Effect of Pion-Pion Interaction on Nuclear Forces*

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The effect of pion-pion interaction on the two-nucleon potential is investigated by using the pion-pion coupling, $H=(4\pi f/c\hbar):U_iU_iU_jU_j:$. It is found that if the coupling constant f is negative, the resulting spin-orbit nuclear force in the nonrelativistic approximation is similar to that produced by the exchange of a neutral meson of mass between 3.3 and 4.1 times the pion mass. A comparison of the theoretical results for the angular dependence of the Wolfenstein coefficient $\bar{C}'$ with those obtained by the phase-shift analysis of the proton-proton scattering at 310 MeV shows reasonable agreement, and gives $f=-0.262$.

1. INTRODUCTION

ALTHOUGH the effect of pion-pion interaction seems to have been observed in several collision processes, the theory of such an interaction is still in a rudimentary stage. The simplest type of pion-pion interaction is given by the interaction energy density

$$H_f=(4\pi f/c\hbar):U_iU_iU_jU_j:, \tag{1}$$

where f is a dimensionless coupling constant. This interaction term was first introduced to carry out the renormlaization of the pseudoscalar pion theory with the pseudoscalar coupling,[1] and it has subsequently been used by several authors.[2] We shall investigate its lowest-order contribution to the nucleon-nucleon scattering to obtain some idea of the effect of pion-pion interaction on nuclear forces, specially with regard to the spin-orbit coupling of nucleons. For this purpose we shall use the usual g coupling, given by

$$H_g=ig:\bar{\psi}\gamma_5\tau_i\psi U_i:, \tag{2}$$

for the creation or annihilation of pions by nucleons, and the f coupling, given by (1), for the interaction of pions with pions.

2. CONTRIBUTION OF PION-PION INTERACTION TO NUCLEON-NUCLEON SCATTERING

Let us consider the scattering of two nucleons, whose propagation four-vectors are p and q in the initial state and p' and q' in the final state. The interaction diagrams of the lowest order in f for this process are shown in Fig. 1. The diagrams b, c, and d represent corrections to the one-pion exchange scattering, and it can be seen from the structure of the matrix elements that, like the one-pion exchange contribution, they do not contain any spin-orbit interaction. We, therefore, concentrate our attention on the diagram a, which represents a correction to the two-pion exchange scattering. The scattering matrix element corresponding to this diagram is

$$S_a=(2ig^4f/\pi^3c^2\hbar^2)\delta(p+q-p'-q')(\delta_{lk}\delta_{mn}+\delta_{km}\delta_{ln}+\delta_{kn}\delta_{lm})$$
$$\times\int dk'\frac{\bar{\psi}^-(\mathbf{p}')\gamma_5\tau_k[i(k'+p)\gamma-\kappa]\gamma_5\tau_l\psi^+(\mathbf{p})}{[(k'+p-p')^2+\lambda^2][(k'+p)^2+\kappa^2][k'^2+\lambda^2]}\times\int dk''\frac{\bar{\psi}^-(\mathbf{q}')\gamma_5\tau_m[i(k''+q)\gamma-\kappa]\gamma_5\tau_n\psi^+(\mathbf{q})}{[(k''+q-q')^2+\lambda^2][(k''+q)^2+\kappa^2][k''^2+\lambda^2]}, \tag{3}$$

where the momentum and energy of a nucleon with the propagation four-vector p are $\hbar\mathbf{p}$ and $c\hbar p_0$, κ and λ are related to the nucleon and pion masses M and μ as $\kappa=Mc/\hbar$ and $\lambda=\mu c/\hbar$, and for the meaning of other symbols we refer to the earlier papers.[3]

Using the usual properties of the propagation four-vectors of real nucleons, and the relation

$$(\delta_{lk}\delta_{mn}+\delta_{km}\delta_{ln}+\delta_{kn}\delta_{lm})\tau_k^{(1)}\tau_l^{(1)}\tau_m^{(2)}\tau_n^{(2)}=15, \tag{4}$$

we simplify the matrix element, and carry out integrations over k' and k''. Thus, we obtain in the center-of-mass system

$$S_a=-480i\pi^3(f\kappa^2/\lambda^4)(g^2/4\pi c\hbar)^2\delta(p+q-p'-q')$$
$$\times[\bar{\psi}^-(\mathbf{q}')\psi^+(\mathbf{q})][\bar{\psi}^-(\mathbf{p}')\psi^+(\mathbf{p})][I(|\mathbf{k}|)]^2, \tag{5}$$

