The expansion of $K_s{}^D$ in terms of $K_s$ is

$$(b|K_s{}^D|a)=(b|K_s{}^\dagger|a)+\left(b\left|K_s{}^\dagger\left[\frac{1-Q_a}{e_a}-\frac{1-Q_b}{e_a{}^0}\right]K_s{}^D\right|a\right). \tag{C.15}$$

The "dispersion term" for the level shift is

$$(a|K_s{}^D-K_s|a)=\left(a\left|(\Omega_{K_s}{}^\dagger-1)e_a{}^0\left[\frac{1}{e_a}-\frac{1}{e_a{}^0}\right]e_a(\Omega_{K_s{}^D}-1)\right|a\right). \tag{C.16}$$

One could approximate $\Omega_{K_s{}^D}$ in Eq. (C.16) by $\Omega_{K_s}{}^\dagger$ or by $\Omega_{K_s}$, leaving correspondingly different correction terms.

The equations as given here contain corrections to terms beyond the second order, which should be included if detailed calculations are extended to third order.

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# Model for the Two-Pion and Three-Pion Resonances

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A relativistic model of $\pi$-$\pi$ interaction is proposed from analogy with field theory. The main assumption in the model is that the interaction kernel is separable in momentum space. A Frazer-Fulco type resonance formula for the isovector $2\pi$ resonance is exactly derived on this model and its parameters are determined from the observed $2\pi$ resonance at 750 MeV.

The $3\pi$ problem is then solved with this model of $\pi$-$\pi$ interaction and the exact isoscalar $3\pi$ wave function expressed in terms of a single-parameter function which satisfies a one-dimensional integral equation for the cases of "scalar" and "axial vector" forms of the wave function. It is found that it is possible to understand the observed energy (780 MeV) of the $3\pi$ resonance, on the above model, only for the case of the axial vector wave function and not for the scalar case. The model thus predicts a vector isoscalar meson at the observed energy.

## 1. INTRODUCTION

THE recent experimental discovery by Maglić *et al.*[1] of an isoscalar three-pion resonance at $787\pm15$ MeV must be welcome to a large number of persons who, for various reasons (electromagnetic structure, resonances in $\pi$-$N$ and $K$-$N$ scattering, spin-orbit potentials, etc.), found it necessary to postulate the existence of an isoscalar vector meson. In the same analysis,[1] these authors have also confirmed the existence of the previously known[2] isovector two-pion resonance at about $750\pm50$ MeV, from which, they feel confident, the $3\pi$ resonance can be distinguished, in spite of the anomalously small separation of the two resonances. Their analysis, based on Dalitz-type plots, though lacking detailed statistics, suggests further that the resonance can be interpreted more likely in terms of an axial vector "matrix element" (corresponding to a vector meson) than as vector or scalar elements (axial vector or pseudoscalar mesons, respectively). While such an interpretation would no doubt be most desirable from the point of view of the electromagnetic structure of the nucleon, there still remains the important question of whether the *mass* of the observed particle fits in with the value needed to explain the isoscalar form factor. For example, the recent findings of Littauer *et al.*[3] on the nucleon model proposed by Bergia *et al.*[4] suggest that the two-pion state should have a mass of about $4.0\,\mu$, and the three-pion state an even lower mass, viz. $2.9\,\mu$—a quasi-bound state. If this interpretation comes to be accepted, the resonances observed by Maglić *et al.*[1] should apparently have nothing to do with the electromagnetic form factors. However, the very fact that they have been *observed* must put their existence on a stronger footing than those of hypothetical particles *deduced* from specific models of the type discussed in reference 4, at least until such time as the latter are also "seen." In the meantime, one hopes that a more careful investigation would be in order before such a delicate question, as to the exact masses of the intermediate $T=1$ and $T=0$ vector mesons needed to explain the electromagnetic form factors, is finally answered.

[1] B. C. Maglić, L. W. Alvarez, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. Letters **7**, 178 (1961).

[2] M. L. Stevenson *et al.*, University of California Radiation Laboratory Report UCRL-9814 (to be published); also E. Pickup, F. Ayer, and E. O. Salant, Phys. Rev. Letters **5**, 161 (1960).

[3] R. M. Littauer, H. S. Schopper, and R. R. Wilson, Phys. Rev. Letters **7**, 144 (1961).

[4] S. Bergia, A. Stangnellini, S. Fubini, and C. Villi, Phys. Rev. Letters **6**, 367 (1961).

While any speculation on the existence of *more* than one isoscalar vector meson[5] must await experimental evidence, it should be of interest to examine if, and to what extent, there is a dynamical connection between the $2\pi$ ($T=1$) and $3\pi$ ($T=0$) resonances that have been observed. Indeed, even before its experimental discovery, Chew[6] had given qualitative arguments to show that a $3\pi$ ($T=0$) state should exist as a dynamical consequence of the existence of a $p$-wave $2\pi$ resonance. Further a $T=0$, $3\pi$ state being a completely saturated unit, the mutual attractions between the various $T=1$ pairs naturally have their full play so that it is even qualitatively plausible that the mass of such a $3\pi$ state is appreciably smaller than what one would normally expect for other (nonsaturated) values of $T$. Yet a dynamical understanding of the *actual* extent of "depression" in the mass (as a result of saturation) is a more complicated problem, which should be difficult to answer in purely qualitative terms. In particular, the existence, on dynamical grounds, of a three-pion state with a mass *less* than that of the $T=1$ two-pion state as envisaged by some authors[3,4] is a matter for only a *detailed* dynamical theory of pion-pion interaction to decide, and cannot just be taken for granted.

