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Field-Theoretical Calculation of the One-Pion-Exchange and Two-Pion-Exchange Contributions to the Phase-Shifts with Higher Angular Momenta for Nucleon-Nucleon Scattering*

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The one-pion-exchange and two-pion-exchange parts of the S matrix for nucleon-nucleon scattering are calculated field-theoretically. The rescattering of virtual pions by nucleons and the pion-pion interaction between virtual pions are taken into account. The S matrix is then decomposed into the partial-wave amplitudes, and the phase shifts are calculated. Numerical evaluations are carried out for the 310-MeV proton-proton scattering, and the results are compared with the phase shifts obtained by analyzing the experimental data. It is found that, without contribution of the pion-pion interaction, the results are far from agreement with experiment because of too strong attraction arising from the contributions of the two-pion-exchange part, but the contribution of the pion-pion resonance in the $I=J=1$ state improves the results considerably by largely cancelling the attraction. It is, however, also found that definite discrepancies still remain between the theory and the experiments, and this suggests that some unknown effects must play important roles in determining the nuclear force in the region of the internucleon distance around the Compton wavelength of the pion.

I. INTRODUCTION

THE present work was started before the double-dispersion representation had been discovered, and the dispersion-theoretic approach to the nucleon-nucleon scattering was very much developed. At that time, the standard meson-theoretic approach to the nucleon-nucleon scattering was, so to speak, the potential approach, in which one first derives the nucleon-nucleon potential from meson theory, and then solves the two-nucleon Schrödinger equation with that potential. It was also desired to derive a potential which treated the nonstatic effects correctly, as the importance of the LS force was recognized. Moreover, it had been shown by Konuma *et al.*[1] that the resonant interaction between nucleons and virtual pions which manifests itself as the 3-3 resonance in pion-nucleon scattering plays an important role in the static nucleon-nucleon potential. It had been recognized, however, that one encounters an ambiguity even in the definition of the potential in addition to technical difficulties, if one tries to derive a nonstatic potential which does not explicitly depend on the energy. Also it was not known how one should calculate a nonstatic potential which would include the effects of the 3-3 resonance. These circumstances led the author to the idea of calculating the S matrix directly, instead of deriving the potential, because, if one tried to calculate the S matrix, it would be easy to completely account for the nonstatic effects, and it would also be possible to include the effect of the 3-3 resonance.

In the present paper we calculate the one-pion-exchange and the two-pion-exchange contributions to the S matrix. The effect of the 3-3 resonance is included by using the dispersion relations for pion-nucleon scattering. We further take the effects of the pion-pion interaction between two virtual pions into account. The S matrix thus obtained contains contributions of both the short-range interaction and the long-range interaction. We are concerned only with the contribution of the long-range interaction because it is insufficient for treating the short-range interaction to retain only the two-pion-exchange part. In order to separate these contributions from each other, we decompose the S matrix into partial-wave amplitudes, and calculate only the phase-shifts corresponding to high angular momenta. From the viewpoint of the potential approach, the phase shifts thus calculated correspond to those calculated from the potential in the Born approximation, because the potential is defined so as to give in the Born approximation the same S matrix as that calculated from field theory when the number of exchanged pions is limited as in the present case. Therefore, among these phase shifts, only the ones whose values turn out to be sufficiently smaller than unity can be meaningfully compared with experiment.

The purpose of the present work is to clarify the role played in the long-range nucleon-nucleon interaction by each of the effects considered here, and to see whether or not we have to include other effects in order to explain the part of the observed data dominated by the long-range interaction. By "long-range" we mean here that the internucleon distance is larger than about $0.7m_\pi^{-1}$.[2] Examining the results of existing analyses of the data by means of phenomenological potentials,[3] we find that the Born approximation is not very bad for almost all the partial waves with impact parameters larger than $0.7m_\pi^{-1}$; the exceptions are only ^3D waves in the energy region between 80 MeV and 300 MeV.

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[1] M. Konuma, H. Miyazawa, and S. Otsuki, Progr. Theoret. Phys. (Kyoto) 19, 17 (1958).

[2] See Suppl. Progr. Theoret. Phys. (Kyoto) 3 (1956).

[3] J. L. Gammel and R. M. Thaler, Phys. Rev. 107, 291; 1337 (1957). R. A. Bryan, Nuovo cimento 16, 895 (1960). T. Hamada, Progr. Theoret. Phys. (Kyoto) 24, 1033 (1960); 25, 247 (1961).

Therefore, our simple method is almost sufficient for our purpose.

In recent years the dispersion theory has been developed, and the applications to nucleon-nucleon scattering have been made.[4] Our method is, in essence, very similar to some dispersion theories of nucleon-nucleon scattering. Indeed, for the simplified case of the scalar nucleon, we have verified that our method is identical to the dispersion theory of Cini and Fubini.[5]

In Sec. 2 we introduce the K matrix, in order to ensure unitarity, and the result of its decomposition into partial waves is given. In Sec. 3 the one-pion-exchange and the two-pion-exchange parts of the S matrix are defined on the basis of the Feynman-Dyson theory. In doing this, the two-pion-exchange part is naturally written as a sum of two terms, one of which does not include the effect of the pion-pion interaction explicitly, the other being purely a result of the pion-pion interaction. These two terms are separately calculated in Secs. 4 and 5, respectively. In Sec. 6 the numerical results for the 310-MeV p-p scattering are given, and comparison with experiment is made.

II. KINEMATICAL PRELIMINARIES

As stated in Sec. I, we calculate the S matrix for nucleon-nucleon scattering up to the two-pion-exchange part. In this approximation, however, the S matrix does not satisfy the unitarity condition. In order to avoid this deficiency, we introduce the K matrix by means of the formula

$$S=(1-iK/2)(1+iK/2)^{-1}. \tag{2.1}$$

The unitarity of S implies the Hermiticity of K. Writing S as

$$S=1-iR, \tag{2.2}$$

and using the Hermiticity of K, we have from (2.1)

$$K=(1/2)\sum_{\nu=0}^{\infty}[R(iR/2)^{\nu}+R^{\dagger}(-iR^{\dagger}/2)^{\nu}]. \tag{2.3}$$

On the basis of this formula we define the n-pion-exchange part K_n of K as follows:

$$K_1=(1/2)(R_1+R_1^{\dagger}),$$
$$K_2=(1/2)(R_2+R_2^{\dagger})+(1/2)[R_1(iR_1/2)+R_1^{\dagger}(-iR_1^{\dagger}/2)],$$

and so on. Here R_n denotes the n-pion-exchange part of R. Since R_1 is Hermitian, we have for K_1 and K_2, which we are here concerned with,

$$K_1=R_1,\quad K_2=(R_2+R_2^{\dagger})/2. \tag{2.4}$$

Owing to Lorentz invariance and charge independence, the matrix element of K between two two-nucleon states may in general be written in the following form:

$$\langle q_1,q_2|K|p_1,p_2\rangle=(2\pi)^4\delta^4(p_1+p_2-q_1-q_2)M^2(p_{10}p_{20}q_{10}q_{20})^{-1/2}$$
$$\times\sum_{i=1}^{5}\{a_i(s,t)[\bar{u}(q_1)\Gamma_iu(p_1)][\bar{u}(q_2)\Gamma_iu(p_2)]+b_i(s,t)[\bar{u}(q_1)\Gamma_i\tau_\alpha u(p_1)][\bar{u}(q_2)\Gamma_i\tau_\alpha u(p_2)]\}, \tag{2.5a}$$

where

$$\Gamma_1=1,\quad \Gamma_2=\gamma_\mu,\quad \Gamma_3=(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)/(2i),\quad \Gamma_4=\gamma_5\gamma_\mu,\quad \Gamma_5=\gamma_5, \tag{2.5b}$$

and

$$s=-(p_1+p_2)^2,\quad t=-(p_1-q_1)^2. \tag{2.5c}$$

$u(p)$ and $\bar{u}(q)=u^{\dagger}(q)\gamma_4$ are the free Dirac spinors normalized as $\bar{u}u=1$.

We write the K matrix given by (2.5) as a sum of the partial-wave amplitudes, then we express the partial-wave amplitudes in terms of $a_i(s,t)$ and $b_i(s,t)$. Since the decomposition of the S matrix into partial-wave amplitudes has already been carried out,[6] we quote only the results in the following.

We first put

$$k_j^I=\tan\delta_j^I,\quad k_{jj}^I=\tan\delta_{jj}^I, \tag{2.6a}$$
$$k_{j-1,j}^I=\tan\delta_{j-1,j}^I\cos^2\epsilon_j^I+\tan\delta_{j+1,j}^I\sin^2\epsilon_j^I, \tag{2.6b}$$
$$k_{j+1,j}^I=\tan\delta_{j-1,j}^I\sin^2\epsilon_j^I+\tan\delta_{j+1,j}^I\cos^2\epsilon_j^I, \tag{2.6c}$$
$$m_j^I=(\tan\delta_{j-1,j}^I-\tan\delta_{j+1,j}^I)\sin2\epsilon_j^I, \tag{2.6d}$$

where δ_j^I denotes the singlet phase shift with angular momentum j, $\delta_{l,j}^I$ denotes the triplet phase shift with orbital angular momentum l and total angular momentum j, and ϵ_j^I denotes the mixing parameter. The superscript I represents the total isotopic spin. The phase shifts are the Blatt-Biedenharn phase shifts. Now let the K matrix be written in the form (2.5) under the temporary assumption that the nucleon with four-momenta p_1 and q_1, and the one with p_2 and q_2 are distinguishable. Then, after the necessary symmetriza-

[4] M. L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. (New York) **2**, 226 (1957). M. Cini, S. Fubini, and A. Stanghellini, Phys. Rev. **114**, 1633 (1959). H. P. Noyes and D. Y. Wong, Phys. Rev. Letters **3**, 191 (1959). Y. Hara and H. Miyazawa, Progr. Theoret. Phys. (Kyoto) **23**, 942 (1960). D. Amati, E. Leader, and B. Vitale, Nuovo cimento **17**, 68 (1960); **18**, 409, 458 (1960). M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960). See also S. Furuichi and S. Machida, Nuovo cimento **19**, 396 (1961); and Y. Hara (to be published).