with

$$I(|\mathbf{k}|)=\int_0^1 du\int_0^u dv\frac{\lambda^2(1-u)}{\kappa^2(1-u)^2+\lambda^2u+\mathbf{k}^2v(u-v)}$$
$$=\frac{\lambda^2}{|\mathbf{k}|}\int_0^1 du\frac{1-u}{z}\ln\frac{2z+|\mathbf{k}|u}{2z-|\mathbf{k}|u}, \tag{6}$$

where

$$\mathbf{k}=\mathbf{p}'-\mathbf{p}, \tag{7}$$

and

$$z=[\kappa^2(1-u)^2+\lambda^2u+(\mathbf{k}^2u^2/4)]^{1/2}. \tag{8}$$

3. SPIN-ORBIT INTERACTION IN THE POTENTIAL FORM

Since the main purpose of this investigation is to see if the spin-orbit coupling of nucleons could possibly be

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[1] A summary of the earliest work on the pion-pion coupling is given by P. T. Matthews and A. Salam, Revs. Modern Phys. **23**, 311 (1951). For the necessity of using the ordered product of the pion field operators U_i, see S. N. Gupta, Phys. Rev. **107**, 1722 (1957).
[2] S. Okubo, Phys. Rev. **118**, 357 (1960); M. Baker and F. Zachariasen, *ibid.* **118**, 1659 (1960); G. F. Chew and S. Mandelstam, *ibid.* **119**, 467 (1960); G. F. Chew, S. Mandelstam, and H. P. Noyes, *ibid.* **119**, 478 (1960).
[3] S. N. Gupta, Phys. Rev. **117**, 1146 (1960); **122**, 1923 (1961).

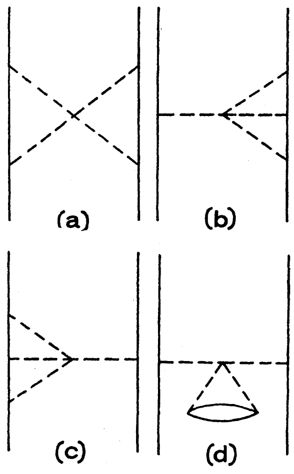

FIG. 1. Diagrams for nucleon-nucleon scattering matrix elements of the lowest order in the pion-pion coupling constant.

accounted for by the pion-pion interaction, we shall confine ourselves to a discussion of the spin-orbit interaction term in (5).

We shall first express the spin-orbit interaction in the form of a potential in the nonrelativistic approximation. Transforming the Dirac spinors appearing in (5) into the Pauli spinors, and treating $\mathbf{p}^2/\kappa^2 \ll 1$, we find that the spin-orbit term in (5) is given by

$$\begin{aligned}S_a = 120 i\pi^3 (f/\lambda^4)(g^2/4\pi c\hbar)^2 \delta(p+q-p'-q') \\ \times \psi_L^{*-}(\mathbf{q}')\psi_L^{*-}(\mathbf{p}')[i(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot\mathbf{k}\times\mathbf{p}] \\ \times[I(|\mathbf{k}|)]^2 \psi_L^{+}(\mathbf{p})\psi_L^{+}(\mathbf{q}),\end{aligned} \tag{9}$$

which corresponds to the nuclear potential

$$V(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}} V(\mathbf{k}), \tag{10}$$

with

$$\begin{aligned}V(\mathbf{k}) = -\left(\frac{g^2}{4\pi c\hbar}\right)^2\left(\frac{15 f c\hbar}{2\pi\lambda^4}\right)[I(|\mathbf{k}|)]^2 \\ \times[i(\mathbf{k}\times\mathbf{p})\cdot(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})].\end{aligned} \tag{11}$$

Now, in the nonrelativistic approximation a scalar[4] or vector[5] neutral meson of mass $n\mu$ gives rise to a spin-orbit potential of the form

$$V'(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}} V'(\mathbf{k}), \tag{12}$$

with

$$V'(\mathbf{k}) = \frac{2\pi V_0}{\lambda^3}\frac{1}{\mathbf{k}^2+n^2\lambda^2}[i(\mathbf{k}\times\mathbf{p})\cdot(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})], \tag{13}$$

where V_0 is a positive constant.[6] The above potential can also be expressed in terms of $x = \lambda|\mathbf{r}|$ as

$$V'(\mathbf{r}) = (V_0/x)(d/dx)(e^{-nx}/x)(\mathbf{L}\cdot\mathbf{S})/\hbar^2. \tag{14}$$

It would be interesting to compare the behavior of (11) and (13). We first observe that (11) will have the same sign as (13) only if f is negative. Further, in order to compare the $\mathbf{k}$ dependence of (11) and (13), it is sufficient to compare $[I(|\mathbf{k}|)/I(0)]^2$ and $n^2\lambda^2/(\mathbf{k}^2+n^2\lambda^2)$, which are obtained by dividing $[I(|\mathbf{k}|)]^2$ and $(\mathbf{k}^2+n^2\lambda^2)^{-1}$ by their values for $\mathbf{k}=0$.