A dynamical theory based on analyticity properties of various amplitudes would no doubt be an ideal choice for investigating a problem of this kind. The techniques developed by Chew and Mandelstam,[7] which are so well developed for treating two-particle states, are however not applicable to three-particle systems. So far the only possible approach in this direction seems to be provided by the graphical techniques of Landau[8] which, as extended by Cutkosky,[9] have already gone a long way towards the development of a consistent mathematical scheme. It is, however, not yet clear to us if such techniques can be applied to the present problem without going into excessive complexities.

The other alternative at this stage is to consider some simple model of $\pi$-$\pi$ interaction and examine its usefulness as a dynamical mechanism for giving rise to a possible isoscalar three-pion resonance. It is this latter view that we have taken in this paper, with the *choice* of our interaction model governed mainly by considerations of mathematical simplicity of such a magnitude as to make the three-body problem subsequently tractable without excessive effort. It is recognized, of course, that an approach based on an empirical model is *not* a substitute for a regular dynamical theory. Still, it is hoped that if it leads to certain reasonable physical consequences, some of its features should be possible to understand in the framework of a more fundamental theory.

In Sec. 2, a plausible relativistic model of the effective $p$-wave $\pi$-$\pi$ interaction ($T=1$) is proposed, and a resonance formula of the Frazer-Fulco[10] type derived with its help. In Sec. 3, this isovector interaction between $\pi$-$\pi$ pairs is employed in a Schrödinger-type wave equation for three pions to derive various forms of the $3\pi$ wave function at a given positive energy, compatible with the basic interaction assumed, viz. scalar, axial vector, and polar vector, respectively. The scalar and axial vector wave functions are expressible in terms of certain single-parameter functions satisfying very similar one-dimensional wave equations. For the polar vector case one has two coupled one-dimensional integral equations. An approximate method of reduction of these integral equations is given in Appendix I.

In Sec. 4 a semiquantitative analysis is given of the dependence of the three-pion resonance on (i) the algebraic structure of the three-pion wave function and (ii) the range and strength parameters of the $\pi$-$\pi$ interaction. The main result is that a $3\pi$ resonance at the observed energy is compatible only with an *axial-vector* form of the $3\pi$ wave function and *not* a scalar form. Its implications in the context of the present experimental situation are briefly discussed.

## 2. ISOVECTOR $\pi$-$\pi$ INTERACTION MODEL

As is well known, the concept of a two-body "potential" in some general sense is one of the most useful tools for the treatment of a three-body problem, insofar as it gives a concrete realization of the interaction *off* the energy shell. The starting point of our approach to the $3\pi$ problem is, therefore, the choice of a model of $\pi$-$\pi$ interaction. Since our main consideration, as already explained, is simplicity of the subsequent $3\pi$ problem, we have been guided by a recent experience[11] with the three-nucleon problem which appeared exactly soluble when treated with the so-called separable potentials[12] operative between pairs. Further, we have found recently that separable potentials can lead to correct analyticity properties of various partial-wave amplitudes and, in particular, to the so-called $N/D$ representations for them.[13] For a treatment of the $3\pi$ problem we were, therefore, led to a search for a suitable generalization of "separable $N$-$N$ potentials" which might be physically applicable to $\pi$-$\pi$ interactions. Now since relativistic effects are extremely important here, any reasonable $\pi$-$\pi$ interaction model must take these effects into account (at least their *kinematical* aspects). A clue to this search is provided by any reasonable approximation scheme within the premises of a rela-

[5] See, e.g., J. Sakurai, Ann. Phys. (New York) **11**, 1 (1960).
[6] G. F. Chew, Phys. Rev. Letters **4**, 142 (1960).
[7] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[8] L. D. Landau, Nuclear Phys. **13**, 181 (1960).
[9] R. E. Cutkosky, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960).
[10] W. Frazer and J. Fulco, Phys. Rev. Letters **2**, 365 (1959).
[11] A. N. Mitra, Nuclear Phys. (to be published).
[12] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).
[13] A. N. Mitra, Phys. Rev. **123**, 1892 (1961).

tivistic field theory (even a perturbation series for that matter). Thus, we know that any "effective" four-pion vertex must, according to the usual plane wave expansions of pion operators, contain the factors $(\omega_1\omega_2\omega_3\omega_4)^{-1/2}$ which in the c.m. system of momenta reduces to $\omega_p^{-1}\omega_{p'}^{-1}$, where $\pm\mathbf{p}$ and $\pm\mathbf{p}'$ are the momenta of the pions before and after the interaction. One could therefore write the "effective" interaction Hamiltonian $H$ in momentum space as[14]

$$(\mathbf{p}|H|\mathbf{p}')=\omega_p^{-1}\omega_{p'}^{-1}(\mathbf{p}|K|\mathbf{p}'), \qquad (2.1)$$

where the detailed structure of $K$ is of course unknown, but at least some kinematical relativistic effects are conveniently represented by the two factors preceding $K$.