[5] M. Cini and S. Fubini, Ann. Phys. (New York) **10**, 352 (1960).

[6] See, for example, P. Cziffra, University of California Radiation Laboratory Report UCRL-9249, 1960 (unpublished).

tion has been made, the k^I's are written as follows:

$$8\pi(E_p/p)k_j{}^I = -(2M^2+p^2)a_{1j}{}^I - 2(M^2+2p^2)a_{2j}{}^I + 12(M^2+p^2)a_{3j}{}^I - 2(3M^2+2p^2)a_{4j}{}^I - p^2 a_{5j}{}^I + p^2 j(2j+1)^{-1}(a_{1j-1}{}^I + 4a_{3j-1}{}^I + a_{5j-1}{}^I) + p^2(j+1)(2j+1)^{-1}(a_{1j+1}{}^I + 4a_{3j+1}{}^I + a_{5j+1}{}^I), \quad (2.7a)$$

$$8\pi(E_p/p)k_{jj}{}^I = -(2M^2+p^2)a_{1j}{}^I - 2(M^2+p^2)a_{2j}{}^I - 4M^2 a_{3j}{}^I + 2(M^2+p^2)a_{4j}{}^I + p^2 a_{5j}{}^I + p^2(j+1)(2j+1)^{-1}(a_{1j-1}{}^I - 2a_{2j-1}{}^I + 2a_{4j-1}{}^I - a_{5j-1}{}^I) + p^2 j(2j+1)^{-1}(a_{1j+1}{}^I - 2a_{2j+1}{}^I + 2a_{4j+1}{}^I - a_{5j+1}{}^I), \quad (2.7b)$$

$$16\pi(E_p/p)k_{j-1,j}{}^I = [(E_p+M)^2 + (E_p-M)^2(2j+1)^{-2}](-a_{1j-1}{}^I - a_{2j-1}{}^I - 2a_{3j-1}{}^I + a_{4j-1}{}^I) + 2(2j+1)^{-1}p^2(-a_{2j-1}{}^I + 2a_{3j-1}{}^I + a_{4j-1}{}^I + a_{5j-1}{}^I) + 2p^2\{a_{1j}{}^I - [3-(2j+1)^{-1}]a_{2j}{}^I - 6[1-(2j+1)^{-1}]a_{3j}{}^I - [1-3(2j+1)^{-1}]a_{4j}{}^I - (2j+1)^{-1}a_{5j}{}^I\} + 4j(j+1)(2j+1)^{-2}(E_p-M)^2 \times(-a_{1j+1}{}^I - a_{2j+1}{}^I - 2a_{3j+1}{}^I + a_{4j+1}{}^I), \quad (2.7c)$$

$$16\pi(E_p/p)k_{j+1,j}{}^I = [(E_p+M)^2 + (E_p-M)^2(2j+1)^{-2}](-a_{1j+1}{}^I - a_{2j+1}{}^I - 2a_{3j+1}{}^I + a_{4j+1}{}^I) + 2(2j+1)^{-1}p^2(a_{2j+1}{}^I - 2a_{3j+1}{}^I - a_{4j+1}{}^I - a_{5j+1}{}^I) + 2p^2\{a_{1j}{}^I - [3+(2j+1)^{-1}]a_{2j}{}^I - 6[1+(2j+1)^{-1}]a_{3j}{}^I - [1+3(2j+1)^{-1}]a_{4j}{}^I + (2j+1)^{-1}a_{5j}{}^I\} + 4j(j+1)(2j+1)^{-2}(E_p-M)^2 \times(-a_{1j-1}{}^I - a_{2j-1}{}^I - 2a_{3j-1}{}^I + a_{4j-1}{}^I), \quad (2.7d)$$

$$4\pi(E_p/p)m_j{}^I = [j(j+1)]^{\frac{1}{2}}(2j+1)^{-1}[p^2(a_{2j-1}{}^I - 2a_{3j-1}{}^I - a_{4j-1}{}^I - a_{5j-1}{}^I - 2a_{2j}{}^I - 12a_{3j}{}^I - 6a_{4j}{}^I + 2a_{5j}{}^I + a_{2j+1}{}^I - 2a_{3j+1}{}^I - a_{4j+1}{}^I - a_{5j+1}{}^I) + (E_p-M)^2(2j+1)^{-1}(a_{1j-1}{}^I + a_{2j-1}{}^I + 2a_{3j-1}{}^I - a_{4j-1}{}^I - a_{1j+1}{}^I - a_{2j+1}{}^I - 2a_{3j+1}{}^I + a_{4j+1}{}^I)], \quad (2.7e)$$

where

$$\begin{aligned} a_{ij}{}^I &= \int_{-1}^{1} d(\cos\theta)P_j(\cos\theta)a_i{}^I[s=4E_p{}^2,\ t=-2p^2(1-\cos\theta)], \\ a_i{}^0(s,t) &= a_i(s,t) - 3b_i(s,t), \\ a_i{}^1(s,t) &= a_i(s,t) + b_i(s,t). \end{aligned} \quad (2.7f)$$

E_p and p are, respectively, the energy and the momentum of either nucleon in the c.m. system. Of course, the k's and m's are zero for the states forbidden by the Pauli principle.

III. DEFINITIONS OF THE ONE-PION-EXCHANGE AND TWO-PION-EXCHANGE PARTS OF THE S MATRIX

We adopt the Feynman-Dyson theory of the S matrix as the basis of our theory. We therefore suppose that the S matrix under consideration is sum of the contributions of all connected Feynman graphs each of which has two open nucleon polygons and no other external lines.

We define the one-pion-exchange part as the sum of contributions of all the Feynman graphs each of which consists of two subgraphs connected by one pion line. Of course each of the two subgraphs has one open nucleon polygon for any Feynman graph with nonvanishing contribution. Then the matrix element of R_1, and hence that of K_1, is written as

$$\langle q_1,q_2|R_1|p_1,p_2\rangle = \langle q_1,q_2|K_1|p_1,p_2\rangle = (2\pi)^4\delta^4(p_1+p_2-q_1-q_2)M^2(p_{10}p_{20}q_{10}q_{20})^{-1/2} \times(1/2)[\bar{u}(q_1)G(t)\tau_\alpha u(p_1)]\Delta_F'(-t)[\bar{u}(q_2)G(t)\tau_\alpha u(p_2)], \quad (3.1)$$

where

$$t = -(p_1-q_1)^2 = -(p_2-q_2)^2.$$

Here $G(t)$ is the renormalized pion-nucleon vertex function, and $\Delta_F'(-t)$ is the renormalized pion propagator including all the radiative corrections.

We now replace $G(t)$ by $G(\mu^2) = g\gamma_5$, and $\Delta_F'(-t)$ by $\Delta_F(-t) \equiv (\mu^2-t)^{-1}$, where g is the renormalized pion-nucleon coupling constant, and μ is the observed mass of the pion. It has been shown by Hoshizaki and Machida[7] that this replacement is allowable in an approximation which neglects the exchange of three or more pions. In fact, the matrix element (3.1) is essentially an analytic function of t with a pole at $t=\mu^2$ and a branch cut extending from $9\mu^2$ to ∞. The contribution of the cut corresponds to an interaction with the range shorter than $(3\mu)^{-1}$. The above replacement therefore

[7] N. Hoshizaki and S. Machida, Progr. Theoret. Phys. (Kyoto) **24**, 1325 (1960).

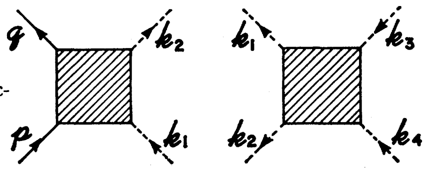


FIG. 1. The irreducible graphs.

retains only the contribution of the pole, neglecting the contribution of the cut.

Comparing (3.1) with (2.5) after the above replacement, we have for the a_i and b_i corresponding to the one-pion-exchange part:

$$b_5(s,t)=(g^2/2)(\mu^2-t)^{-1};$$
$$\text{all the other } a_i \text{ and } b_i=0. \quad (3.2)$$

We now proceed to the two-pion-exchange part. We first define a two-pion-exchange graph as follows: Consider a connected Feynman graph which has two open nucleon polygons and no other external lines. We refer to this graph as a two-pion-exchange graph when it can be divided by opening two pion lines into two connected subgraphs each of which has one of the open nucleon polygons, but it cannot be divided in such a way by opening one-pion line. The two-pion-exchange part of the S matrix is defined as the sum of contributions of all the two-pion-exchange graphs.

It is convenient to introduce the irreducible subgraphs before we proceed further. Consider a connected graph of the type of Fig. 1(a). We call this graph irreducible when it cannot be divided by opening one or two internal pion lines into two connected portions one of which has the external nucleon lines p and q, and the other has the external pion lines k_1 and k_2. Similarly, for a connected graph of the type of Fig. 1(b), we call this graph irreducible when it cannot be divided by opening one or two internal pion lines into two portions, one of which has the external pion lines k_1 and k_2, and the other has the external pion lines k_3 and k_4. We denote the sum of contributions of all the irreducible graphs of the type of Fig. 1(a) [Fig. 1(b)] by $T(q,k_2;p,k_1)$ [$\mathcal{K}(k_1,k_2;k_3,k_4)$]. In this statement the irreducible graph really means that part of the graph which does not contain the external lines. In counting up the different irreducible graphs, the labeling of the external pion lines should be included in the feature of the graph.

