Nucleon Form Factors in the Helicity Representation*

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The cross section for electron–nucleon scattering should be expressed in terms of form factors which are proportional to helicity amplitudes. Then electron–nucleon scattering and pion–nucleon scattering are related even more closely by the π–π interaction in $I=1$, $J=1$. In particular the S-wave charge-exchange scattering data complement data on the electric isovector form factor. A consistent picture of the $I=1$, $J=1$ π–π interaction requires the narrowness of the 750-MeV resonance.

I. INTRODUCTION

THE analytic properties of scattering amplitudes[1] have already revealed several structural aspects of the strong interactions. Bowcock and collaborators,[2] working within this framework, were able to show that the pion–pion interaction (in that state characterized by unit values for both isospin and total angular momentum) can account for the isospin dependence of the S-wave pion–nucleon interaction. An important ingredient of their success was the lesson from Frazer and Fulco[3] that helicity amplitudes[4] should be used to represent the channel $\bar{\pi}+\pi\rightarrow\bar{N}+N$. More detailed analyses by Hamilton and co-workers[5] strengthen the footing beneath this important structural link.

Invariants without kinematic singularities[6] do not provide the most direct route from one experiment to another; helicity amplitudes do. Electron–nucleon scattering measures helicity amplitudes with scant error correlation.[7] Pion–nucleon scattering also measures helicity amplitudes directly. So, as the substitution law invokes a contribution from the channel $\bar{\pi}+\pi\rightarrow\bar{N}+N$, helicity amplitudes represent naturally what must be common to isovector electron–nucleon scattering and to isospin-dependent pion–nucleon scattering. Why use a poor representation and propagate errors through off-diagonal elements of the S matrix when studying the pion–pion interaction?

II. ELECTRON-NUCLEON SCATTERING

An attractive form for the Rosenbluth formula is that one in which form factors contribute incoherently to the cross section.[7] With the momentum transfer represented by

$$t=2q^2(1-\cos\phi), \tag{2.1}$$

and the form factors given by

$$G_M(-t)=F_1(-t)+2MKF_2(-t), \tag{2.2}$$

$$G_E(-t)=F_1(-t)+(t/4M^2)2MKF_2(-t), \tag{2.3}$$

the elastic cross section becomes

$$\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}\times\left\{\frac{G_E{}^2-(t/4M^2)G_M{}^2}{1-t/4M^2}-\frac{t}{4M^2}G_M{}^2 2\tan^2\left(\frac{\theta}{2}\right)\right\}. \tag{2.4}$$

The channel $\bar{\pi}+\pi\rightarrow\bar{N}+N$ contributes significantly to isovector electron–nucleon scattering[8] and to isospin-dependent pion–nucleon scattering.[6] If the variables are such that this channel is physical, then

$$t=(2E)^2=4(p^2+M^2)=4(k^2+\mu^2), \tag{2.5}$$

$$s=-p^2-k^2+2pk\cos\theta_3, \tag{2.6}$$

$$\bar{s}=2M^2+2\mu^2-s-t, \tag{2.7}$$

where p is the barycentric nucleon momentum and k is the barycentric pion momentum.

Frazer and Fulco calculated[3] the contribution to $\mathrm{Im}G_{EV}$ and $\mathrm{Im}G_{MV}$ from two-pion intermediate states by applying unitarity to the process $\gamma\rightarrow\bar{N}+N$. The result is couched in terms of the pion form factor F_π and helicity amplitudes for the process $\bar{\pi}+\pi\rightarrow\bar{N}+N$. Equations (2.2) and (2.3) of this reference and some algebraic endeavor reveal that

$$\mathrm{Im}G_{MV}=\tfrac{1}{2}eF_\pi{}^*(k^3/E)(f_-/\sqrt{2}), \tag{2.8}$$

$$\mathrm{Im}G_{EV}=\tfrac{1}{2}eF_\pi{}^*(k^3/E)(f_+/M), \tag{2.9}$$

where the helicity amplitudes, of course, correspond to $J=1$. The Rosenbluth cross section shows clearly that G_M determines the backward scattering. Under these conditions the nucleon helicity must reverse; the identification makes sense physically.

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1741, 1752 (1959).