By further analogy with field theory,[14,15] the $\pi$-$\pi$ wave function $\phi(\mathbf{p})$ can be pictured as satisfying a Schrödinger-type equation (in the c.m. system) which, using (2.1), reads as

$$(E-2\omega_p)\phi(\mathbf{p})=\int(\omega_p\omega_{p'})^{-1}d\mathbf{p}'(\mathbf{p}|K|\mathbf{p}')\phi(\mathbf{p}'). \qquad (2.2)$$

This equation can be analyzed into equations for individual partial waves in the standard manner by writing

$$(\mathbf{p}|K|\mathbf{p}')=\sum_0^\infty K_l(p,p')P_l(\cos\theta). \qquad (2.3)$$

In this form, we can give a precise and concrete meaning to a "separable interaction model," viz. one in which (i) a *finite* number of terms in (2.3) are taken and (ii) each $K_l$ is *assumed* to have the structure

$$K_l(p,p')=-\lambda_l v_l(p)v_l(p'). \qquad (2.4)$$

It may be emphasized that a kernel like (2.4), operative in a single state $l$, cannot by any means be called a "potential" in the conventional sense of the term, yet it gives a useful representation for the $\pi$-$\pi$ interaction which is *not* incompatible with dispersion theory, since the amplitudes can be made "analytic"[13] with the form (2.4). Instead, therefore, of calling our model one of separable potential, as in low-energy nuclear physics, we might call it one of separable *kernel*. It may be noted that such an emphasis on the kernel in momentum space, rather than on a conventional potential, goes much beyond the limits of validity of the latter, insofar as the relativistic effects are being almost fully incorporated. Of course such a model for the $\pi$-$\pi$ interaction (like any other model of two-particle interaction without second quantization) is incapable of describing processes like pion production, etc., except in an over-all fashion through the appearance of the so-called "resonance widths" which collectively represent the probabilities of all the inelastic channels. While this limitation may be serious at very high energies, there is still a fairly large energy region (much larger than one in which a simple potential concept would be valid) for which a model such as the present one is likely to be useful.

Having so decided on a model of the $\pi$-$\pi$ interaction, we now restrict our considerations to only $s$ and $p$ waves. At the high energies we are considering here we might, without significant error, take only the $p$-wave interaction. One argument which might be given in favor of such a simplification is that in the $p$-wave dominant solution of Chew, Mandelstam, and Noyes, the effect of the (large) $p$-wave amplitudes on the (small) $s$-wave amplitudes is so significant that even the sign of the latter is uncertain. Of course, in our interaction of the type (2.4), there is no mutual interference between the various $l$ terms, so that each partial wave interaction is of an effective nature, already incorporating these feedback effects. Another, perhaps more important, argument stems from a consideration of the nature of the problem at hand. Since we are here interested in studying an *isoscalar* $3\pi$ amplitude, the momentum-space wave function must be *totally* antisymmetric in the charge coordinates, and likewise (by Bose statistics) in the three momenta. On the other hand, an $s$-wave interaction between a pair of pions gives rise to a state which is symmetric in the pair. The condition of total antisymmetry should, therefore, largely project out the effects of the $s$-state interaction between $\pi$-$\pi$ pairs.

With these assumptions, the isovector $\pi$-$\pi$ interaction kernel can, according to Eqs. (2.1)–(2.4), be taken as

$$(\mathbf{p}|H|\mathbf{p}')=-\lambda\omega_p^{-1}\omega_{p'}^{-1}\rho(1)v_1(p)v_1(p')(\hat{p}\cdot\hat{p}'), \qquad (2.5)$$

where $\rho(T)$ stands for the projection operator for a total isospin $T$ and is given, for $T=1$, by

$$\rho_{ij}(1)=1-\tfrac{1}{2}(\mathbf{T}_i\cdot\mathbf{T}_j)-\tfrac{1}{2}(\mathbf{T}_i\cdot\mathbf{T}_j)^2, \qquad (2.6)$$

$\mathbf{T}_i$ and $\mathbf{T}_j$ representing the isospin operators for the two pions ($i, j=1, 2, 3$). To make the form (2.5) consistent with effective-range theory, the threshold behavior of the $\pi$-$\pi$ amplitude demands that $v_1(p)$ should vanish as $p$ at the origin. Thus we set

$$v_1(p)=pv(p), \qquad (2.7)$$

where $v(p)$ remains finite at $p=0$. The $\pi$-$\pi$ wave function $\phi(\mathbf{p})$ corresponding to $T=1$ and total c.m. energy $2\omega_k$ is then seen to satisfy the Schrödinger equation

$$(2\omega_p-2\omega_k)\phi(\mathbf{p})=\lambda\int d\mathbf{p}'\,v(p)v(p')(\mathbf{p}\cdot\mathbf{p}')\phi(\mathbf{p}'), \qquad (2.8)$$

where the isospin factor $\rho(1)$ in (2.5) has been replaced by its eigenvalue unity. From this the $p$-wave scattering amplitude $f(\mathbf{p})$, which is related to $\phi(\mathbf{p})$ by the

[14] See, e.g., A. N. Mitra and R. P. Saxena, Phys. Rev. **108**, 1082 (1957).

[15] F. J. Dyson, M. Ross, E. E. Salpeter, S. S. Schweber, M. K. Sundaresan, W. M. Visscher, and H. A. Bethe, Phys. Rev. **95**, 1644 (1954).

normalization

$$\phi(\mathbf{p})=\omega_k\delta(\mathbf{k}-\mathbf{p})+\tfrac{3}{2}\pi^{-2}(2\omega_p-2\omega_k-i\epsilon)^{-1}f(\mathbf{p}), \quad (2.9)$$

is obtained in the form

$$e^{i\delta}\sin\delta/k=f(k)=N(k)/D(k), \quad (2.10)$$

where

$$N(k)=f_{\mathrm{Born}}(k)=2\pi^2\lambda k^2v^2(k)/\omega_k, \quad (2.11)$$

and

$$D(k)=1-\pi^{-1}\int_0^\infty p^2dp\, N(p)\omega_p^{-1}(\omega_p-\omega_k-i\epsilon)^{-1}. \quad (2.12)$$