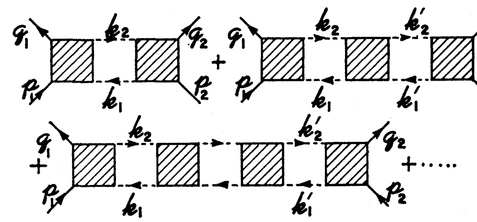


FIG. 2. Series representation of the two-pion-exchange part of the nucleon-nucleon S matrix.

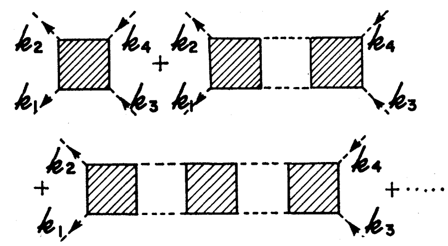


FIG. 3. Series representation of the two-pion Green's function.

Any two-pion-exchange graph consists of a certain number of the irreducible subgraphs in the manner shown by Fig. 2. Correspondingly the two-pion-exchange part of the S matrix is written as a series in which each term is the sum of contributions of all the two-pion-exchange graphs consisting of a definite number of irreducible subgraphs. This series is graphically represented by Fig. 2. Summation of this series reduces to the summation of another series which is graphically represented by Fig. 3. We denote the sum of the latter series by $G(k_1,k_2;k_3,k_4)$. It is evident from Fig. 3 that G obeys the integral equation

$$G(k_1,k_2;k_3,k_4)=\mathcal{K}(k_1,k_2;k_3,k_4)-4i(2\pi)^{-4}\int d^4k_1'd^4k_2'\,\delta^4(k_1+k_2-k_1'-k_2')$$
$$\times\mathcal{K}(k_1,k_2;k_1',k_2')\Delta_F'(k_1'^2)\Delta_F'(k_2'^2)G(k_1',k_2';k_3,k_4). \quad (3.3)$$

G is the two-pion Green's function, and it becomes the scattering amplitude for pion-pion scattering when all the four-momenta carried by the external lines are on the mass shell. It satisfies the crossing symmetry owing to the definition of $\mathcal{K}$. Carrying out the summation of the series represented by Fig. 2 with the aid of G, we have the following expression for the two-pion-exchange part of the R matrix:

$$R_2=R_2^{(0)}+R_2^{(\pi)}, \quad (3.4a)$$

$$\langle q_1,q_2|R_2^{(0)}|p_1,p_2\rangle=i(1/4)\delta^4(p_1+p_2-q_1-q_2)M^2(p_{10}p_{20}q_{10}q_{20})^{-1/2}\int d^4k_1d^4k_2\,\delta^4(p_1+k_1-q_1-k_2)$$
$$\times[\bar{u}(q_1)T(q_1,k_2;p_1,k_1)u(p_1)]\Delta_F'(k_1^2)\Delta_F'(k_2^2)[\bar{u}(q_2)T(q_2,k_1;p_2,k_2)u(p_2)], \quad (3.4b)$$

$$\langle q_1,q_2|R_2^{(\pi)}|p_1,p_2\rangle=(2\pi)^{-4}\delta^4(p_1+p_2-q_1-q_2)M^2(p_{10}p_{20}q_{10}q_{20})^{-1/2}\int d^4k_1d^4k_2d^4k_1'd^4k_2'\,\delta^4(p_1+k_1-q_1-k_2)$$
$$\times\delta^4(p_2+k_2'-q_2-k_1')[\bar{u}(q_1)T(q_1,k_2;\,p_1,k_1)u(p_1)]\Delta_F'(k_1^2)\Delta_F'(k_2^2)G(k_1,\,-k_2;\,k_1',\,-k_2')$$
$$\times\Delta_F'(k_1'^2)\Delta_F'(k_2'^2)[\bar{u}(q_2)T(q_2,k_1';\,p_2,k_2')u(p_2)].\quad(3.4\text{c})$$

Now, owing to Lorentz invariance and charge independence, T can be written in the following form (here we write the charge indices explicitly):

$$T_{\beta\alpha}(q,k_2;\,p,k_1)=\delta_{\beta\alpha}T^{(+)}(q,k_2;\,p,k_1)+(1/2)[\tau_\beta,\tau_\alpha]T^{(-)}(q,k_2;\,p,k_1),\quad(3.5\text{a})$$
$$T^{(\pm)}(q,k_2;\,p,k_1)=-A^{(\pm)}(q,k_2;\,p,k_1)+(i/2)\gamma(k_1+k_2)B^{(\pm)}(q,k_2;\,p,k_1),\quad(3.5\text{b})$$

where $A^{(\pm)}$ and $B^{(\pm)}$ are the functions of the scalar products formed by k_1, k_2, p, and q. We try to express T in terms of the quantities related to pion-nucleon scattering. For this purpose, we consider the quantity defined by

$$T'(q,k_2;\,p,k_1)=T(q,k_2;\,p,k_1)-4i(2\pi)^{-4}\int d^4k_1'd^4k_2'\,\delta^4(p+k_1'-q-k_2')$$
$$\times T(q,k_2';\,p,k_1')\Delta_F'(k_1'^2)\Delta_F'(k_2'^2)G(k_1',\,-k_2';\,k_1,\,-k_2).\quad(3.6)$$

This reduces to the pion-nucleon scattering amplitude when k_1 and k_2 are on the mass shell. Therefore, if $A^{(\pm)\prime}$ and $B^{(\pm)\prime}$, respectively, denote the quantities corresponding to $A^{(\pm)}$ and $B^{(\pm)}$ in the expression of T' corresponding to (3.5), they can be written in the following dispersion form[8]

$$A^{(\pm)\prime}(q,k_2;\,p,k_1)=\pi^{-1}\int_{(M+\mu)^2}^{\infty}dW^2\,\mathrm{Im}A^{(\pm)\prime}(W^2,t)\{[(p+k_1)^2+W^2-i\epsilon]^{-1}\pm[(p-k_2)^2+W^2-i\epsilon]^{-1}\},\quad(3.7\text{a})$$

$$B^{(\pm)\prime}(q,k_2;\,p,k_1)=g^2\{[(p+k_1)^2+M^2-i\epsilon]^{-1}\mp[(p-k_2)^2+M^2-i\epsilon]^{-1}\}$$
$$+\pi^{-1}\int_{(M+\mu)^2}^{\infty}dW^2\,\mathrm{Im}B^{(\pm)\prime}(W^2,t)\{[(p+k_1)^2+W^2-i\epsilon]^{-1}\mp[(p-k_2)^2+W^2-i\epsilon]^{-1}\},\quad(3.7\text{b})$$

with

$$t=-(p-q)^2=-(k_1-k_2)^2,$$

provided that $k_1^2=k_2^2=-\mu^2$. Here $\mathrm{Im}A^{(\pm)\prime}(W^2,t)$ and $\mathrm{Im}B^{(\pm)\prime}(W^2,t)$ are the imaginary parts of $A^{(\pm)\prime}$ and $B^{(\pm)\prime}$ for the total c.m. energy W and the squared momentum transfer $-t$. Owing to unitarity, they can be expressed as the sum of contributions of the relevant intermediate states. Let $\mathrm{Im}A_{\mathrm{el}}^{(\pm)}$ and $\mathrm{Im}B_{\mathrm{el}}^{(\pm)}$ denote those parts of $\mathrm{Im}A^{(\pm)\prime}$ and $\mathrm{Im}B^{(\pm)\prime}$ which are contributed by the intermediate states consisting only of one nucleon and one pion, and consider the quantities which are obtained from (3.7) by replacing $\mathrm{Im}A^{(\pm)\prime}$ and $\mathrm{Im}B^{(\pm)\prime}$ by $\mathrm{Im}A_{\mathrm{el}}^{(\pm)}$ and $\mathrm{Im}B_{\mathrm{el}}^{(\pm)}$. Then we see that these quantities are not contributed by the second term in the right-hand side of (3.6), and therefore they are entirely included by $A^{(\pm)}$ and $B^{(\pm)}$, because the second term in the right-hand side of (3.6) is contributed by only the graphs in which any intermediate state for the channel of pion-nucleon scattering involves at least one nucleon and two pions. This is illustrated by Fig. 4.[9] Of course $A^{(\pm)}$ and $B^{(\pm)}$ contain the other contributions, for example, those corresponding to the graph in Fig. 5. We neglect these contributions because the inelastic cross section for the pion-nucleon scattering seems to be mainly due to the pion-pion interaction,[10] and we have already taken this

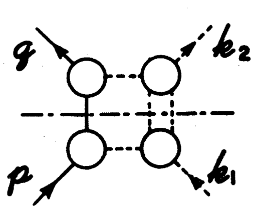


FIG. 4. The graph with the lowest-mass intermediate state which contributes to the second term in the right-hand side of (3.6).

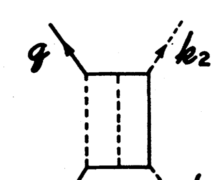


FIG. 5. An example of a graph with an inelastic absorptive part which contributes to $T(q,k_2;\,p,k_1)$.

[8] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[9] In this connection, see also M. Cini and S. Fubini, reference 5.

