which gives for Mellin transform of the discontinuity function

$$\frac{1}{\pi}M^{-2l}\Gamma(q)/(l+p)^q\int_{M^2}^{\infty}\frac{ds'}{s'-s}(s/M^2)^{-p}[\ln(s/M^2)]^{q-1},$$

which has a cut at $l=-p$ for nonintegral q.)

Even for the proof of holomorphy for $\mathrm{Re}\,l\geqq 1$, $\nu<-m^2$, a weak form of the unitarity condition has been used. It is also clear that the poles in the l plane for one channel are the result of highly inelastic processes in the crossed channels. This can be seen from the fact that these poles dominate the high-energy behavior of the amplitude in crossed channels and correctly predict a purely imaginary forward scattering amplitude. Therefore, a complete proof of analyticity in l must, we feel, make use of the full unitarity condition. Because the unitarity condition couples all channels, such a proof must await further developments in the S-matrix theory of multiparticle processes. Also perturbation theory is no guide in establishing the analyticity in l, since a finite number of diagrams does not give energy-dependent poles. The proof given in this section may indicate the direction along which a future proof should proceed.

If one invokes a principle of maximal analyticity in angular momentum then the previous discussion shows that the only singularities required by unitarity are poles, throughout the entire l plane. However, in order to avoid the introduction of a new postulate, one would like to derive the principle of maximal analyticity in l as a consequence of maximal analyticity in linear momentum.

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Rate Difference between a Clock in an Artificial Satellite and a Clock on the Earth*

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Taking account of the diurnal rotation and the oblateness of the earth, the average relative rate difference is calculated for an arbitrary inclination of the satellite orbit. For eccentricities less than 0.1, an accuracy of order 10^{-14} is obtained.

IT is well known that[1,2] due to the general theory of relativity, the rate of a clock on the earth will be different from the rate of a similar clock in an artificial satellite. For an elliptic satellite orbit the average relative rate difference is[3,4]

$$\bar{\Delta}\equiv\frac{\tau_s-\tau_E}{\tau_E}=\frac{GM_E}{c^2r_0}\left(1-\frac{3r_0}{2a}\right)=6.96\times10^{-10}\left(1-\frac{3r_0}{2a}\right). \quad (1)$$

τ_E and τ_s are the periodic times read on the earth clock and the satellite clock, respectively; G is the gravitational constant, M_E is the mass of the earth, r_0 is the radius of the earth, and $2a$ is the major axis of the satellite orbit. In this derivation the diurnal rotation and the oblateness of the earth are not taken into account. Hoffmann[5] has calculated the influence on $\bar{\Delta}$ from these two perturbations, and finds for a circular orbit in the equatorial plane corrections terms in $\bar{\Delta}$ of order 10^{-12}. In the present paper we will calculate the correction terms for an elliptic orbit with an arbitrary inclination. The rate of a standard clock in a satellite and on the earth, respectively, is given by[6,7]

$$dt_s=(1+2\chi_s/c^2-u_s^2/c^2)^{\frac{1}{2}}dt, \quad (2)$$

$$dt_E=(1+2\chi_E/c^2-u_E^2/c^2)^{\frac{1}{2}}dt, \quad (3)$$

t_E, t_s, and t being the times read on the earth clock, the satellite clock, and a coordinate clock, respectively u_E and u_s being the velocity of the earth clock and the satellite, respectively, relative to the center of the earth, and χ_E and χ_s being the scalar gravitational potentials on the earth and on the satellite, respectively. Neglecting terms of order χ^2/c^4, $\chi u^2/c^4$, and u^4/c^4, we get from Eqs. (2) and (3)

$$dt_s=[1+c^{-2}(\chi_s-\chi_E-\tfrac{1}{2}u_s^2+\tfrac{1}{2}u_E^2)]dt_E. \quad (4)$$

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[7] We have ignored the vector potential because the fractional error in $\bar{\Delta}$ due to this is of order u_E^2/c^2.

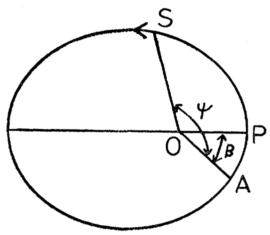

FIG. 1. The orbit of the satellite S round the center of the earth, O.

Integrating (4) over one period, we obtain

$$\bar{\Delta}=c^{-2}[\langle\chi_s-\tfrac{1}{2}u_s^2\rangle_{\mathrm{av}}-\chi_E+\tfrac{1}{2}u_E^2], \tag{5}$$

where $\langle\chi_s-\frac{1}{2}u_s^2\rangle_{\mathrm{av}}$ is the time average of $(\chi_s-\frac{1}{2}u_s^2)$ over one period. We have, in accordance with Hoffmann, found that the correction due to the rotation of the earth is equal to the special relativistic correction due to the velocity of the earth clock. The problem is now to compute $\langle\chi_s-\frac{1}{2}u_s^2\rangle_{\mathrm{av}}$ for a satellite moving round the oblate earth, where χ_s is given by

$$\chi_s=c^2[-k/r_s-(Kk/2r_s^3)(1-3\cos^2\theta)], \tag{6}$$

r_s being the distance from the center of the earth to the satellite, and where $k=GM_E/c^2$ and $K=4.40\times10^4$ km^2.[8] From King-Hele,[9] we get

$$\begin{aligned}\frac{1}{r_s}=L\{1&+e\cos(\psi-\beta)\\&+\tfrac{3}{2}KL^2[\tfrac{1}{2}(5\cos^2\alpha-3)+\tfrac{1}{6}\sin^2\alpha\cos2\psi\\&+(5/24)e\sin^2\beta\cos(3\psi-\beta)+O(e^2)]\},\end{aligned} \tag{7}$$