It is easily seen that (2.10) satisfies the unitarity condition and leads trivially to the effective-range formula

$$k^3\cot\delta/\omega_k=(2\pi^2\lambda)^{-1}v^{-2}(k)[1-2\pi\lambda\mathfrak{F}(k)], \quad (2.13)$$

where

$$\mathfrak{F}(k)=P\int_0^\infty p^4dp\, v^2(p)\omega_p^{-2}(\omega_p-\omega_k)^{-1}. \quad (2.14)$$

It is interesting to note that Eq. (2.10) can be put in the form of a Frazer-Fulco[10] type formula, viz.,

$$\omega_k k^{-3}\sin\delta e^{i\delta}=\Gamma(k)[k_0^2-k^2-i\Gamma(k)k^3\omega_k^{-1}]^{-1}, \quad (2.15)$$

where $k_0$ is the resonance momentum defined by the equation [cf. (2.13)]

$$1-2\pi\lambda\mathfrak{F}(k_0)=0, \quad (2.16)$$

and $\Gamma(k)$ (which may be termed the "reduced width" at momentum $k$) is given by[16]

$$\pi/\Gamma(k)=v^{-2}(k)(k_0^2-k^2)^{-1}[\mathfrak{F}(k_0)-\mathfrak{F}(k)]. \quad (2.17)$$

While (2.17) is, of course, nonsingular at $k=k_0$, a little caution is necessary in its evaluation in the limit $k\rightarrow k_0$. Thus, no attempt should be made to explicitly factor out $(\omega_0-\omega_k)$ from the brackets in (2.17) by using the representation (2.14), *before* evaluating the principal-value integrals (2.14); otherwise one would encounter in the integrand a factor such as $(\omega_p-\omega_0)^{-2}$ which would make the integral *divergent*, and $\Gamma(k_0)$ zero. Indeed, from the way $\Gamma(k)$ is defined in (2.17), it might be of either sign at resonance. A negative sign for $\Gamma(k_0)$ means, of course, that the phase shift *decreases* with energy near resonance. However, this should not necessarily indicate the existence of a bound state, for the antisymmetry of the $p$-wave $2\pi$ state should effectively inhibit any such possibility.

So far we have said nothing about the form of the interaction which is essentially represented by the structure of $N(k)$, according to (2.11). As in reference (13), we could make (2.10) conform to dispersion-theoretic requirements by demanding, in addition, that $N(k)$ be expressible as an integral over the left-hand branch cut in the variable $s=k^2$, with the cut starting at $s=-\mu^2$. Such a form would, of course, be essential for mathematical consistency for processes whose *physical* energies are situated on or near the left-hand side of the complex $s$ plane. However, the energies involved in the present investigation being sufficiently large (viz. 5–6 $\mu$), the region of physical interest in the $s$ plane is far removed from the left branch cut. One therefore expects a pole-type approximation to be adequate. One may even go further and assert that it should not be essential to take the usual first-order pole approximation in preference to, say, a *second*-order pole which happens to make the subsequent integrals (to be encountered) more easily convergent. This leads finally to the choice

$$v(p)=(\beta^2+p^2)^{-1}, \quad (2.18)$$

where $\beta$ is left as a free parameter of the order of the pion mass.

In this connection we note that an interesting suggestion has recently been made by Greenberger and Margolis[17] about the use of conformal mappings to "localize" the left-hand branch cut and then approximate it by a pole in the transformed plane. In the language of their paper, our Eq. (2.18) should read as

$$v(p)=(\beta+\omega_p)^{-1}, \quad \beta>\mu. \quad (2.19)$$

While such a form would have been quite adequate to represent an $s$-wave interaction, it is insufficient for our $p$-wave case, because of the presence of an additional factor $p$ in $v_1(p)$ [see Eq. (2.7)] to ensure correct threshold behavior. To maintain the form (2.19) would therefore necessitate a subtraction for the convergence of the dispersion relation (2.12). Since, on the other hand, our model does not allow for such *ad hoc* subtraction without a number of supplementary assumptions, we prefer to use the form (2.18) for the present investigation.

Some numerical values of $\beta$ and $\lambda$ used in this investigation are listed in Sec. 4. The following feature of our model (2.18) may, however, be noted here. For a given c.m. energy $2\omega_k$, $\lambda$ can be determined from Eq. (2.16) corresponding to a preassigned value of $\beta$. It is found that $\lambda$ is positive or negative according as $\beta$ is larger or smaller than a "critical" value $\beta_c$ (for $\omega_k=2.68\,\mu$, $\beta_c$ happens to be $\approx 1.8\,\mu$). While a positive value of $\lambda$ in this model implies the existence of a quasi-bound state[13] (not necessarily attainable though) in addition to the positive-energy resonance, a negative value of $\lambda$ allows *only* the positive-energy resonance. Thus, a "long-range" $\pi$-$\pi$ interaction ($\beta<\beta_c$) in this model precludes the existence of any quasi-bound state.

We now turn to the solution of the $3\pi$ problem, with the help of this model of $\pi$-$\pi$ interaction.

### 3. THE $3\pi$ PROBLEM

Taking the momenta of the pions as $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ where

$$\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3=0, \quad (3.1)$$

[16] It may be observed that Eq. (2.15) is an *exact* result in our model, unlike the case in reference 10 where a "pole approximation" was needed to derive an analogous form.

