (b) is required in order to understand and interpret the quantum theory which we might obtain, and to see if any new or unfamiliar difficulties arise due to our employing independent dynamical variables defined on a characteristic surface. We are particularly encouraged by the striking similarity of structure which all of the above-mentioned theories exhibit in the Penrose formalism.

We should mention in passing that associated with the weak conservation law, Eq. (4), we have a strongly (i.e., identically) conserved vector field $D^\rho(r)$:

$$D^\rho(r)\equiv(r_{\alpha\beta}C^{\alpha\beta\gamma\rho})_{;\gamma}. \tag{19}$$

Therefore, the conservation theorems derived in this paper are particularly suitable for a study of the equations of motion of singular regions and regions which contain sources of the gravitational field.[7]

[7] J. N. Goldberg, Phys. Rev. **89**, 263 (1953).

V. APPENDIX

In this section we shall briefly sketch the Penrose formalism[1] and define the Penrose function. Although all the quantities of interest can be expressed in terms of the usual tensor notation, the expressions are often very cumbersome. For this reason we shall confine ourselves to Penrose's spinor presentation. Greek indices shall run from 1 to 4 and refer to the usual tensor indices. Capital Latin indices run from 1 to 2 and refer to the spin space. Greek indices are raised and lowered by means of the usual metric $g_{\mu\nu}$. Capital Latin indices are raised and lowered by means of the alternating symbols ϵ_{AB} and $\epsilon_{A'B'}$. Primed indices[8] refer to the complex conjugate spin space. Tensor indices may be translated into pairs of Hermitian spin indices by means of the Pauli spin matrices $\sigma^\mu{}_{AB'}$, and vice versa. For example, covariant differentiation can be denoted by

$$\partial_{AB'}\equiv\sigma^\mu{}_{AB'}\partial_\mu. \tag{A1}$$

An affine connection is understood in the spin space such that ϵ_{AB}, $\epsilon_{A'B'}$, and $\sigma^\mu{}_{AB'}$ are covariantly constant.

To every bivector can be associated a symmetric spinor, and conversely to every symmetric spinor can be associated a bivector. This is accomplished by means of the relation:

$$F_{\mu\nu}=\sigma_\mu{}^{AB'}\sigma_\nu{}^{CD'}(\psi_{AC}\epsilon_{B'D'}+\psi_{B'D'}\epsilon_{AC}), \tag{A2}$$

where

$$\psi_{AB}=\psi_{BA}. \tag{A3}$$

In particular a spinor ξ_A can be related to the null bivector:

$$N_{\mu\nu}=\sigma_\mu{}^{AB'}\sigma_\nu{}^{CD'}(\xi_A\xi_C\epsilon_{B'D'}+\xi_{B'}\xi_{D'}\epsilon_{AC}). \tag{A4}$$

If $F_{\mu\nu}$ satisfies the homogeneous Maxwell field equations, then from Eq. (A2) we find

$$\partial_{AB'}\psi^{AC}=0. \tag{A5}$$

Conversely, if Eq. (A5) is satisfied $F_{\mu\nu}$ is a solution of the homogeneous Maxwell equations. The Penrose function for the Maxwell field is

$$\psi\equiv\psi_{AB}\xi^A\xi^B, \tag{A6}$$

which is evidently a projection of the Maxwell tensor into the null cone. Penrose has shown that knowledge of ψ on a null cone provides a unique, independent, nonredundant characterization of solutions of the Maxwell equations.

For gravitation theory, Penrose has shown that when the Ricci tensor vanishes the Riemann tensor is fully equivalent to the completely symmetric spinor ψ_{ABCD}, the relationship being effected by

$$C_{\mu\nu\sigma\rho}=\sigma_\mu{}^{AB'}\sigma_\nu{}^{CD'}\sigma_\sigma{}^{EF'}\sigma_\rho{}^{GH'} \times(\psi_{ACEF}\epsilon_{B'D'}\epsilon_{F'H'}+\psi_{B'D'F'H'}\epsilon_{AC}\epsilon_{EG}). \tag{A7}$$

The Bianchi identities take the form [analogous to

[8] We use primed indices rather than the more usual dotted indices for typographical reasons.

Eq. (A5)]:

$$\partial_{AB'}\psi^{ACDE}=0. \tag{A8}$$

The Penrose function for gravitational field is

$$\psi\equiv\psi_{ABCD}\xi^A\xi^B\xi^C\xi^D. \tag{A9}$$

Knowledge of ψ on a null cone provides a unique, independent, nonredundant characterization of solutions of the Einstein equations.