[2] J. Bowcock, W. N. Cottingham, and D. Lurié, Nuovo cimento **16**, 918 (1960); **19**, 142 (1961).

[3] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603, 1609 (1960).

[4] M. Jacob and G. C. Wick, Ann. Phys. (New York) **7**, 404 (1959).

[5] J. Hamilton and T. D. Spearman, Ann. Phys. (New York) **12**, 172 (1961); J. Hamilton, P. Menotti, T. D. Spearman, and W. S. Woolcock, Nuovo cimento **20**, 519 (1961); J. Hamilton, T. D. Spearman, and W. S. Woolcock, Ann. Phys. (New York) **17**, 1 (1962).

[6] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[7] L. N. Hand, D. G. Miller, and R. Wilson, Phys. Rev. Letters **8**, 110 (1962).

[8] P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958).

To achieve the correct phase for the helicity amplitude, write[2]

$$f_\pm(t)=f_{\pi\pi}(t)\int_{-\infty}^{a}\frac{\mathrm{Im}f_\pm(t')dt'}{f_{\pi\pi}(t')(t'-t)},\tag{2.10}$$

where $f_{\pi\pi}$ is the $I=1$, $J=1$ pion–pion scattering amplitude. On the basis of recent data[9] on pion correlations following antiproton-proton annihilation, it becomes attractive to assume that the structure of $f_\pm$ is dominated by a narrow pion–pion resonance, i.e.,

$$f_{\pi\pi}\simeq\gamma/[(t_r-t)-i\gamma k^3],\tag{2.11}$$

and, consequently,

$$f_\pm\simeq N_\pm/[(t_r-t)-i\gamma k^3].\tag{2.12}$$

The pion form factor also has the same phase as the pion–pion scattering amplitude. It must satisfy

$$F_\pi(t)=1+\frac{t}{\pi}\int_{(2\mu)^2}^{\infty}\frac{\mathrm{Im}F_\pi(t')dt'}{t'(t'-t)},\tag{2.13}$$

which has the solution

$$F_\pi(t)=(t_r+\gamma)/(t_r-t-i\gamma k^3).\tag{2.14}$$

Now Eq. (2.9) can be used to obtain

$$\mathrm{Im}G_{EV}(t)=\frac{e}{2E_r}\frac{t_r+\gamma}{\gamma}\frac{N_+}{M}\pi\delta(t'-t).\tag{2.15}$$

This simplifies the integral for the form factor.

$$G_{EV}=\frac{e}{2}+\frac{t}{\pi}\int_{(2\mu)^2}^{\infty}\frac{\mathrm{Im}G_{EV}dt'}{t'(t'-t)}=\frac{e}{2}\left\{1-a+\frac{a}{1-t/t_r}\right\},\tag{2.16}$$

where

$$a=(2/\gamma t_r^{1/2})(N_+/M).\tag{2.17}$$

G_{MV} can be evaluated in a similar manner.

III. PION-NUCLEON SCATTERING

When the variables are such that the physical channel is pion–nucleon scattering, t again becomes the momentum transfer and s represents the square of the barycentric energy

$$t=-2q^2(1-\cos\theta),\tag{3.1}$$

$$s=W^2=[(M^2+q^2)^{1/2}+(\mu^2+q^2)^{1/2}]^2.\tag{3.2}$$

The barycentric differential cross section can be written[6]

$$\frac{d\sigma}{d\Omega}=\sum_{\text{spins}}|(f|f_1+(\sigma\cdot q_f)(\sigma\cdot q_i)f_2|i)|^2,\tag{3.3}$$

where the matrix element is taken between two-component spinors. Any pion–pion contribution to $\pi^-+p\to\pi^0+n$ must be $I=1$ and, by the exclusion principle, $J=1$ also. So the $I=1$, $J=1$ pion–pion interaction can be seen clearly in the $L=0$ charge-exchange amplitude.

$$f_S^{\pi\pi}(s)=\frac{1}{2}\int_{-1}^{1}d(\cos\theta)[f_1^{\pi\pi}+f_2^{\pi\pi}\cos\theta].\tag{3.4}$$