[17] D. Greenberger and B. Margolis, Phys. Rev. Letters **6**, 310 (1961).

the wave function $\Psi(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3)$ for a total energy $E$ satisfies the Schrödinger equation

$$(E-\omega_1-\omega_2-\omega_3)\Psi(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3) = \sum \int (\mathbf{P}_i\mathbf{P}_j|H_{ij}|\mathbf{P}_i'\mathbf{P}_j')\Psi_{ij}(\mathbf{P}_i'\mathbf{P}_j') \times d\mathbf{P}_i'd\mathbf{P}_j', \quad (3.2)$$

where

$$\omega_i=\omega(P_i), \quad 2\mathbf{p}_{ij}=\mathbf{P}_i-\mathbf{P}_j, \quad (3.3)$$

and

$$(\mathbf{P}_i\mathbf{P}_j|H_{ij}|\mathbf{P}_i'\mathbf{P}_j') = \delta(\mathbf{P}_i+\mathbf{P}_j-\mathbf{P}_i'-\mathbf{P}_j')(\mathbf{p}_{ij}|H_{ij}|\mathbf{p}_{ij}'). \quad (3.4)$$

$\Psi_{ij}(\mathbf{P}_i\mathbf{P}_j)$ is the functional form of $\Psi$ when it is expressed in terms of the two momenta $\mathbf{P}_i$, $\mathbf{P}_j$ with the help of (3.1). $H_{ij}$ is the interaction Hamiltonian for the pair of indices $i$ and $j$ defined for each pair by (2.5)–(2.7) and the summation in (3.2) is over the three possible cyclic permutations of these indices.

The wave function $\Psi$ as defined by (3.2) includes, of course, the charge coordinates of the pions. Its dependence on the charge coordinates can, however, be eliminated in a trivial manner by noting that an isoscalar ($T=0$) amplitude in three pions must be totally antisymmetric. Thus, if $\alpha_i$, $\beta_i$, $\gamma_i$ are the isopin functions of the $i$th pion corresponding to charges 1, 0, $-1$, respectively, the isospin part of $\Psi$ is obviously given by the Slater determinant

$$6^{-\frac{1}{2}}\begin{vmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{vmatrix}. \quad (3.5)$$

Now since, as noted already in Sec. 1, this can go only with a totally antisymmetric spatial function, the effect of the isospin projection operator $\rho(1)$ in the expression (2.5) for $H$ is automatically incorporated here simply by demanding that $\Psi$ be *totally antisymmetric* in $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. With this understanding the presence of $\rho(1)$ in (2.5) can be ignored and subsequent calculations made without further explicit considerations of isospin.

A first step towards the solution of (3.2) lies in deducing various possible algebraic structures of $\Psi$ (scalar, vector, and pseudovector), by substituting the form (2.5) for $H$ [without $\rho(1)$]. Noting that $\Psi$ depends only on two independent momenta which can be chosen in three different ways, we take these independent momenta to be the c.m. and relative momenta of two pions, according to the definitions

$$\mathbf{P}_1+\mathbf{P}_2=\mathbf{Q}_3(\equiv-\mathbf{P}_3), \quad 2\mathbf{p}_3=\mathbf{P}_1-\mathbf{P}_2, \quad (3.6)$$

and two similar sets obtained by cyclic permutations. In terms of these sets it is easy to deduce by substituting (2.5) on the right of (3.2) that the only *scalar* form of $\Psi$ compatible with our model of $\pi$-$\pi$ interaction is

$$\Psi_S=\sum_{i=1}^{3} v(p_i)\omega^{-1}(p_i)(\mathbf{p}_i\cdot\mathbf{Q}_i)g_i(Q_i), \quad (3.7)$$

$$D(E)=\omega_1+\omega_2+\omega_3-E, \quad (3.7\text{a})$$

where the $g_i$'s are three arbitrary scalar functions of the respective magnitudes $Q_i$. It is now seen immediately that the condition of total antisymmetry with respect to $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ demands, e.g.

$$g_1\equiv g_2\equiv g_3=g_s. \quad (3.8)$$

Substitution of the structure (3.7) and (3.8) in (3.2) leads to an integral equation for $g_s$ which, after some algebraic simplifications, reduces finally to

$$[1-2\pi\lambda\mathfrak{F}_s(Q)]g_s(Q) = -2\pi\lambda\int d\mathbf{p}\, K(\mathbf{Q},\mathbf{p})f_s(\mathbf{Q},\mathbf{p})g_s(p). \quad (3.9)$$

Here

$$K(\mathbf{Q},\mathbf{p})=\{2\pi[\omega_p+\omega_Q+\omega(\mathbf{p}+\mathbf{Q})-E]\}^{-1} \times v(\mathbf{p}+\tfrac{1}{2}\mathbf{Q})v(\mathbf{Q}+\tfrac{1}{2}\mathbf{p})\omega^{-1}(\mathbf{p}+\tfrac{1}{2}\mathbf{Q})\omega^{-1}(\mathbf{Q}+\tfrac{1}{2}\mathbf{p}); \quad (3.10)$$

$$f_s(\mathbf{Q},\mathbf{p})=6Q^{-2}[\mathbf{Q}\cdot(\mathbf{p}+\tfrac{1}{2}\mathbf{Q})][\mathbf{p}\cdot(\mathbf{Q}+\tfrac{1}{2}\mathbf{p})]; \quad (3.11)$$

$$\mathfrak{F}_s(Q)=\frac{1}{2\pi}\int d\mathbf{p}\frac{v^2(p)}{\omega^2(p)} \times\frac{3p^2\cos^2\theta}{\omega_Q+\omega(\mathbf{p}+\frac{1}{2}\mathbf{Q})+\omega(\mathbf{p}-\frac{1}{2}\mathbf{Q})-E}. \quad (3.12)$$