The vector field $C^\rho(r)$ of Eq. (2) can be expressed in spinor notation by means of the Hermitian spinor

$$C^{AF'}=\psi^{ABCD}\partial_B{}^{F'}r_{CD}+\psi^{F'B'C'D'}\partial_{B'}{}^{A}r_{C'D'}, \tag{A10}$$

where r_{CD} is the symmetric spinor associated with the arbitrary bivector $r_{\mu\nu}$ by means of Eq. (A2). It is easily confirmed that

$$\partial_{AB'}C^{AB'}=0. \tag{A11}$$

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Determination of π-π Cross Sections by the Chew-Low Extrapolation Method*

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(Received March 15, 1962)

A discussion of the Chew-Low conjecture applied to reactions of the type $\pi+p\rightarrow\pi+p+\pi^0$ is given. The extrapolation technique and the so-called "physical-region-plot method" are discussed. These methods are then applied to a sample of 1684 interactions of the type $\pi^++p\rightarrow\pi^++p+\pi^0$ and 411 interactions of the type $\pi^-+p\rightarrow\pi^-+p+\pi^0$ at 1.25-BeV/c incident pion momentum. In Sec. II, we present details of the way in which this sample was obtained and processed. The $\sigma_{\pi^+\pi^0}$ physical-region plots for the π^+ data confirm the existence of the now well-established $T=J=1$ $\pi-\pi$ resonance at 725±25 MeV. On the other hand, the physical-region plot of the π^- data shows strong deviations from the results expected on the basis of this same resonance.

Extrapolation results are presented in detail for seven different $\pi-\pi$ energy regions. The π^- and π^+ data (although very different in the physical region) extrapolate to the same $\pi-\pi$ cross-section values for energy regions around the resonance. This fact and the general character of the π^- extrapolations indicate that the π^- physical-region distortions should be attributed to final-state interactions. Although the errors are large, the $\pi-\pi$ cross-section curve obtained by extrapolation on the combined $\pi^+\pi^-$ sample, is in full agreement with the existence of a $\pi-\pi$ P-wave resonance at 725±25 MeV.

I. THE CHEW-LOW CONJECTURE

During the past few years—mainly under the impulse of the dispersion-relation development—a considerable amount of study has been devoted to detecting the presence and the location of singularities in S matrices. In most cases, the existence and the location of these singularities was actually conjectured on the basis of plausibility arguments supplied by perturbation calculations using conventional field theory; in some special instances, proofs were given.

Singularities in the momentum-transfer variables have been utilized for practical purposes following a proposal by Chew[1] to determine the pion-nucleon coupling constant from either nucleon-nucleon scattering data[2] or photoproduction data.[3] Using the same principles (actually, generalizing them), Chew and Low made a conjecture opening the possibility of determining cross sections on targets not available in the laboratory.[4] The conjecture states that:

a. Each S-matrix element corresponding to a definite number of particles (when considered as a function of the independent invariants of the problem) has poles at points related to the masses of the single-particle states which can occur as intermediate states between *two* subgroups of particles formed from the total group.

b. The residue of such an S-matrix pole is given by the product of two (dimensionally smaller) S-matrix elements, each of which connects a subgroup to the intermediate particle. These submatrix elements can correspond to processes not realizable in the laboratory. They now become "measurable" by means of a residue evaluation of the over-all S-matrix element at the pole in question. Since the poles discussed here lie in regions of the variables that are not physically allowed, this residue evaluation is connected with extrapolation procedures.

The Chew-Low conjecture, stated specifically with respect to the poles in the *momentum-transfer variable*, has been used in an "orthodox" way (i.e., by actually

* This work was done under the auspices of the U. S. Atomic Energy Commission.

† Partially based on work submitted to the graduate division of the University of California in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

‡ On leave of absence from the Inter-University Institute for Nuclear Sciences, Brussels, Belgium.

[1] Geoffrey F. Chew, Phys. Rev. **112**, 1380 (1958).

[2] P. Cziffra and M. J. Moravcsik, Phys. Rev. **116**, 226 (1959).

[3] J. G. Taylor, M. J. Moravcsik, and J. L. Uretsky, Phys. Rev. **113**, 689 (1959).

[4] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).