As the Cini–Fubini approximation[10] is invoked to calculate the amplitude $[f_1^{\pi\pi}+f_2^{\pi\pi}\cos\theta]$ from the discontinuities in the helicity amplitudes, the invariants without kinematic singularities[6] (A and B) will appear only as a pause in an unattractive representation. At this juncture, however, A and B provide convenient trial markers in the woods. These invariants were related to the conventional scattering amplitudes when first used.[6]

$$f_1^{\pi\pi}=\frac{(W+M)^2-\mu^2}{4W^2 4\pi}[(W-M)B^{\pi\pi}+A^{\pi\pi}],\tag{3.5}$$

$$f_2^{\pi\pi}=\frac{(W-M)^2-\mu^2}{4W^2 4\pi}[(W+M)B^{\pi\pi}+A^{\pi\pi}].\tag{3.6}$$

In their turn, Frazer and Fulco[3] related these invariants to the helicity amplitudes. Only the $J=1$ helicity amplitude will interest us.

The resonant form of Eq. (2.12) leads to a delta function in the spectral integral. Then the evaluation is easy. With the definition

$$V=W-M+(q^2\cos\theta/2M),\tag{3.7}$$

we obtain

$$A^{\pi\pi}=12\pi V\frac{1}{\pi}\int_{(2\mu)^2}^{\infty}\left[\frac{4M^2}{t'-4M^2}\right]\times\left[\frac{\mathrm{Im}f_-(t')}{\sqrt{2}}-\frac{\mathrm{Im}f_+(t')}{M}\right]\frac{dt'}{t'-t}$$
$$=\frac{12\pi}{t_r-t}V\frac{4M^2}{t_r-4M^2}\left[\frac{N_-}{\sqrt{2}}-\frac{N_+}{M}\right],\tag{3.8}$$

and

$$B^{\pi\pi}=12\pi\frac{1}{\sqrt{2}\pi}\int_{(2\mu)^2}^{\infty}\frac{\mathrm{Im}f_-(t')dt'}{t'-t}=\frac{12\pi}{t_r-t}\frac{N_-}{\sqrt{2}}.\tag{3.9}$$

In the approximation $q^2\ll 4M^2$, we have

$$f_2^{\pi\pi}\simeq\frac{3}{t_r-t}\frac{q^2}{4M^2}(W+M)\frac{N_-}{\sqrt{2}}\tag{3.10}$$

and the spin-flip pion–nucleon scattering amplitudes

$$\frac{(2L+1)}{2L(L+1)}\int_{-1}^{1}d(\cos\theta)\sin^2\theta\frac{dP_L(\cos\theta)}{d(\cos\theta)}f_2\tag{3.11}$$

[9] J. Button, G. R. Kalbfleisch, G. R. Lynch, B. C. Maglić, A. H. Rosenfeld and M. L. Stevenson, University of California Radiation Laboratory Report UCRL-9814 (unpublished).

[10] M. Cini and S. Fubini, Ann. Phys. (New York) 3, 352 (1960).

relate[2] directly to G_{MV} data. In this same approximation, the corresponding result for the non-spin-flip amplitude is

$$f_1^{\pi\pi}+f_2^{\pi\pi}\cos\theta\simeq\frac{3V}{t_r-t}\left(\frac{t_r(N_-/\sqrt{2})-4M^2(N_+/M)}{t_r-4M^2}\right). \quad (3.12)$$

So in the further approximation, $t_r \ll 4M^2$, the non-spin-flip scattering amplitudes,

$$\frac{2L+1}{2}\int_{-1}^{1} d(\cos\theta)P_L(\cos\theta)(f_1+f_2\cos\theta), \quad (3.13)$$

relate directly to G_{EV} data.

Now the angular integral in Eq. (3.4) can be evaluated:

$$f_S^{\pi\pi}\simeq\frac{3N_+}{M}[(\mu^2+q^2)^{1/2}+q^2/2M] \times\frac{\ln(1+4q^2/t_r)}{4q^2}\frac{4M^2}{4M^2-t_r}. \quad (3.14)$$

An isospin projection[6] yields the corresponding contribution to the s-wave amplitude for charge-exchange scattering.

$$f_{\pi\pi}^{I=1/2}-f_{\pi\pi}^{I=3/2}=9\frac{N_+}{M}[(\mu^2+q^2)^{1/2}+q^2/2M] \times\frac{\ln(1+4q^2/t_r)}{4q^2}\frac{4M^2}{4M^2-t_r}. \quad (3.15)$$