In a similar way, the totally antisymmetric axial vector form of $\Psi$ compatible with our $\pi$-$\pi$ interaction model is given by

$$\Psi_A=\sum_{i=1}^{3} v(p_i)\omega^{-1}(p_i)(\mathbf{p}_i\times\mathbf{Q}_i)g_A(Q_i), \quad (3.13)$$

where $g_A$ satisfies the integral equation

$$[1-2\pi\lambda\mathfrak{F}_A(Q)]g_A(Q) = 2\pi\lambda\int d\mathbf{p}\, K(\mathbf{Q},\mathbf{p})f_A(\mathbf{Q},\mathbf{p})g_A(p). \quad (3.14)$$

Here

$$\mathfrak{F}_A(\mathbf{Q},\mathbf{p})=3p^2\sin^2\theta, \quad (3.15)$$

and $\mathfrak{F}_A(Q)$ is given by (3.12) with the replacement

$$\cos^2\theta\rightarrow\tfrac{1}{2}\sin^2\theta \quad (3.16)$$

inside the integral.

Finally, for the sake of completeness we list here the totally antisymmetric vector structure of $\Psi$ compatible with our model, viz.,

$$\Psi_V=\sum_{i=1}^{3}\frac{v(p_i)}{\omega(p_i)}\left[\frac{(\mathbf{p}_i\cdot\mathbf{Q}_i)\mathbf{Q}_i}{Q_i^2}g_{1V}(Q_i) + \frac{\mathbf{Q}_i\times(\mathbf{p}_i\times\mathbf{Q}_i)}{Q_i^2}g_{2V}(Q_i)\right], \quad (3.17)$$

where $g_{1V}$ and $g_{2V}$ satisfy the equations

$$[1-2\pi\lambda\mathfrak{F}_S(Q)]g_{1V}(Q) = -2\pi\lambda\int d\mathbf{p}\, K(\mathbf{Q},\mathbf{p})[f_1 g_{1V}(p)+f_2 g_{2V}(p)], \quad (3.18)$$

$$[1-2\pi\lambda\mathfrak{F}_A(Q)]g_{2V}(Q) = -2\pi\lambda\int d\mathbf{p}\, K(\mathbf{Q},\mathbf{p})[F_1 g_{1V}(p)+F_2 g_{2V}(p)], \quad (3.19)$$

and the functions inside the integrals are defined by

$$\begin{aligned} f_1&=6(\tfrac{1}{2}p+Q\cos\theta)(\tfrac{1}{2}Q+p\cos\theta)\cos\theta,\\ f_2&=6\sin^2\theta(\tfrac{1}{2}Q^2+pQ\cos\theta),\\ F_1&=3\sin^2\theta(\tfrac{1}{2}p^2+pQ\cos\theta),\\ F_2&=-3pQ\sin^2\theta\cos\theta. \end{aligned} \quad (3.20)$$

### 4. NUMERICAL RESULTS AND DISCUSSION

In this paper we do not propose to give any detailed numerical results for the solution of any one of the integral equations derived in Sec. 3. Rather, our emphasis is on certain general aspects of these equations which seem to have a bearing (positive or negative) on the observed $3\pi$ resonance. Keeping this in mind, we start to discuss some qualitative features of Eqs. (3.9) and (3.14) for the scalar and axial vector cases, respectively, which, incidentally, have rather similar algebraic structures. The vector wave function will not be discussed any further.

First of all, it may be noted that the appearance of the factor $[1-2\pi\lambda\mathfrak{F}_{S,A}(Q)]$ on the left of these equations corresponds to the "resonating state" of a pair of pions in the presence of the third (momentum $-\mathbf{Q}$). Indeed, a comparison of Eq. (3.12) with Eq. (2.13) shows clearly their similarity, except for the kinematical effects of the third pion exhibited by (3.12). Further, if the angular correlation between $\mathbf{p}$ and $\mathbf{Q}$ in the energy denominator of (3.12) is neglected [an approximation which derives its strength from the presence of *both* $\omega(\mathbf{p}+\frac{1}{2}\mathbf{Q})$ and $\omega(\mathbf{p}-\frac{1}{2}\mathbf{Q})$ in the denominator], one can interpret $\pm\mathbf{p}$ as just the c.m. momenta of the two pions, and $\omega_Q$ as the energy of the third pion in the overall c.m. system. In this approximation, one can easily deduce the condition for a "bound state" of the three pions due to their individual resonances. This condition is simply that the value of $\omega_Q$ obtained from the equation

$$\omega_Q+2(\mu^2+\tfrac{1}{4}Q^2+p^2)^{1/2}-E=0 \quad (4.1)$$

should be *less* than $\mu$, when $p$ is put equal to the two-pion resonance momentum, viz. $k_0$. This leads trivially to the inequality

$$E<2\omega(k_0)+\mu\equiv E_0+\mu, \quad (4.2)$$

which happens to be consistent with the observed value of 787 MeV for the $3\pi$ resonance.[1]