[10] W. A. Perkins, J. C. Caris, R. W. Kenney, E. A. Knapp, and V. Perez-Mendez, Phys. Rev. Letters **3**, 56 (1959). L. S. Rodberg, Phys. Rev. Letters **3**, 58 (1959).

interaction into account in (3.4). We therefore put

$$A^{(\pm)}(q,k_2;p,k_1)=\pi^{-1}\int_{(M+\mu)^2}^{\infty}dW^2\,\mathrm{Im}A_{\mathrm{el}}^{(\pm)}(W^2,t)\{[(p+k_1)^2+W^2-i\epsilon]^{-1}\pm[(p-k_2)^2+W^2-i\epsilon]^{-1}\},\quad(3.8\mathrm{a})$$

$$B^{(\pm)}(q,k_2;p,k_1)=g^2\{[(p+k_1)^2+M^2-i\epsilon]^{-1}\mp[(p-k_2)^2+M^2-i\epsilon]^{-1}\}$$
$$+\pi^{-1}\int_{(M+\mu)^2}^{\infty}dW^2\,\mathrm{Im}B_{\mathrm{el}}^{(\pm)}(W^2,t)\{[(p+k_1)^2+W^2-i\epsilon]^{-1}\mp[(p-k_2)^2+W^2-i\epsilon]^{-1}\}.\quad(3.8\mathrm{b})$$

The right-hand sides of (3.8) are the values of the actual $A^{(\pm)}$ and $B^{(\pm)}$ at $k_1^2=k_2^2=-\mu^2$. We nevertheless use (3.8) in evaluating the integrals in (3.4) where the regions of integration extend off the mass shell, because, owing to the pion propagators in the integrands, the values of T on the mass shell will give the most important contributions to the integrals. It seems probable that the differences between the actual $A^{(\pm)}$ and $B^{(\pm)}$ and the right-hand sides of (3.8) are appreciable only when the $-k_i^2$ are close to or greater than $(3\mu)^2$, and therefore these differences represent a short-range interaction which is beyond the scope of the present paper.

In (3.4) we replace $\Delta_F'(k^2)$ by $\Delta_F(k^2)=(k^2+\mu^2-i\epsilon)^{-1}$, because the radiative correction can be written as a superposition of the propagators with masses not smaller than 3μ, so that the inclusion of it corresponds to calculating contributions of the exchange of four or more pions.

Finally we calculate the two-pion Green's function G. Writing the charge indices of pions explicitly, we can write G and $\mathcal{K}$ in terms of their diagonal elements with respect to the isotopic spin as follows

$$G(k_1\alpha,k_2\beta;k_3\gamma,k_4\delta)=\sum_{I=0}^{2}P_{\alpha\beta,\gamma\delta}{}^{I}G_I(k_1,k_2;k_3,k_4),\quad(3.9\mathrm{a})$$

$$\mathcal{K}(k_1\alpha,k_2\beta;k_3\gamma,k_4\delta)=\sum_{I=0}^{2}P_{\alpha\beta,\gamma\delta}{}^{I}\mathcal{K}_I(k_1,k_2;k_3,k_4),\quad(3.9\mathrm{b})$$

where $P_{\alpha\beta,\gamma\delta}{}^{I}$ are the projection operators to the eigenstates of the total isotopic spin with the eigenvalue I, and are given by

$$P_{\alpha\beta,\gamma\delta}{}^{0}=(1/3)\delta_{\alpha\beta}\delta_{\gamma\delta},$$
$$P_{\alpha\beta,\gamma\delta}{}^{1}=(1/2)(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma}),$$
$$P_{\alpha\beta,\gamma\delta}{}^{2}=(1/2)(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})-P_{\alpha\beta,\gamma\delta}{}^{0}.$$

Using (3.9), we have from (3.3) the integral equations for G_I which are of the same form as (3.3) except that $\mathcal{K}$ is replaced by $\mathcal{K}_I$ there. In order to solve these equations, we have to specify the kernels $\mathcal{K}_I$. Now from the definition of $\mathcal{K}$ we find that $\mathcal{K}$ has the following symmetry property

$$\mathcal{K}(k_1\alpha,k_2\beta;k_3\gamma,k_4\delta)=\mathcal{K}(k_2\beta,k_1\alpha;k_3\gamma,k_4\delta)$$
$$=\mathcal{K}(-k_3\gamma,-k_4\delta;-k_1\alpha,-k_2\beta).$$

It follows from this that

$$\mathcal{K}_I(k_1,k_2;k_3,k_4)=(-1)^I\mathcal{K}_I(k_2,k_1;k_3,k_4)$$
$$=\mathcal{K}_I(-k_3,-k_4;-k_1,-k_2).\quad(3.10)$$

As the simplest ansatz to satisfy (3.10), we put

$$\mathcal{K}_0(k_1,k_2;k_3,k_4)=-5\lambda_0,$$
$$\mathcal{K}_1(k_1,k_2;k_3,k_4)=-(3\lambda_1/2\mu^2)(k_1-k_2)(k_3-k_4).\quad(3.11)$$

Solving the integral equations with these kernels is to calculate G_I by the so-called chain approximation. This has been done by Miyamoto.[11] The results are written as

$$G_0(k_1,k_2;k_3,k_4)=4\pi f_0(t),\quad(3.12\mathrm{a})$$

$$G_1(k_1,k_2;k_3,k_4)=3\pi\mu^{-2}(k_1-k_2)(k_3-k_4)f_1(t),\quad(3.12\mathrm{b})$$

where

$$t=-(k_1+k_2)^2=-(k_3+k_4)^2,$$

and, for $t<0$, $f_I(t)$ are given by

$$f_0(t)=\left\{\frac{1}{a_0}+\frac{2}{\pi}\left(\frac{t-4\mu^2}{t}\right)^{\frac{1}{2}}\ln\left[\left(1-\frac{t}{4\mu^2}\right)^{\frac{1}{2}}+\left(\frac{-t}{4\mu^2}\right)^{\frac{1}{2}}\right]\right\}^{-1},\quad(3.13\mathrm{a})$$

$$f_1(t)=\left\{\frac{1}{a_1}+\frac{-t+4\mu^2}{4\pi\mu^2}\left[\ln\frac{\Lambda}{\mu}+\frac{5}{3}-2\left(\frac{t-4\mu^2}{t}\right)^{\frac{1}{2}}\ln\left(\left(1-\frac{t}{4\mu^2}\right)^{\frac{1}{2}}+\left(\frac{-t}{4\mu^2}\right)^{\frac{1}{2}}\right)\right]\right\}^{-1}.\quad(3.13\mathrm{b})$$

Lambda in (3.13b) is the cutoff momentum. For $t>4\mu^2$, $f_I(t)$ are given by the analytic continuations of (3.13). On the other hand, for $t>4\mu^2$, $f_0(t)$ and $f_1(t)$ are found to be the partial-wave amplitudes for the pion-pion scattering for the states $I=J=0$ and $I=J=1$, respectively, and they are expressed in terms of the respective phase shifts η_0 and η_1 as follows

$$f_0(t)=t^{\frac{1}{2}}(t-4\mu^2)^{-\frac{1}{2}}e^{i\eta_0}\sin\eta_0,\quad(3.14\mathrm{a})$$

$$f_1(t)=4\mu^2t^{\frac{1}{2}}(t-4\mu^2)^{-\frac{3}{2}}e^{i\eta_1}\sin\eta_1.\quad(3.14\mathrm{b})$$

[11] Y. Miyamoto, Progr. Theoret. Phys. (Kyoto) 24, 840 (1960). See also S. Okubo, Phys. Rev. 118, 357 (1960).

Comparing the analytic continuations of (3.13) with (3.14), we find that a_I in (3.13) are just the scattering lengths. It has been shown by Miyamoto[11] that $f_1(t)$ defined by (3.13b) gives a resonance in the p-wave pion-pion scattering, and that we can make the position and width of the resonance close to those which are required to fit the data of the electromagnetic form factors of nucleon by a reasonable choice of the values of a_1 and Λ.

In later calculations, however, we do not use the explicit forms of the solutions obtained here. In calculating the contributions of the p wave pion-pion interaction, we will replace the relevant quantities and expressions by some phenomenological parameters, and try to use the experimental information as much as possible. As for the s-wave pion-pion interaction, we will replace $f_0(t)$ by a constant, because we are not sure that the solution (3.13a) correctly represents the matter.

IV. TWO-PION-EXCHANGE PART NOT INCLUDING CONTRIBUTIONS OF THE PION-PION INTERACTION

In this section we calculate the matrix element given by (3.4b). Using (3.5), with (3.8) for $A^{(\pm)}$ and $B^{(\pm)}$, we can rewrite (3.4b) as

$$
\begin{aligned}
&\langle q_1,q_2|R_2^{(0)}|p_1,p_2\rangle \\
&\quad = i(2\pi^2)^{-1}\delta^4(p_1+p_2-q_1-q_2)M^2(p_{10}p_{20}q_{10}q_{20})^{-1/2}\int_0^\infty dW_1^2\int_0^\infty dW_2^2\int d^4k_1d^4k_2\,\delta^4(p_1+k_1-q_1-k_2) \\
&\qquad \times(k_1^2+\mu^2-i\epsilon)^{-1}(k_2^2+\mu^2-i\epsilon)^{-1}\{[(p_1+k_1)^2+W_1^2-i\epsilon]^{-1}[(p_2-k_1)^2+W_2^2-i\epsilon]^{-1} \\
&\qquad \times[3\bar{u}(q_1)F^{(+)}(W_1^2,t,-k_1-k_2)u(p_1)\bar{u}(q_2)F^{(+)}(W_2^2,t,k_1+k_2)u(p_2) \\
&\qquad -2\bar{u}(q_1)F^{(-)}(W_1^2,t,-k_1-k_2)\tau_\alpha u(p_1)\bar{u}(q_2)F^{(-)}(W_2^2,t,k_1+k_2)\tau_\alpha u(p_2)] \\
&\qquad +[(p_1-k_2)^2+W_1^2-i\epsilon]^{-1}[(p_2-k_1)^2+W_2^2-i\epsilon]^{-1}[3\bar{u}(q_1)F^{(+)}(W_1^2,t,k_1+k_2)u(p_1) \\
&\qquad \times\bar{u}(q_2)F^{(+)}(W_2^2,t,k_1+k_2)u(p_2)+2\bar{u}(q_1)F^{(-)}(W_1^2,t,k_1+k_2)\tau_\alpha u(p_1)\bar{u}(q_2)F^{(-)}(W_2^2,t,k_1+k_2)\tau_\alpha u(p_2)]\}, \quad (4.1\text{a})
\end{aligned}
$$

with the abbreviations

$$F^{(\pm)}(W^2,t,Q)=U^{(\pm)}(W^2,t)+(i/2)(\gamma Q)V^{(\pm)}(W^2,t), \quad (4.1\text{b})$$