IV. THE π–π INTERACTION IN $I=1$, $J=1$

From the foregoing analysis, it is clear that the $I=1$, $J=1$ π–π interaction receives complementary illumination from data on the isovector electric form factor, from data on s-wave charge-exchange scattering, and from data on the two-pion correlations following $\bar{p}+p\to n\pi$. Neither electron-nucleon scattering nor pion-nucleon scattering can at present locate t_r precisely. From the narrow Berkeley two-pion resonance at (750±20) MeV,[9] the parameters $t_r=29.2\pm0.8$ and $\gamma=0.048\pm0.009$ are obtained.

The form factor G_{EV} is poorly known, for inelastic electron–deuteron scattering[11] studies G_{En}^2 indirectly and it is small. On the basis of these data alone, even the isospin assignments for the electric form factors are uncertain.[7]

Fortunately, the neutron-electron interaction[12] fixes the difference between the limiting slopes.

$$[dG_{ES}/d(-t)](0)-[dG_{EV}/d(-t)](0)=0.021\ \mathrm{F}^2. \quad (4.1)$$

[11] F. Bumiller, M. Croissiaux, E. Dally, and R. Hofstadter, Phys. Rev. **124**, 1623 (1961); Stanford University High-Energy Physics Laboratory Report HEPL-248 (unpublished); R. M. Littauer, H. F. Schopper and R. R. Wilson, Phys. Rev. Letters **7**, 144 (1961).

[12] L. L. Foldy, Revs. Modern Phys. **30**, 471 (1958).

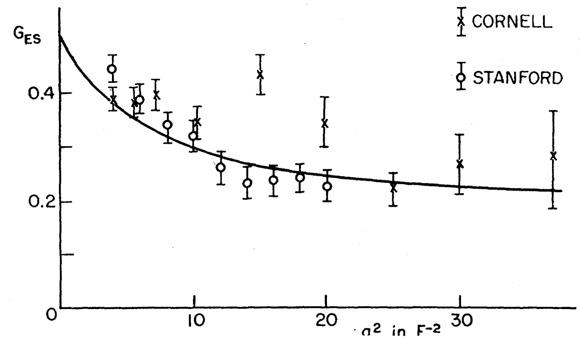

FIG. 1. The $I=0$ electric form factor.

Precise measurements of G_{Ep} are available from Orsay[13] at $(-t)=1.05$ and 3.00. A quadratic fit to these measurements determines

$$[dG_{Ep}/d(-t)](0)=-(0.116\pm0.012)\ \mathrm{F}^2.$$

On this basis

$$[dG_{ES}/d(-t)](0)=-(0.047\pm0.006)\ \mathrm{F}^2,$$

and

$$[dG_{EV}/d(-t)](0)=-(0.068\pm0.006)\ \mathrm{F}^2.$$

The latter corresponds to $a=0.20\pm0.02$ in Eq. (2.16).

These limiting slopes accommodate the data on inelastic electron-deuteron scattering because the errors are large below $(-t)=4\ \mathrm{F}^{-2}$. Figure 1 shows G_{ES} fit by an ω and an η resonant contribution,

$$G_{ES}=0.50\left\{0.30+\frac{0.80}{1-t/15.6}+\frac{(-0.10)}{1-t/31.6}\right\}. \quad (4.2)$$

Elastic electron–deuteron scattering[14] supports this interpolation of G_{ES}. On the other hand, electroproduction of positive pions from protons at $(-t)=3\ \mathrm{F}^{-2}$ is difficult to reconcile with any simple interpolation of G_{EV}.[15] The electroproduction data suggests that $G_{EV}>G_{ES}$ at $(-t)=3\ \mathrm{F}^{-2}$.

Perhaps s-wave change-exchange scattering can extend our knowledge of this helicity amplitude away

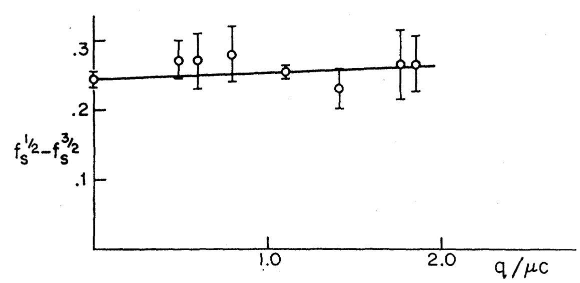

FIG. 2. The S-wave charge-exchange amplitude.