In the center-of-mass system $0<|\mathbf{k}|<2|\mathbf{p}|$, and thus when the kinetic energy of the incident nucleon in the laboratory system is about 310 MeV, $|\mathbf{k}|$ varies from 0 to about 5.5λ. We have numerically[7] evaluated the quantity $[I(|\mathbf{k}|)/I(0)]^2$ for various values of $|\mathbf{k}|$ in the above range by taking[8] $M/\mu=6.8$ and $\mu c^2=138$ MeV, and the results are given in Table I. We have also evaluated $n^2\lambda^2/(\mathbf{k}^2+n^2\lambda^2)$ for various values of $|\mathbf{k}|$ by taking several different values of n, and the results for $n=3.3$ and 4.1 are also given in Table I. Since the values in column 2 lie between those in columns 3 and 4, it follows that the spin-orbit contribution due to the pion-pion interaction is similar to that produced by a neutral meson of mass between $3.3\,\mu$ and $4.1\,\mu$.

4. SPIN-ORBIT INTERACTION IN THE *M*-MATRIX FORM

We shall now discuss the spin-orbit contribution due to the pion-pion interaction more precisely by expressing it in the M-matrix[9] form. Following the same procedure as in Sec. 2, but including also the term obtained by interchanging the roles of the two nucleons in the final state, we find that the M matrix for the diagram a of

TABLE I. Comparison of spin-orbit nuclear potential due to the pion-pion interaction with that due to the exchange of a neutral meson of mass $n\mu$. The values of $[I(|\mathbf{k}|)]^2$ can be obtained from this table by using the result $[I(0)]^2=6.7825\times10^{-4}$.

$\dfrac{\lvert\mathbf{k}\rvert}{\lambda}$	$\left[\dfrac{I(\lvert\mathbf{k}\rvert)}{I(0)}\right]^2$	$\left[\dfrac{n^2\lambda^2}{\mathbf{k}^2+n^2\lambda^2}\right]_{n=3.3}$	$\left[\dfrac{n^2\lambda^2}{\mathbf{k}^2+n^2\lambda^2}\right]_{n=4.1}$
0.0	1.0000	1.0000	1.0000
0.5	0.9784	0.9776	0.9853
1.0	0.9208	0.9159	0.9439
1.5	0.8430	0.8288	0.8820
2.0	0.7592	0.7314	0.8078
2.5	0.6785	0.6354	0.7290
3.0	0.6047	0.5475	0.6513
3.5	0.5393	0.4706	0.5785
4.0	0.4821	0.4050	0.5123
4.5	0.4322	0.3497	0.4536
5.0	0.3890	0.3034	0.4021
5.5	0.3513	0.2647	0.3572

[4] S. N. Gupta, Phys. Rev. Letters **2**, 124 (1959).

[5] G. Breit, Proc. Nat. Acad. Sci. U. S. **46**, 746 (1960); Phys. Rev. **120**, 287 (1960).

[6] Although both the scalar and vector neutral mesons give rise to an attractive spin-orbit potential, the accompanying static force is attractive in the scalar case and repulsive in the vector case. The advantage of the repulsive static force has been discussed by Breit in reference 5.

[7] The numerical integration was carried out on an IBM 650 digital computer at the University's Computing Center.

[8] The value $M/\mu=6.8$ is obtained by taking the average of the proton and neutron masses for M and the average of π^+, π^-, and π^0 masses for μ.

[9] L. Wolfenstein, Phys. Rev. **96**, 1654 (1954).

TABLE II. Comparison of the theoretical values of the coefficient $\bar{C}'$ in units of 10^{-13} cm with those obtained by the phase-shift analysis at 310 MeV.