L being a constant, e the eccentricity of the orbit, and α the inclination. β and ψ are given in Fig. 1, where A is the point of minimum θ, P is perigee, S is the satellite, and O is the center of the earth. However, the orbital plane is not constant, but precesses on the earth's axis with an angular velocity

$$\dot{\Omega}\equiv d\Omega/dt=10[(r_0/a)^{3.5}+O(e^2)]\cos\alpha\ \text{deg/day}. \tag{8}$$

Further, the major axis rotates in the orbital plane with an angular velocity

$$\dot{\beta}=5(r_0/a)^{3.5}[5\cos^2\alpha-1+O(e)]\ \text{deg/day}. \tag{9}$$

Making use of the virial theorem, we get for one period

$$\langle u_s^2\rangle_{\mathrm{av}}=\left\langle\frac{\partial\chi_s}{\partial r_s}r_s\right\rangle_{\mathrm{av}}+\frac{2}{\tau}[F(\tau)-F(0)], \tag{10}$$

where τ is the periodic time and F is equal to $\mathbf{u}_s\cdot\mathbf{r}_s$. We now let one period correspond to a variation of 2π in $(\psi-\beta)$. Then F will be unaltered after one period. Hence, for one period,

$$\langle u_s^2\rangle_{\mathrm{av}}=\left\langle\frac{\partial\chi_s}{\partial r_s}r_s\right\rangle_{\mathrm{av}}. \tag{11}$$

From Eqs. (5), (6), and (11) we then obtain

$$\bar{\Delta}=-\frac{3}{2}k\left\langle\frac{1}{r_s}\right\rangle_{\mathrm{av}}-\frac{5}{4}Kk\left\langle\frac{1-3\cos^2\theta}{r_s^3}\right\rangle_{\mathrm{av}}-(\chi_E-\tfrac{1}{2}u_E^2). \tag{12}$$

Using the relation $\cos\theta=\sin\alpha\cos\psi$, it is easily shown that

$$\left\langle\frac{1-3\cos^2\theta}{r_s^3}\right\rangle_{\mathrm{av}}=\left\langle\frac{1}{r_s}\right\rangle_{\mathrm{av}}^3[1-\tfrac{3}{2}\sin^2\alpha+O(e^2)]. \tag{13}$$

As pointed out by Hoffmann, $(\chi_E-\frac{1}{2}u_E^2)$ is independent of the latitude of the earth clock, provided we correct for the height above the sea level (geoid), because the geoid is an equipotential surface of this combined gravitational and centrifugal potential. Therefore, $\bar{\Delta}$ is independent of the latitude of the earth clock. For a clock at the equator we have[10]

$$\chi_E=\chi_e=-k/r_e-Kk/2r_e^3,\quad u_e^2/2c^2=1.204\times10^{-12}. \tag{14}$$

From Eq. (12), we then obtain

$$\begin{aligned}\bar{\Delta}=\frac{k}{r_e}\left(1-\frac{3}{2}\left\langle\frac{r_e}{r_s}\right\rangle_{\mathrm{av}}\right)+\frac{u_e^2}{2c^2}\\+\frac{kK}{2r_e^3}\left[1-\frac{5}{2}\left\langle\frac{r_e}{r_s}\right\rangle_{\mathrm{av}}^3(1-\tfrac{3}{2}\sin^2\alpha)+O(e^2)\right].\end{aligned} \tag{15}$$

For $\alpha=0$ and $r_s=$constant, we obtain Hoffmann's result. Numerically, $\bar{\Delta}$ turns out to be

$$\begin{aligned}\bar{\Delta}=6.9535\times10^{-10}\left(1-\frac{3}{2}\left\langle\frac{r_e}{r_s}\right\rangle_{\mathrm{av}}\right)-9.40\times10^{-13}\\\times\left[\left\langle\frac{r_e}{r_s}\right\rangle_{\mathrm{av}}^3(1-\tfrac{3}{2}\sin^2\alpha)+O(e^2)\right]+1.580\times10^{-12}.\end{aligned} \tag{16}$$

For $e\leq0.1$, the accuracy in $\bar{\Delta}$ should be of order 10^{-14}.

[8] Following Hoffmann, we use the constant K which is related to the constant J used by King-Hele and Sterne [T. Sterne, *An Introduction to Celestial Mechanics* (Interscience Publishers, Inc., New York, 1960), p. 37], by $K=\frac{2}{3}Jr_e^2$, r_e being the equatorial radius. Hoffmann gives $K=4.47\times10^4$ km^2. However, from Sterne we find $K=4.40\times10^4$ km^2, which is used in the present paper.

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[10] Hoffmann gives $u_e^2/2c^2=1.24\times10^{-12}$ which is not quite correct.