[13] P. Lehmann, R. Taylor, and R. Wilson, Phys. Rev. **126**, 1183 (1962).

[14] N. K. Glendenning and G. Kramer, Phys. Rev. Letters **7**, 471 (1961).

[15] L. N. Hand (private communication).

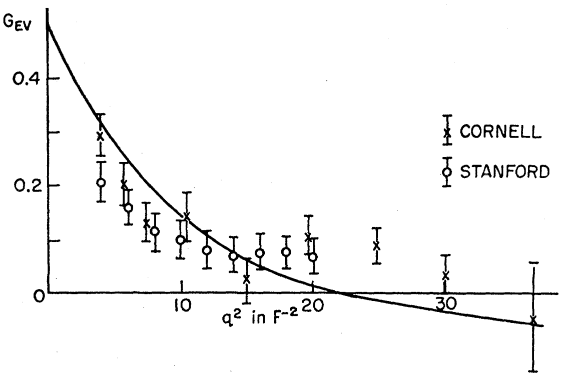

FIG. 3. The $I=1$ electric form factor.

from the static limit in a reliable manner. High-quality measurements of $f_S^{1/2}-f_S^{3/2}$ are scarce. However, Rochester[16] and Liverpool[17] have made significant recent contributions. Table I shows the precision of the data used.

Only one nearby discontinuity in the unphysical region makes a significant contribution to the S-wave scattering amplitude. Hamilton and co-workers[5] show that the angular integration makes all long-range

TABLE I. S-wave charge-exchange amplitudes.

$q/\mu c$	$(\sin 2\alpha_1-\sin 2\alpha_3)/2q$
0	0.245±0.01
0.483	0.272±0.025
0.597	0.271±0.04
0.696	0.280±0.04
1.107	0.253±0.01
1.42	0.229±0.03
1.77	0.266±0.05
1.86	0.264±0.04

[16] K. Miyake, K. F. Kinsey, and D. E. Knapp, University of Rochester Report NYO-9545 (unpublished).
[17] S. Frank (private communication).

contributions negligible except the circle contribution from the process $\bar{\pi}+\pi\rightarrow\bar{N}+N$. Any uncalculated contribution must correspond to a singularity at least 60 μ^2 distant along the real s axis, which could only cause a short-range contribution, $3D/(s-0)$, having little structure. The circle contribution is nearly real.[5] In fact our approximation has just the energy dependence expected from a circle contribution.

Figure 2 shows a fit to the form:

$$[f_S^{1/2}-f_S^{3/2}]_{\exp}=f_{\pi\pi}^{1/2}-f_{\pi\pi}^{3/2}+3D/(s-0), \quad (4.3)$$

with $t_r=29.2\mu^2$, $N_+/M=0.21\pm0.03$, and $D=3.3\pm0.4$.

On the basis of this fit,

$$a=(2/\gamma\sqrt{t_r})(N_+/M)=1.6\pm0.4. \quad (4.4)$$

This compares with the static value of $a=2.0\pm0.2$ obtained from electron–proton scattering and the neutron-electron interaction. If $a=1.8$ is chosen, the electric isovector form factor becomes

$$G_{EV}=0.50\{-0.80+1.80/(1-t/29.2)\}. \quad (4.5)$$

It is compared with the data in Fig. 3. Evidently a simple interpolation for G_{EV} finds support from the role played by the $I=1$, $J=1$ pion–pion interaction in S-wave charge-exchange scattering.

V. CONCLUSIONS

Several recent developments contribute to a more precise picture of the $I=1$, $J=1$ π–π interaction. Most important, its location and resonant width have been measured. In addition, helicity amplitudes measured in electron–nucleon scattering prove useful for directly relating pion–pion contributions to various amplitudes. As an illustration, recent data on charge exchange scattering and electron–nucleon scattering are applied to the $I=1$, $J=1$ π-π interaction. The results are quite different from original insights.[3]