Next we observe that under the condition (4.2) the integrals (3.12) or (3.16) are algebraically *greater* than the integral (2.13) for $k=k_0$. This means that the left-hand sides of Eqs. (3.9) and (3.14) are positive and negative for $\beta<\beta_c$ and $\beta>\beta_c$, respectively (see end of Sec. 3). For a consistent solution of these equations for the $3\pi$ resonance energy $E$, their right-hand sides should therefore have at least corresponding signs. Now, since the right-hand integrands fall off very rapidly with momentum (as we shall see more explicitly below), most of the contributions to these integrals arise out of *low* momenta for which the energy denominator $\omega_p+\omega_Q+\omega(\mathbf{p}+\mathbf{Q})-E$ is *negative.* Therefore the integrals multiplying the factors $\mp 2\pi\lambda$ on the right of (3.9) and (3.14) are *negative.* This fact, coupled with our earlier remarks on the signs of the left-hand sides of these equations, clearly leads to the conclusion that the "axial-vector" equation (3.9) is at least consistent with the *sign* requirements, whereas the scalar equation (3.14) is not. This statement holds for *both* types of interaction, (i) $\beta>\beta_c$ ($\lambda>0$) and (ii) $\beta<\beta_c$ ($\lambda<0$), as is quite evident from the presence of the factors $\mp 2\pi\lambda$ multiplying the right-hand integrals of (3.9) and (3.14). This fact we consider a rather nontrivial consequence of our model depending entirely on the algebraic structure of the integral equations, and independent of any detailed numerical evaluation. The model is in harmony with an isoscalar *vector* meson and inconsistent with a pseudoscalar one, at the observed energies of the $2\pi$ and $3\pi$ resonances.

Next, a semiquantitative analysis of Eq. (3.14) has been carried out by using some approximations discussed in the Appendix, by means of which it can be reduced to the form

$$G(Q)=3\lambda\int d\mathbf{p}\, L(p)\sin^2\theta[\omega_p+\omega_Q+\omega(\mathbf{p}+\mathbf{Q})-E]^{-1}G(p), \quad (4.3)$$

where

$$L(p)=[1-2\pi\lambda\mathfrak{F}_A(p)]^{-1}\times[\beta^2+(5/4)p^2]^{-2}[\mu^2+(5/8)p^2]^{-2}, \quad (4.4)$$

and $G(Q)$ is given in terms of $g_A(Q)$ by Eq. (A5). Further, (4.3) shows that $G(Q)$ approaches a nonzero constant for $Q\to 0$ and falls off like $Q^{-1}$ for large $Q$. For a numerical evaluation of (4.3) it is therefore convenient to represent $G(Q)$ by the structure

$$G(Q)=A\omega_Q^{-1}+BQ\omega_Q^{-2}+CQ^2\omega_Q^{-3}, \quad (4.5)$$

which has the above behavior with $Q$, term by term. The constants $A$, $B$, $C$ may be determined through the solution of three simultaneous equations for reasonably *low* values of $Q$, viz., $Q=0$, $\mu-2\mu$, $2\mu-4\mu$, say.

The following procedure was adopted for the numerical solution. For a preassigned value of $\beta$, $\lambda$ was determined from Eq. (2.16). A second estimate of $\lambda$ was then made from a solution of Eq. (4.3) with the

$3\pi$ energy corresponding to 787 MeV (i.e., $E=5.62\,\mu$) and compared with the earlier value. The results of evaluation of $2\pi\lambda$ for some typical values of $\beta$ are given by Table I. This table suggests that on the "short-range" side of $\beta$, viz., $\beta=2\,\mu$, the three-pion resonance needs a higher value of $\lambda$ than is provided by the two-pion resonance. The picture is just the opposite on the "long-range" side, the value $\beta=(2)^{-1/2}\mu$ representing an extreme case of how the two-pion resonance value of $\lambda$ is too large (numerically) for providing the three-pion resonance at the observed energy. The "matching" seems to be provided in our model by a value of $\beta$ near $(2)^{1/2}\mu$, which happens to be on the "long-range" side, as the negative sign of $\lambda$ indicates. It may be emphasized once again that this demarcation between "long" and "short" ranges is purely a function of the two-pion resonance energy, and that the dividing line shifts with the value of the latter.

A drawback of our model is the prediction of rather large resonance widths for the two-pion resonance. Thus, for $\beta=2\mu$, the value of $\Gamma(k_0)$ defined by Eq. (2.17) becomes 1.2, corresponding to a resonance width of $\sim$200 MeV at $E_0=750$ MeV. The three-pion resonance width, which could be calculated in principle by regarding Eq. (3.14) [or equivalently (4.3)] as an eignevalue equation in the complex energy $E$ and determining its imaginary part, has, however, not been evaluated.

TABLE I. Values of $2\pi\lambda$ for some typical values of $\beta$.

| $\beta$ | $2\,\mu$ | $(2)^{1/2}\mu$ | $(2)^{-1/2}\mu$ |
|---|---|---|---|
| Two-pion estimate | $+25\,\mu^2$ | $-41\,\mu^2$ | $-4.1\,\mu^2$ |
| Three-pion estimate | $+40\,\mu^2$ | $-42\,\mu^2$ | $-0.8\,\mu^2$ |

It may be of some interest to speculate on the possibility of understanding on this model the resonances at considerably lower masses as suggested, e.g., by Littauer *et al.*,[3] viz., $E_0=4\mu$ and $E=2.9\,\mu$ (in our notation), to fit the isoscalar and isovector form factors of the nucleon. Using exactly the same reasoning as given earlier in this section, it is clear that the axial-vector wave function (i.e., vector meson) does *not* fit in with this, low three-pion mass. For the integral multiplying $2\pi\lambda$ on the right of Eq. (3.14) is now *positive* as a result of the energy denominator being always positive, so that the signs of the two sides of (3.14) do not agree for this case. On the other hand, the signs of the two sides would now agree for the scalar wave function satisfying (3.9), so that there is, in principle, a possibility in our model (with different parameters of course) of predicting a low-mass $T=0$ three-pion state which is pseudoscalar. Of course, insofar as such low-mass models have been proposed to explain the electromagnetic form factors, there is clearly no use, from this point of view, in their being pseudoscalar.