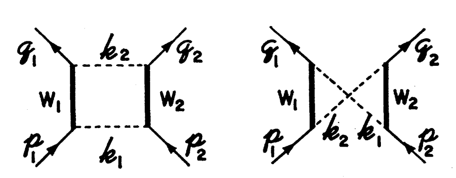
where $U^{(\pm)}$ and $V^{(\pm)}$ are defined by

$$
\begin{aligned}
U^{(\pm)}(W^2,t)&=\mathrm{Im}A_{\mathrm{el}}^{(\pm)}(W^2,t) && \text{for}\quad W^2\geq(M+\mu)^2 \\
&=0, && \text{for}\quad W^2<(M+\mu)^2,
\end{aligned}
\quad (4.2\text{a})
$$

$$
\begin{aligned}
V^{(\pm)}(W^2,t)&=\pi g^2\delta(W^2-M^2)+\mathrm{Im}B_{\mathrm{el}}^{(\pm)}(W^2,t) && \text{for}\quad W^2\geq(M+\mu)^2 \\
&=\pi g^2\delta(W^2-M^2) && \text{for}\quad W^2<(M+\mu)^2.
\end{aligned}
\quad (4.2\text{b})
$$

The integrals over k_1 and k_2 in (4.1) essentially correspond to the Feynman graphs in Fig. 6. Using Feynman's method, we transform these integrals into parameter integrals. Then the matrix element of $K_2^{(0)}$ (Hermitian part of $R_2^{(0)}$) can be written in the form of (2.5a), and the resulting a_i and b_i are found to be as follows

$$
\begin{aligned}
\left.\begin{matrix}\tfrac{2}{3}a_1(s,t)\\ b_1(s,t)\end{matrix}\right\} &= (2\pi)^{-4}\int_0^\infty dW_1^2\int_0^\infty dW_2^2\{-U^{(\pm)}(W_1^2,t)U^{(\pm)}(W_2^2,t)K_{11}^{(\pm)}(W_1^2,W_2^2,s,t)+U^{(\pm)}(W_1^2,t)V^{(\pm)}(W_2^2,t)(8M)^{-1} \\
&\quad \times[(2s+t+4M^2)K_{23}^{(\pm)}(W_1^2,W_2^2,s,t)+(2s+t-12M^2)K_{32}^{(\pm)}(W_1^2,W_2^2,s,t)] \\
&\quad -V^{(\pm)}(W_1^2,t)V^{(\pm)}(W_2^2,t)(16)^{-1}[(2s+t)K_{45}^{(\pm)}(W_1^2,W_2^2,s,t) \\
&\qquad +(2s+t-8M^2)K_{54}^{(\pm)}(W_1^2,W_2^2,s,t)]\}, \quad (4.3\text{a})
\end{aligned}
$$



(1) (2)

FIG. 6. The graphs corresponding to the integrals in (4.1).

$$\left.\begin{array}{l}\frac{2}{3}a_2(s,t)\\ b_2(s,t)\end{array}\right\}=(2\pi)^{-4}\int_0^\infty dW_1^2\int_0^\infty dW_2^2\{U^{(\pm)}(W_1^2,t)V^{(\pm)}(W_2^2,t)(M/2)[K_{23}^{(\pm)}(W_1^2,W_2^2,s,t)+K_{32}^{(\pm)}(W_1^2,W_2^2,s,t)]$$
$$+V^{(\pm)}(W_1^2,t)V^{(\pm)}(W_2^2,t)(16)^{-1}[8K_{00}^{(\pm)}(W_1^2,W_2^2,s,t)-(2s+t)K_{45}^{(\pm)}(W_1^2,W_2^2,s,t)$$
$$+(2s+t-8M^2)K_{54}^{(\pm)}(W_1^2,W_2^2,s,t)]\}, \quad (4.3b)$$

$$\left.\begin{array}{l}\frac{2}{3}a_3(s,t)\\ b_3(s,t)\end{array}\right\}=(2\pi)^{-4}\int_0^\infty dW_1^2\int_0^\infty dW_2^2\{-U^{(\pm)}(W_1^2,t)V^{(\pm)}(W_2^2,t)(16M)^{-1}t$$
$$\times[K_{23}^{(\pm)}(W_1^2,W_2^2,s,t)+K_{32}^{(\pm)}(W_1^2,W_2^2,s,t)]+V^{(\pm)}(W_1^2,t)V^{(\pm)}(W_2^2,t)(t/32)$$
$$\times[K_{45}^{(\pm)}(W_1^2,W_2^2,s,t)+K_{54}^{(\pm)}(W_1^2,W_2^2,s,t)]\}, \quad (4.3c)$$

$$\left.\begin{array}{l}\frac{2}{3}a_4(s,t)\\ b_4(s,t)\end{array}\right\}=(2\pi)^{-4}\int_0^\infty dW_1^2\int_0^\infty dW_2^2V^{(\pm)}(W_1^2,t)V^{(\pm)}(W_2^2,t)(t/16)$$
$$\times[-K_{45}^{(\pm)}(W_1^2,W_2^2,s,t)+K_{54}^{(\pm)}(W_1^2,W_2^2,s,t)], \quad (4.3d)$$

$$\left.\begin{array}{l}\frac{2}{3}a_5(s,t)\\ b_5(s,t)\end{array}\right\}=(2\pi)^{-4}\int_0^\infty dW_1^2\int_0^\infty dW_2^2\{U^{(\pm)}(W_1^2,t)V^{(\pm)}(W_2^2,t)(8M)^{-1}(2s+t-4M^2)$$
$$\times[K_{23}^{(\pm)}(W_1^2,W_2^2,s,t)+K_{32}^{(\pm)}(W_1^2,W_2^2,s,t)]-V^{(\pm)}(W_1^2,t)V^{(\pm)}(W_2^2,t)(16)^{-1}$$
$$\times[(2s+t)K_{45}^{(\pm)}(W_1^2,W_2^2,s,t)+(2s+t-8M^2)K_{54}^{(\pm)}(W_1^2,W_2^2,s,t)]\}, \quad (4.3e)$$

with the abbreviations

$$K_{ij}^{(\pm)}(W_1^2,W_2^2,s,t)=I_i^{(2)}(W_1^2,W_2^2,s,t)\mp(-1)^{\min(i,j)}I_j^{(1)}(W_1^2,W_2^2,s,t), \quad (4.3f)$$

where $I_i^{(1)}$ and $I_i^{(2)}$ are the parameter integrals which are given by

$$I_0^{(a)}(W_1^2,W_2^2,s,t)=\int_0^1dx\int_0^1dy\int_0^1dz\,z(1-z)D^{(a)}(x,y,z,W_1^2,W_2^2,s,t)^{-1}, \quad (4.4a)$$

$$I_i^{(a)}(W_1^2,W_2^2,s,t)=\int_0^1dx\int_0^1dy\int_0^1dz\,z(1-z)f_i(y,z)D^{(a)}(x,y,z,W_1^2,W_2^2,s,t)^{-2}, \quad (i=1,\cdots,5), \quad (4.4b)$$

with

$$f_1(y,z)=1,\quad f_2(y,z)=z,\quad f_3(y,z)=(2y-1)z,\quad f_4(y,z)=z^2,\quad f_5(y,z)=(2y-1)^2z^2, \quad (4.4c)$$

$$D^{(1)}(x,y,z,W_1^2,W_2^2,s,t)=-x(1-x)(1-z)^2t-y(1-y)z^2s+(1-z)\mu^2-z(1-z)M^2+yzW_1^2+(1-y)zW_2^2, \quad (4.4d)$$

$$D^{(2)}(x,y,z,W_1^2,W_2^2,s,t)=-x(1-x)(1-z)^2t+y(1-y)z^2(s+t-4M^2)$$
$$+(1-z)\mu^2-z(1-z)M^2+yzW_1^2+(1-y)zW_2^2. \quad (4.4e)$$

The integrals $I_i^{(1)}$ and $I_i^{(2)}$, respectively, correspond to the graphs Fig. 6 (1) and (2). Since $D^{(1)}$ has zero points in the region of integration, the integrals $I_i^{(1)}$ should be understood to be the principal-value integrals.

It is convenient for numerical calculations to transform $I_i^{(a)}$ into the following dispersion form:

$$I_i^{(a)}(W_1^2,W_2^2,s,t)=\int_{-\infty}^{\infty}\frac{\rho_i^{(a)}(W_1^2,W_2^2,s,t')}{t'-t}dt'. \quad (4.5)$$