θ	$\bar{C}'$ (phase-shift analysis)	$\bar{C}'$ (theoretical)
1°	0.602	0.602
2°	0.602	0.602
5°	0.600	0.600
15°	0.584	0.585
30°	0.535	0.546
60°	0.400	0.473

Fig. 1 is

$$M_{r's',rs}=\left(-\frac{p_0}{4\pi c\hbar}\right)\left(\frac{30f\kappa^2c\hbar}{\pi\lambda^4}\right)\left(\frac{g^2}{4\pi c\hbar}\right)^2 \times\{[\bar{u}_{r'}(\mathbf{p}')u_r(\mathbf{p})][\bar{u}_{s'}(-\mathbf{p}')u_s(-\mathbf{p})][I(|\mathbf{k}|)]^2 -[\bar{u}_{s'}(-\mathbf{p}')u_r(\mathbf{p})][\bar{u}_{r'}(\mathbf{p}')u_s(-\mathbf{p})][I(|\mathbf{s}|)]^2\}, \quad (15)$$

where the spin indices r, s, r', and s' can take the values 1 and 2, $u_r(\mathbf{p})$ is the Dirac spinor normalized as

$$u_r^*(\mathbf{p})u_s(\mathbf{p})=\delta_{rs}, \quad (16)$$

and

$$\mathbf{s}=\mathbf{p}'+\mathbf{p}. \quad (17)$$

Transforming the u_r into the normalized Pauli spinors v_r, we can express the spin-orbit term in (15) as

$$M_{r's',rs}=v_{r'}^*(\mathbf{p}')v_{s'}^*(-\mathbf{p}')Mv_s(-\mathbf{p})v_r(\mathbf{p}) \quad (18)$$

with

$$M=C(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot\mathbf{n}, \quad (19)$$

where the indices r and r' refer to $\boldsymbol{\sigma}^{(1)}$, s and s' refer to $\boldsymbol{\sigma}^{(2)}$, $\mathbf{n}$ is a unit vector along $\mathbf{p}'\times\mathbf{p}$, and

$$C=\frac{15if\kappa^2}{8\pi^2\lambda^4}\left(\frac{g^2}{4\pi c\hbar}\right)^2\frac{|\mathbf{p}'\times\mathbf{p}|}{p_0}\left\{[I(|\mathbf{k}|)]^2\left[1+\frac{\frac{1}{2}\mathbf{k}^2-\mathbf{p}^2}{(\kappa+p_0)^2}\right] \pm[I(|\mathbf{s}|)]^2\left[1+\frac{\frac{1}{2}\mathbf{s}^2-\mathbf{p}^2}{(\kappa+p_0)^2}\right]\right\}, \quad (20)$$

the plus and minus signs corresponding to the isospin triplet and singlet states, respectively.

Hence, for proton-proton scattering the coefficient

$$\bar{C}'=iC\mathbf{p}^2/|\mathbf{p}'\times\mathbf{p}|, \quad (21)$$

is given by

$$\bar{C}'=-\frac{15f\kappa^2}{8\pi^2\lambda^4}\left(\frac{g^2}{4\pi c\hbar}\right)^2\frac{\mathbf{p}^2}{p_0}\left\{[I(|\mathbf{k}|)]^2\left[1+\frac{\frac{1}{2}\mathbf{k}^2-\mathbf{p}^2}{(\kappa+p_0)^2}\right] +[I(|\mathbf{s}|)]^2\left[1+\frac{\frac{1}{2}\mathbf{s}^2-\mathbf{p}^2}{(\kappa+p_0)^2}\right]\right\}, \quad (22)$$

where $\mathbf{k}^2$ and $\mathbf{s}^2$ can be expressed in terms of the scattering angle θ as

$$\mathbf{k}^2=2\mathbf{p}^2(1-\cos\theta), \quad \mathbf{s}^2=2\mathbf{p}^2(1+\cos\theta). \quad (23)$$

We have evaluated $\bar{C}'$ for various values of θ, and compared our theoretical values with those obtained by MacGregor, Moravcsik, and Stapp[10] by the phase-shift analysis of the proton-proton scattering at 310 MeV, which corresponds to $\mathbf{p}^2/\lambda^2=7.63$. The two sets of values for $\bar{C}'$ are given in Table II, where we have taken $(g^2/4\pi c\hbar)=14$, and chosen f in such a way that the two values of $\bar{C}'$ agree at $\theta=0$, which gives us

$$f=-0.262. \quad (24)$$

Considering the simplicity of our treatment of the pion-pion interaction, the agreement between the theoretical results and those obtained by the phase-shift analysis is reasonable. The sign and order of magnitude of f are also in reasonable agreement with other estimates[11] of this coupling constant.

[10] M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **116**, 1248 (1959). See solution 1 in Table V.

[11] B. R. Desai, Phys. Rev. Letters **6**, 497 (1961); B. H. Bransden and J. W. Moffat, *ibid.* **6**, 708 (1961).