Very recently, it came to our notice that a second three-pion resonance at an energy of about 560 MeV has been observed by Pevsner *et al.*[18] This lower mass resonance (should it turn out to be a $1^-$ state) must be particularly encouraging in the context of the observed isoscalar form factor of the nucleon, and it seems to bring Sakurai's vector theory of strong interactions[5,19] rather close to realization. However, since the spin and parity of this resonance have yet to be measured, we prefer at this stage not to discuss this resonance in the context of our model, except to point out that only a $0^-$ assignment of this lower energy state is in qualitative accord with our model.

After this work had been written up, an analogous approach by Schiff[20] on the two-pion and three-pion resonances came to the author's notice. Schiff has considered an attractive square-well potential which is strong enough to bind the di-pion and which satisfactorily accounts for the observed $p$-wave $\pi$-$\pi$ resonance. However, Schiff has pointed out that such an interaction between all pairs produces far too much binding for the $3\pi$ system to agree with observation, and this has led him to infer the condition of near saturation for the pair interactions.

Our approach differs from Schiff's in two distinct ways. First, in an approach based on configuration space potentials (e.g., square well), it usually becomes necessary to take the "adiabatic approximation" for the relative motion of the $2\pi$ system and this may cause substantial kinematical errors in the presence of a third pion, particularly when the total energy of the system is large. In our approach, on the other hand, kinematical effects are almost exactly incorporated. It is, however, difficult to compare offhand a momentum space interaction with a conventional local interaction, without reference to the various physically measurable quantities which can, of course, be calculated with both types of interaction.

A second point of difference between Schiff's approach and ours is that, while a saturation condition arises in a natural manner in the former, it does not seem to emerge from our theory. The answer to this apparent anomaly is simply that since the interaction considered by us is operative *only* in $p$ states, it *effectively incorporates* the saturation requirement. On the other hand, a conventional local potential operates in all angular momentum states, so that if a saturation condition should become necessary, it has to be imposed separately, and it appears from Schiff's work that it *is* necessary for the $3\pi$ resonance.

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[18] A. Pevsner, R. Kraemer, M. Nussbaum, C. Richardson, P. Schlein, R. Strand, T. Toohig, M. Block, A. Engler, R. Gessaroli, and C. Meltzer, Phys. Rev. Letters **7**, 421 (1961).
[19] See also, J. J. Sakurai, Phys. Rev. Letters **7**, 355 (1961).
[20] L. I. Schiff, Phys. Rev. **125**, 777 (1962).

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## APPENDIX I

Our approximation scheme for a reduction of the kernel of Eq. (3.14) consists of two parts: (A) averaging over angular correlations in most of the factors and (B) expressing these factors in an approximately 'separable' form.

Approximation (A) gives the following result:

$$\langle v(\mathbf{p}+\tfrac{1}{2}\mathbf{Q})v(\mathbf{Q}+\tfrac{1}{2}\mathbf{p})\omega^{-1}(\mathbf{p}+\tfrac{1}{2}\mathbf{Q})\omega^{-1}(\mathbf{Q}+\tfrac{1}{2}\mathbf{p})\rangle_{\mathrm{av}} \approx (\beta^2+p^2+\tfrac{1}{4}Q^2)^{-1}(\beta^2+Q^2+\tfrac{1}{4}p^2)^{-1} \times(\mu^2+p^2+\tfrac{1}{4}Q^2)^{-\frac{1}{2}}(\mu^2+Q^2+\tfrac{1}{4}p^2)^{-\frac{1}{2}}. \quad \text{(A1)}$$

The main justification for this approximation lies in the recognition that each factor on the left is positive definite. The more sensitive factors in the kernel, viz.,

$$3\sin^2\theta(\omega_p+\omega_Q+\omega(\mathbf{p}+\mathbf{Q})-E)^{-1}, \quad \text{(A2)}$$

are, however, left undisturbed.

For approximation (B), we note the following:

$$(\beta^2+p^2+\tfrac{1}{4}Q^2)(\beta^2+Q^2+\tfrac{1}{4}p^2) \equiv(\beta^2+\tfrac{5}{4}p^2)(\beta^2+\tfrac{5}{4}Q^2)+\tfrac{1}{4}(p^2-Q^2)^2. \quad \text{(A3)}$$

If, therefore, we neglect the last term in (A3), the resulting expression on the right of (A1) is an *overestimate* for *large* $p$ or $Q$. However, for large $p$ or $Q$, the contribution to the integral is much smaller than for small values of these momenta. Therefore, our approximation consists in neglecting the last term in (A3). In the other two factors in (A1) it would of course be desirable to compensate partially for the above, by making the opposite approximation. Thus, if we write on analogous lines

$$(\mu^2+p^2+\tfrac{1}{4}Q^2)^{1/2}(\mu^2+Q^2+\tfrac{1}{4}p^2)^{1/2} \approx(\mu^2+\tfrac{5}{8}p^2)(\mu^2+\tfrac{5}{8}Q^2), \quad \text{(A4)}$$

where an obvious binomial expansion for small momenta has been used in the left-hand side of (A4), the resulting expression in (A1) would clearly be an underestimate for large $p$ or $Q$. It is, therefore, very reasonable to expect that the product of the two appromations (A3) and (A4) is a fairly faithful representation of the right-hand side of (A1). Using these approximations and defining a new function $G(Q)$ by the equation

$$[1-2\pi\lambda\mathfrak{F}_A(Q)]^{-1}G(Q) =[\beta^2+(5/4)Q^2][\mu^2+(5/8)Q^2]g_A(Q), \quad \text{(A5)}$$

we obtain Eqs. (4.3) and (4.4) of the text.