Then, from (4.4a, d, e) we immediately have

$$\rho_0^{(a)}(W_1^2,W_2^2,s,t)=\epsilon^{(a)}(t)\int_0^1dx\int_0^1dy\int_0^1dz\,z(1-z)\delta[D^{(a)}(x,y,z,W_1^2,W_2^2,s,t)], \quad (4.6)$$

where $\epsilon^{(1)}(t)=1$, $\epsilon^{(2)}(t)=t/|t|$. The integrations in (4.6) can be carried out analytically. We find that, for $\rho_0^{(1)}$, the right-hand side of (4.6) becomes sum of two terms, one of which is nonzero only for $t\geq 4\mu^2$, and the other is nonzero for whole region of t. Since, however, we can prove the latter term gives only vanishing contribution to the integral in (4.5), we drop this term. $\rho_0^{(2)}$ is found to be nonzero only for $t\geq 4\mu^2$ and $t\leq -s$, as is expected. We discard the contribution from the region $t'<-s$ to the integral in (4.5), because it does not affect the higher partial waves significantly. Thus we can replace the lower limit of the integral in (4.5) by $4\mu^2$. For $4\mu^2<t<4M^2$, $\rho_0^{(a)}$

are given by

$$\rho_0^{(a)}(W_1^2,W_2^2,s,t)=\sigma_0^{(a)}(W_1^2,W_2^2,s,t)+\sigma_0^{(a)}(W_2^2,W_1^2,s,t), \tag{4.7a}$$

$$\begin{aligned}
-t^{\frac12}su\sigma_0^{(1)}(W_1^2,W_2^2,s,t)
&=(4M^2-t)^{-\frac12}[\zeta(W_1^2,t)s-(W_1^2-W_2^2)(4M^2-t)]\tan^{-1}[(4M^2-t)^{\frac12}(t-4\mu^2)^{\frac12}\zeta(W_1^2,t)^{-1}]\\
&\quad-(t-4\mu^2)^{-\frac12}\xi(W_1^2,W_2^2,s,t)^{\frac12}(1/2)\ln\left[\frac{s(t-4\mu^2)+(W_1^2-W_2^2)\zeta(W_1^2,t)+\xi(W_1^2,W_2^2,s,t)^{\frac12}}{s(t-4\mu^2)+(W_1^2-W_2^2)\zeta(W_1^2,t)-\xi(W_1^2,W_2^2,s,t)^{\frac12}}\right]\\
&\qquad\text{when}\quad \xi(W_1^2,W_2^2,s,t)>0,
\end{aligned} \tag{4.7b}$$

$$\begin{aligned}
-t^{\frac12}su\sigma_0^{(1)}(W_1^2,W_2^2,s,t)
&=(4M^2-t)^{-\frac12}[\zeta(W_1^2,t)s-(W_1^2-W_2^2)(4M^2-t)]\tan^{-1}[(4M^2-t)^{\frac12}(t-4\mu^2)^{\frac12}\zeta(W_1^2,t)^{-1}]\\
&\quad-(t-4\mu^2)^{-\frac12}[-\xi(W_1^2,W_2^2,s,t)]^{\frac12}\tan^{-1}\left\{\frac{s(t-4\mu^2)+(W_1^2-W_2^2)\zeta(W_1^2,t)}{[-\xi(W_1^2,W_2^2,s,t)]^{\frac12}}\right\}\quad\text{when}\quad \xi(W_1^2,W_2^2,s,t)<0,
\end{aligned}$$

$$\begin{aligned}
-t^{\frac12}su\sigma_0^{(2)}(W_1^2,W_2^2,s,t)
&=(4M^2-t)^{-\frac12}[\zeta(W_1^2,t)u-(W_1^2-W_2^2)(4M^2-t)]\tan^{-1}[(4M^2-t)^{\frac12}(t-4\mu^2)^{\frac12}\zeta(W_1^2,t)^{-1}]\\
&\quad+(t-4\mu^2)^{-\frac12}\eta(W_1^2,W_2^2,s,t)^{\frac12}(1/2)\ln\left|\frac{-u(t-4\mu^2)-(W_1^2-W_2^2)\zeta(W_1^2,t)+\eta(W_1^2,W_2^2,s,t)^{\frac12}}{-u(t-4\mu^2)-(W_1^2-W_2^2)\zeta(W_1^2,t)-\eta(W_1^2,W_2^2,s,t)^{\frac12}}\right|\\
&\qquad\text{when}\quad \eta(W_1^2,W_2^2,s,t)>0,
\end{aligned} \tag{4.7c}$$

$$\begin{aligned}
-t^{\frac12}su\sigma_0^{(2)}(W_1^2,W_2^2,s,t)
&=(4M^2-t)^{-\frac12}[\zeta(W_1^2,t)u-(W_1^2-W_2^2)(4M^2-t)]\tan^{-1}[(4M^2-t)^{\frac12}(t-4\mu^2)^{\frac12}\zeta(W_1^2,t)^{-1}]\\
&\quad+(t-4\mu^2)^{-\frac12}[-\eta(W_1^2,W_2^2,s,t)]^{\frac12}\tan^{-1}\left\{\frac{-u(t-4\mu^2)-(W_1^2-W_2^2)\zeta(W_1^2,t)}{[-\eta(W_1^2,W_2^2,s,t)]^{\frac12}}\right\}\quad\text{when}\quad \eta(W_1^2,W_2^2,s,t)<0,
\end{aligned}$$

$$\xi(W_1^2,W_2^2,s,t)=-s(t-4\mu^2)\{(s+2\mu^2-2M^2-W_1^2-W_2^2)^2+u[s-2W_1^2-2W_2^2+(W_1^2-W_2^2)^2s^{-1}]\},$$

$$\eta(W_1^2,W_2^2,s,t)=-u(t-4\mu^2)\{(u+2\mu^2-2M^2-W_1^2-W_2^2)^2+s[u-2W_1^2-2W_2^2+(W_1^2-W_2^2)^2u^{-1}]\},$$

$$\zeta(W^2,t)=t+2W^2-2M^2-2\mu^2,\quad u=4M^2-s-t.$$

The weight functions for $t>4M^2$ do not contribute to the higher partial waves. It is evident from (4.4) that $\rho_i^{(a)}$ $(i=1,\cdots,5)$ can be obtained as appropriate combinations of the derivatives of $\rho_0^{(a)}$ with respect to s, μ^2, M^2, W_1^2, and W_2^2.

Introducing (4.5) into (4.3), we write a_i and b_i in the similar dispersion form. Introducing them into (2.7), we can write k_j^I, k_{lj}^I, and m_j^I as linear combinations of the integrals of the form

$$\int_{-1}^{1}d(\cos\theta)P_j(\cos\theta)\int_{4\mu^2}^{\infty}\frac{\rho(s,t')dt'}{t'+2p^2(1-\cos\theta)}=\frac{1}{p^2}\int_{4\mu^2}^{\infty}\rho(s,t)Q_j\left(1+\frac{t}{2p^2}\right)dt, \tag{4.8}$$

where Q_j is the Légendre function of the second kind.

$U^{(\pm)}$ and $V^{(\pm)}$ can be expressed in terms of the renormalized coupling constant and the partial-wave amplitudes for the elastic pion-nucleon scattering. We retain only the 3-3 amplitude in $\mathrm{Im}A_{\mathrm{el}}^{(\pm)}$ and $\mathrm{Im}B_{\mathrm{el}}^{(\pm)}$, and make the narrow-resonance approximation to the 3-3 amplitude.[8] Then $U^{(\pm)}$ and $V^{(\pm)}$ can be written in the following form

$$U^{(+)}(W^2,t)=-2U^{(-)}(W^2,t)=\pi\mu g^2\delta(W^2-W_r^2)(\alpha_1+\alpha_2\mu^{-2}t), \tag{4.9a}$$

$$V^{(+)}(W^2,t)=\pi g^2[\delta(W^2-M^2)-\delta(W^2-W_r^2)(\beta_1-\beta_2\mu^{-2}t)], \tag{4.9b}$$

$$V^{(-)}(W^2,t)=\pi g^2[\delta(W^2-M^2)+(1/2)\delta(W^2-W_r^2)(\beta_1-\beta_2\mu^{-2}t)]. \tag{4.9c}$$

We take as the value of W_r

$$W_r=M+2\mu. \tag{4.10}$$

Then the numerical values of α_i and β_i are found to be

$$\alpha_1=3.02,\quad \alpha_2=0.147,\quad \beta_1=1.12,\quad \beta_2=0.0095.$$

V. CONTRIBUTION OF THE PION-PION INTERACTION

In this section we calculate the matrix element (3.4c). Substituting (3.5a) and (3.9a) into (3.4c), and using (3.12), we have

$$R_2^{(\pi)}=R_2^{(\pi0)}+R_2^{(\pi1)}, \tag{5.1a}$$

$$\begin{aligned}\langle q_1,q_2|R_2^{(\pi0)}|p_1,p_2\rangle &=6(2\pi)^{-3}\delta^4(p_1+p_2-q_1-q_2)M^2(p_{10}p_{20}q_{10}q_{20})^{-1/2}f_0(t)\int d^4k_1d^4k_2\,\delta^4(p_1+k_1-q_1-k_2)(k_1^2+\mu^2-i\epsilon)^{-1}\\ &\times(k_2^2+\mu^2-i\epsilon)^{-1}[\bar{u}(q_1)T^{(+)}(q_1,k_2;\,p_1,k_1)u(p_1)]\int d^4k_1'd^4k_2'\,\delta^4(p_2+k_1'-q_2-k_2')(k_1'^2+\mu^2-i\epsilon)^{-1}\\ &\times(k_2'^2+\mu^2-i\epsilon)^{-1}[\bar{u}(q_2)T^{(+)}(q_2,k_2';\,p_2,k_1')u(p_2)],\end{aligned} \tag{5.1b}$$

$$\begin{aligned}\langle q_1,q_2|R_2^{(\pi1)}|p_1,p_2\rangle &=(2\pi)^{-4}\delta^4(p_1+p_2-q_1-q_2)M^2(p_{10}p_{20}q_{10}q_{20})^{-1/2}3\pi\mu^{-2}f_1(t)\int d^4k_1d^4k_2d^4k_1'd^4k_2'\,(k_1+k_2)\cdot(k_1'+k_2')\\ &\times\delta^4(p_1+k_1-q_1-k_2)(k_1^2+\mu^2-i\epsilon)^{-1}(k_2^2+\mu^2-i\epsilon)^{-1}[\bar{u}(q_1)(1/2)[\tau_\beta,\tau_\alpha]T^{(-)}(q_1,k_2;\,p_1,k_1)u(p_1)]\\ &\times\delta^4(p_2+k_2'-q_2-k_1')(k_1'^2+\mu^2-i\epsilon)^{-1}(k_2'^2+\mu^2-i\epsilon)^{-1}[\bar{u}(q_2)(1/2)[\tau_\alpha,\tau_\beta]T^{(-)}(q_2,k_1';\,p_2,k_2')u(p_2)].\end{aligned} \tag{5.1c}$$

We first calculate (5.1b) which is the contribution of the pion-pion interaction in the $I=J=0$ state. Substituting (3.5b) into (5.1b), with (3.8) for $A^{(\pm)}$ and $B^{(\pm)}$, and transforming the integrals which arise into parameter integrals, we can write $\langle q_1,q_2|K_2^{(\pi0)}|p_1,p_2\rangle$, where $K_2^{(\pi0)}$ is the Hermitian part of $R_2^{(\pi0)}$, in the form of (2.5), and find that the resulting a_i and b_i are zero except a_1 and that a_1 is given by

$$\begin{aligned}a_1(s,t)=-(3/\pi)(2\pi)^{-4}f_0(t)\bigg\{\int_0^\infty dW^2\int_0^1 dy\int_0^1 dz\, z[-U^{(+)}(W^2,t)+(1-z)MV^{(+)}(W^2,t)]\\ \times[z\mu^2+(1-z)W^2-z(1-z)M^2-y(1-y)z^2t]^{-1}\bigg\}^2.\end{aligned} \tag{5.2}$$

We now proceed to the calculation of the matrix element given by (5.1c) which is the contribution of the pion-pion interaction in the $I=J=1$ state. If we substitute (3.5b) into (5.1c), with (3.8) for $A^{(\pm)}$ and $B^{(\pm)}$, we encounter divergent integrals. Therefore the result of the calculation will not be reliable. Such a calculation has been made by Fujii.[12] He first wrote the integrals in the dispersion form, and then cut them off at the threshold for the nucleon-antinucleon pair formation, as suggested by a unitarity argument. He has, however, noticed that his results are not completely satisfactory. We try to avoid the difficulty by introducing information from other experiments. Recently Bowcock *et al.*[13] have tried to explain the data for the low-energy pion-nucleon scattering and the electromagnetic form factors of the nucleon simultaneously by introducing terms representing the pion-pion resonance in the dispersion relations for the pion-nucleon scattering. They expressed these terms in terms of some phenomenological parameters. We try to express our matrix element in terms of the same parameters. For this purpose we consider the pion-nucleon amplitude given by (3.6). If G is replaced by (3.9a) with (3.12), and only the part with $I=1$ is retained, (3.6) becomes

$$\begin{aligned}T_{\beta\alpha}'(q,k_2;\,p,k_1)=T_{\beta\alpha}(q,k_2;\,p,k_1)-4i(2\pi)^{-4}3\pi\mu^{-2}f_1(t)\int d^4k_1'd^4k_2'(k_1+k_2)\cdot(k_1'+k_2')\\ \times\delta^4(p+k_1'-q-k_2')(k_1'^2+\mu^2-i\epsilon)^{-1}(k_2'^2+\mu^2-i\epsilon)^{-1}(1/2)[\tau_\beta,\tau_\alpha]T^{(-)}(q,k_2';\,p,k_1').\end{aligned} \tag{5.3}$$

Following Bowcock *et al.*, we make the narrow-resonance approximation to the pion-pion scattering amplitude. Then we have, for $t<4\mu^2$,

$$f_1(t)=\gamma t_r^{\frac{1}{2}}(2\mu)^{-1}(t_r-t)^{-1}, \tag{5.4}$$

where t_r is the square of the total energy at the resonance, and γ is the width of the resonance. Equation (5.4) has been derived by using the dispersion relation for $f_1(t)$ inferred from the expression (3.13b). If we use the ρ-meson theory of Itabashi *et al.*[14] to calculate $f_1(t)$, we obtain the same result.

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[12] Y. Fujii, Progr. Theoret. Phys. (Kyoto) **25**, 441 (1961).
[13] J. Bowcock, W. N. Cottingham, and D. Lurié, Nuovo cimento **16**, 918 (1960); **19**, 142 (1961).
[14] K. Itabashi, M. Kato, K. Nakagawa, and G. Takeda, Progr. Theoret. Phys. (Kyoto) **24**, 529 (1960).

Bowcock *et al.* wrote the pion-nucleon scattering amplitude in the following form

$$T_{\beta\alpha}'(q,k_2;p,k_1)=T_{\beta\alpha}(q,k_2;p,k_1)-(1/2)[\tau_\beta,\tau_\alpha]6\pi(t_r-t)^{-1}[C_2(p+q)\cdot(k_1+k_2)+i\gamma\cdot(k_1+k_2)(C_1+2MC_2)], \quad (5.5)$$

where C_1 and C_2 are adjustable parameters. Comparing (5.3) with (5.5), and taking account of (5.4), we have

$$\int d^4k_1d^4k_2\,\delta^4(p+k_1-q-k_2)(k_1+k_2)_\mu(k_1^2+\mu^2-i\epsilon)^{-1}(k_2^2+\mu^2-i\epsilon)^{-1}T^{(-)}(q,k_2;p,k_1)$$
$$=-i(2\pi)^4\mu^3\gamma^{-1}t_r^{-1/2}[C_2(p+q)_\mu+i\gamma_\mu(C_1+2MC_2)]. \quad (5.6)$$

Inserting (5.6) into (5.1c), we have

$$\langle q_1,q_2|R_2^{(\pi1)}|p_1,p_2\rangle$$
$$=\langle q_1,q_2|K_2^{(\pi1)}|p_1,p_2\rangle=-(2\pi)^4\delta^4(p_1+p_2-q_1-q_2)M^2(p_{10}p_{20}q_{10}q_{20})^{-1/2}3\pi\mu^3\gamma^{-1}t_r^{-1/2}(t_r-t)^{-1}$$
$$\times\bar{u}(q_1)[C_2(p_1+q_1)_\mu+i\gamma_\mu(C_1+2MC_2)]\tau_\alpha u(p_1)\bar{u}(q_2)[C_2(p_2+q_2)_\mu+i\gamma_\mu(C_1+2MC_2)]\tau_\alpha u(p_2). \quad (5.7)$$

We can readily rewrite this in the form of (2.5), and find that

$$\begin{aligned}
&a_i(s,t)=0,\quad i=1,\cdots,5,\\
&b_1(s,t)=g(t)(2M)^{-1}(2s+t-4M^2)C_2(C_1+4MC_2),\\
&b_2(s,t)=g(t)(C_1+2MC_2)(C_1+4MC_2),\\
&b_3(s,t)=-g(t)t(4M)^{-1}C_2(C_1+2MC_2),\\
&b_4(s,t)=0,\\
&b_5(s,t)=g(t)(2M)^{-1}(2s+t-4M^2)C_2(C_1+2MC_2),
\end{aligned} \quad (5.8a)$$

where

$$g(t)=3\pi\mu^3\gamma^{-1}t_r^{-1/2}(t_r-t)^{-1}. \quad (5.8b)$$

The values of the parameters C_1, C_2, t_r, and γ have been determined by Bowcock *et al.* as follows: From the comparison of the isovector electric form factor with experiments, one has

$$t_r=22.4\mu^2,\quad C_1\mu^3\gamma^{-1}t_r^{-1/2}=-0.6. \quad (5.9a)$$

From the fact that the isovector magnetic form factor has the same form as the electric form factor, one has

$$MC_2=1.85C_1. \quad (5.9b)$$

From the low-energy pion-nucleon scattering, one has

$$C_1=-1.0. \quad (5.9c)$$

The value of t_r in (5.9a) is considerably smaller than that which has been found from the analyses of inelastic pion-nucleon scattering.[15] In the present work, however, we are concerned with the region $t<0$, which is the unphysical region for pion-pion scattering, as in the case of form factors. Also we have adopted the approximation (5.4) which fit the data of form factors with the value (5.9a); we therefore take (5.9a).

VI. NUMERICAL RESULTS FOR THE 310-MeV p-p SCATTERING AND DISCUSSION

In this section we compute the phase shifts for the proton-proton scattering at 310-MeV laboratory kinetic energy by the method described in the preceding sections.

In the preceding sections we have expressed the total K matrix as a sum of the one-pion-exchange contribution K_1 and the two-pion-exchange contribution K_2. K_2 has been divided into three parts: $K_2^{(0)}$, $K_2^{(\pi0)}$, and $K_2^{(\pi1)}$, where $K_2^{(0)}$ was the part not containing explicit effects of the pion-pion interaction, and $K_2^{(\pi0)}$ and $K_2^{(\pi1)}$ were the parts coming from the pion-pion interactions in the $I=J=0$ state and $I=J=1$ state, respectively. Here we further divide $K_2^{(0)}$ into two terms $K_2^{(B)}$ and $K_2^{(R)}$, where $K_2^{(B)}$ is that part of $K_2^{(0)}$ which is obtained by the renormalized fourth-order perturbation theory, and $K_2^{(R)}$ is the correction to it due to the 3-3 resonance. Therefore $K_2^{(B)}$ is defined as that part of $K_2^{(0)}$ which is obtained from (4.1) by putting

$$U^{(\pm)}(W^2,t)=0,\quad V^{(\pm)}(W^2,t)=\pi g^2\delta(W^2-M^2),$$

and $K_2^{(R)}$ is defined by $K_2^{(R)}=K_2^{(0)}-K_2^{(B)}$. Thus we have

$$K=K_1+K_2^{(B)}+K_2^{(R)}+K_2^{(\pi1)}+K_2^{(\pi0)}. \quad (6.1)$$

We tabulate in Table I contributions of the terms in the right-hand side of (6.1) to the partial-wave amplitudes k_j, k_{lj}, and m_j which were defined in terms of nuclear Blatt-Biedenharn phase shifts by (2.6). The columns 1, 2B, 2R, π1, and π0 correspond, respectively, to K_1, $K_2^{(B)}$, $K_2^{(R)}$, $K_2^{(\pi1)}$, and $K_2^{(\pi0)}$. The figures in the columns 1, 2B, and 2R have been obtained by taking $g^2/4\pi=14.4$. In computing the columns 2B and 2R, we have cut off the integrals of the type (4.8) at $t=132\mu^2$, since a table of the functions Q_j was not available beyond this value. This cutoff affects only k_0, k_{10}, k_{11}, and k_{12}, and therefore does not affect the phase shifts for $l\geqq 2$. In calculating the contributions of $K_2^{(\pi0)}$, we have treated the function $f_0(t)$, the pion-pion scattering amplitude in the $I=J=0$ state, as a constant for the reason mentioned in Sec. 1. Moreover we evaluated the integral appearing in (5.2) in a very crude approximation, so that the results in the column π_0 should not be taken seriously. The figures in this column are probably too large, in particular, for higher partial waves. We should have cut off the integral in

[15] J. A. Anderson, Vo X. Bang, P. G. Burke, D. D. Carmony, and N. Schmitz, Phys. Rev. Letters **6**, 365 (1961). D. Stonehill, C. Baltay, H. Courant, W. Fickinger, E. C. Fowler, H. Kraybill, J. Sandweiss, J. Sandford, and H. Taft, Phys. Rev. Letters **6**, 624 (1961). A. R. Erwin, R. March, W. D. Walker, and E. West, Phys. Rev. Letters **6**, 628 (1961).

(5.2) after writing it in the dispersion form, in order to ensure the unitarity for the channel $n+\bar{n}\rightarrow 2\pi$ in the S state.[16]

We see from Table I, $2B$ that the fourth-order perturbation theory gives a strong attractive force in the singlet even and triplet odd states. Such an attractive force has been derived by the present author previously ln a calculation of a semistatic potential.[17] That calcuiation shows that the attractive force is a nonstatic effect and it disappears in the static limit. The column $2R$ shows that the effect of the 3-3 resonance strengthens the attraction for the states other than S and P states. The column $\pi 1$ shows that the pion-pion resonance in the $I=J=1$ state gives a strongly repulsive central force. It also gives an attractive LS force. This is seen from the fact that $k_{l,l+1}>k_{l,l}>k_{l,l-1}$. These features are also observed in the Fujii's results.[12] The pion-pion interaction in the $I=J=0$ state gives a simple central force as is seen from the column $\pi 0$. This force is attractive or repulsive according to whether f_0 is positive or negative.

In Table II we tabulate the phase shifts for $l\geq 2$ calculated from the values in Table I, together with the phenomenological phase shifts. The columns 1, $1+2B$, $1+2B+2R$, and $1+2B+2R+\pi 1$ correspond to K_1, $K_1+K_2^{(B)}$, $K_1+K_2^{(B)}+K_2^{(R)}$, and $K_1+K_2^{(B)}+K_2^{(R)}+K_2^{(\pi 1)}$, respectively. In the columns CMMS and MMS we list the results of the phase-shift analyses made by Cziffra *et al.*[18] and MacGregor *et al.*[19] respectively. In the CMMS analysis phase shifts for the partial waves up through H waves are taken as adjustable phase shifts, and all the partial-wave amplitudes beyond H waves are replaced by one-pion-exchange contributions. On the other hand, in the MMS analysis phase shifts up through F phase shifts are taken as adjustable ones, all the partial-wave amplitudes beyond F waves being replaced by one-pion-exchange contributions. In view of the fact that our results for G and H phase shifts appreciably differ from the one-pion-exchange contributions, and that CMMS results have been obtained by taking the same value of the pion-nucleon coupling constant as ours,

TABLE I. One-pion-exchange and two-pion-exchange contributions to the partial-wave amplitudes. k_j, k_{lj}, and m_j in the first column are the partial-wave amplitudes defined by (2.6) in the text. Column 1 is the one-pion-exchange contribution. Column $2B$ is that part of the two-pion-exchange contribution which does not contain effects of 3-3 resonance and pion-pion interaction. Columns $2R$, $\pi 1$, and $\pi 0$ are corrections to the two-pion-exchange contribution due to 3-3 resonance, pion-pion interaction in the $I=J=1$ state, and that in the $I=J=0$ state, respectively.

	1	2B	2R	π1	π0
k_0	−1.1993	19.183	−8.1504	−39.809	165.1 f_0
k_2	0.0428	0.5454	0.7057	−0.9875	89.9 f_0
k_4	0.0160	0.0410	0.0184	−0.0328	29.8 f_0
k_6	0.0065	0.0046	0.0014	−0.0012	10.2 f_0
k_{11}	−0.6374	2.7313	−1.9816	−6.9924	130.6 f_0
k_{33}	−0.0825	0.1345	0.2340	−0.1936	51.8 f_0
k_{55}	−0.0229	0.0130	0.0045	−0.0067	16.9 f_0
k_{10}	1.1993	2.2617	−4.1281	−9.0790	128.6 f_0
k_{12}	0.0973	2.6488	−1.0692	−5.4808	134.5 f_0
k_{32}	0.0506	0.0614	0.3107	−0.2262	49.8 f_0
m_2	−0.2287	0.0546	0.0517	0.0729	−0.05 f_0
k_{34}	0.0161	0.1401	0.0767	−0.1640	54.5 f_0
k_{54}	0.0096	0.0085	0.0037	−0.0078	16.2 f_0
m_4	−0.0574	0.0022	0.0015	0.0021	−0.01 f_0
k_{56}	0.0041	0.0137	0.0043	−0.0058	17.8 f_0
k_{76}	0.0026	0.0012	0.0004	−0.0003	6.6 f_0
m_6	−0.0198	0.00018	0.0001	0.00007	−0.003 f_0

TABLE II. Calculated phase shifts for $l\geq 2$ in comparison with the phenomenological phase shifts. The second to fifth columns are results of the present theory. 1, $2B$, $2R$, and $\pi 1$ in the captions of these columns correspond to those in Table I. CMMS, MMS, and H are sets of the phenomenological phase shifts given by Cziffra *et al.* (reference 18), MacGregor *et al.* (reference 19), and Hamada (reference 20), respectively. Entries are nuclear bar phase shifts in degrees.

					CMMS		MMS		
	1	1+2B	1+2B+2R	1+2B +2R+π1	Set 1	Set 2	Solution 1	Solution 2	H
$^1D_2$	2.45	30.47	52.30	17.04	12.1	4.4	11.87±0.49	4.78±0.54	13.06
$^1G_4$	0.92	3.26	4.31	2.44	1.2	1.1	0.77	0.85	1.20
$^1I_6$	0.37	0.64	0.72	0.65					0.11
$^3F_2$	2.82	6.25	22.83	11.12	1.3	0.1	1.21±0.70	−0.49±0.89	0.54
$^3F_3$	−4.72	2.98	15.96	5.28	−4.6	−0.5	−3.53±0.66	−0.03±0.39	−3.61
$^3F_4$	0.92	8.88	13.11	3.94	3.2	2.9	3.54±0.35	3.33±0.56	3.35
$^3H_4$	0.55	1.03	1.24	0.80	1.7	2.4	0.49	0.54	0.55
$^3H_5$	−1.31	−0.57	−0.31	−0.69	−0.4	−1.5	−1.12	−1.24	−1.09
$^3H_6$	0.24	1.02	1.27	0.94	1.2	1.4	0.21	0.23	0.11
ϵ_4	−1.64	−1.56	−1.50	−1.47	−1.4	−1.7	−1.40	−1.55	−1.52
ϵ_6	−0.57	−0.56	−0.56	−0.56					−0.57

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[18] P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **114**, 880 (1959).
[19] M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **116**, 1248 (1959).

we think that the CMMS phase shifts are more adequate to be compared with our results than the MMS phase shifts. However differences between the CMMS phase shifts and the MMS ones will give some measure for uncertainties in the phenomenological phase shifts. Cziffra *et al.* do not give the errors associated with their results. But the errors are expected to be a little bit larger in the CMMS phase shifts than in the MMS ones, because Cziffra *et al.* use a larger number of adjustable parameters than MacGregor *et al.* As another example of an acceptable set of the phenomenological phase shifts, we list in Table II, Column H the phase shifts which Hamada[20] has given in his analysis by means of a phenomenological potential. With these phase shifts he has obtained quite a good fit. At 310 MeV, the impact parameter for the D wave is $0.7\mu^{-1}$. Therefore we have calculated only the phase shifts for $l\geq 2$ according to what was mentioned in Sec. 1.

We see from Table II that, without the pion-pion interaction, the calculated phase shifts for the singlet states and ^3F states are extremely large compared with the experimental values, and that the inclusion of the pion-pion interaction in the $I=J=1$ state very much improves the results. For the singlet phase shifts, we cannot say that the remaining discrepancies between theory and experiments are definite, if we consider that the calculated values are the result of mutual cancellation of large quantities each of which has an uncertainty due to various approximations. The ^3H phase shifts are not very much affected by the pion-pion resonance. This is due to the largeness of the resonance energy t_r. Our final values $(1+2B+2R+\pi 1)$ of the ^3H phase shifts lie between CMMS, set 1 and MMS, solution 1, and also between CMMS, set 1 and H. We see, however, that definite discrepancies still remain for the ^3F phase shifts. These discrepancies cannot be removed by a simple central force such as that due to the S-state pion-pion interaction. This leads us to the conclusion that there must be some other effects which play an important role in the long-range nucleon-nucleon interaction. It is probable that the three-pion resonance discovered recently[21] is one of such effects.

[20] T. Hamada, Progr. Theoret. Phys. (Kyoto) **24**, 1033 (1960).

As for the S-state pion-pion interaction, we cannot even draw any conclusion concerning its sign. If the three-pion resonance gives a significant contribution to the nuclear force, it will probably give a repulsive central force as the P-state pion-pion resonance does, because the spin and parity of the three-pion resonant state are expected to be the same as those of the two-pion resonance.[22] It is possible that the S-state pion-pion interaction is required to be attractive (f_0 positive) in order to cancel the repulsive force due to the three-pion resonance.

From the above results we observe that, in the nucleon-nucleon interaction, there is no single effect which dominates over all the others. This is partly due to the fact that the nucleon-nucleon scattering amplitude cannot be expressed as a combination of a small number of partial-wave amplitudes, when it is continued to the nucleon-anti-nucleon scattering channel. We further observe that there is a large cancellation between the contributions of various effects. It seems to the present author that the recognition of these facts is important for further developments.

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