Fundamental Properties of Perturbation-Theoretical Integral Representations

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The integral representations proposed previously are extended to the case of general transition amplitudes. The analyticity domain of any term in these representations is found explicitly. It is proved that any function analytic in this domain has an integral representation of the above type and then the weight function in this representation is uniquely determined. It is also proved in the case of two-particle scattering that the amplitude uniquely determines the three weight functions in its integral representation if they satisfy certain conditions.

I. INTRODUCTION

MUCH recent work on strong interactions has been done on the basis of the Mandelstam representation.[1] However, the latter has two major difficulties, namely, its validity is not assured even in the equal-mass case and it cannot be extended to production amplitudes. In previous works of the present author[2-4] new integral representations free from these difficulties were proposed on the basis of perturbation theory. For instance, the two-particle scattering amplitude can be represented as

$$\int d\alpha \int_0^1 dz \frac{\rho_{12}(\alpha,z)}{\alpha - zs-(1-z)t-i\epsilon} + \int d\beta \int_0^1 dz \frac{\rho_{23}(\beta,z)}{\beta - zt-(1-z)u-i\epsilon} + \int d\gamma \int_0^1 dz \frac{\rho_{31}(\gamma,z)}{\gamma - zu-(1-z)s-i\epsilon}, \quad (1.1)$$

in unsubtracted form. This representation seems to be useful as a substitute for the Mandelstam representation.

Now, in order that the weight functions in (1.1) shall represent physically significant quantities, it will be required that they are *uniquely* determined by the amplitude. For example, when one combines (1.1) with unitarity, such a uniqueness property is necessary to obtain integral equations for the weight functions. The main purpose of the present work is to prove the uniqueness of the weight functions.

In the next section we shall give integral representations for general transition amplitudes. In Sec. III, the analyticity domain of any term in our integral representations will be found explicitly. Section IV will be devoted to showing that any function analytic in the domain given in Sec. III has an integral representation of the above type, provided that a boundedness condition is satisfied. In Sec. V it will be proved that the weight function in this representation is uniquely determined. In the final section the uniqueness theorem will be proved for the two-particle scattering amplitude (1.1) under some conditions. The uniqueness of the weight functions for general amplitudes will be shown to be very plausible but not proven as yet.

II. GENERAL FORM OF THE INTEGRAL REPRESENTATIONS

Consider a graph having l external lines $A_1, A_2, \cdots, A_l$. We denote the external momentum corresponding to A_j by k_j. The conservation law is

$$\sum_{I=1}^{l} k_I = 0. \quad (2.1)$$

Now, in a previous paper[4] we have shown that at least one external line must be dealt with asymmetrically for the production amplitude. So we deal with k_l asymmetrically from the beginning. Namely, using (2.1) we eliminate k_l and define $2^{l-1}-1$ squares as follows:

$$\begin{aligned} s_I &= k_I^2, \quad (I=1, 2, \cdots, l-1) \\ s_{IJ} &= (k_I+k_J)^2, \quad (I<J<l) \\ s_{IJK} &= (k_I+k_J+k_K)^2, \quad (I<J<K<l) \\ &\cdots \\ s_{12\cdots(l-1)} &= (\sum_{I=1}^{l-1} k_I)^2 = s_l. \end{aligned} \quad (2.2)$$

In (2.2), l squares s_I $(I=1, 2, \cdots, l)$ are fixed on mass shells.

The S-matrix element corresponding to our graph is written as a Feynman parametric integral, whose denominator function is given by[2]

$$V = \sum_{i=1}^{N} x_i m_i^2 - \sum_h \zeta_h s_h, \quad (2.3)$$

with

$$\zeta_h \geqq 0, \quad (2.4)$$

where s_h stands for anyone of the squares defined in (2.2).

Theorem I. The denominator function V can always

[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1741, 1752 (1959).
[2] N. Nakanishi, Prog. Theor. Phys. **26**, 337 (1961).
[3] N. Nakanishi, Prog. Theor. Phys. **26**, 927 (1961).
[4] N. Nakanishi, J. Math. Phys. (to be published).

be rewritten as

$$V=\sum_{i=1}^{N} x_i m_i^2-\sum_{I=1}^{l} \zeta_I' s_I-\sum_{I<J<l} \zeta_{IJ}' s_{IJ}, \tag{2.5}$$

with

$$\zeta_{IJ}'\geqq 0. \tag{2.6}$$

Proof. Since

$$\begin{aligned} s_{IJ\cdots K}&=(k_I+k_J+\cdots+k_K)^2 \\ &=k_I^2+k_J^2+\cdots+k_K^2+2(k_Ik_J+\cdots) \end{aligned} \tag{2.7}$$

and

$$2k_Ik_J=s_{IJ}-s_I-s_J, \tag{2.8}$$

any square $s_{IJ\cdots K}$ involves s_{IJ} with the positive sign. Hence, (2.6) follows from (2.4). Q.E.D.

Only one linear identity,

$$\sum_{I<J<l} s_{IJ}=(l-3)\sum_{I=1}^{l-1} s_I+s_l, \tag{2.9}$$

holds between s_{IJ} $(I<J<l)$ because of the last equation of (2.2). Since any momentum has four dimensions, any five momenta k_I $(I=1, 2, 3, 4, 5)$ cannot be linearly independent. Their Gram determinant must vanish. When $l\geqq 6$, $(l-4)(l-5)/2$ nonlinear identities hold between s_{IJ} $(I<J<l)$ on account of the above reason. If one eliminates redundant squares by using these nonlinear identities, the linearity in the denominator function will be lost. So in the following we shall always neglect them.

Now, when all particles are scalar, the Feynman integral is

$$\text{const}\int_0^1 dx_1\cdots\int_0^1 dx_N\,\delta(1-\sum_{i=1}^{N} x_i)\frac{1}{U^2(V-i\epsilon)^k}, \tag{2.10}$$

where U is a non-negative function of x_i and V is given by (2.5). Using an identity,

$$\begin{aligned} \sum_{i=1}^{n+1} \theta(y_1-y_i)\cdots\theta(y_{i-1}-y_i) \\ \times\theta(y_{i+1}-y_i)\cdots\theta(y_{n+1}-y_i)=1, \end{aligned} \tag{2.11}$$

for $y_i=\zeta_{IJ}'$, we can rewrite (2.10) as follows[2]:

$$\int_{-\infty}^{\infty} d\alpha\int_0^1 dz_1\cdots\int_0^1 dz_{n+1}\left[\sum_{i=1}^{n+1}\delta(z_i)\right]\delta(1-\sum_{i=1}^{n+1} z_i) \times\frac{\rho(\alpha,z_1,\cdots,z_{n+1})}{\alpha-\sum_{i=1}^{n+1} z_it_i-i\epsilon}, \tag{2.12}$$

where t_i stands for s_{IJ} $(I<J<l)$ and $n+1=(l-1)(l-2)/2$. When $l=4$ or $l=5$, our integral representation (2.12) coincides with the previous ones.[2,4]

III. ANALYTICITY DOMAIN

In this section we investigate the analyticity domain of a function [cf. a term of (2.12)]

$$f(t_1,\cdots,t_n)=\int_{-\infty}^{\infty} d\alpha\int_0^1 dz_1\cdots\int_0^1 dz_n\,\delta(1-\sum_{i=1}^{n} z_i) \times\frac{\rho(\alpha,z_1,\cdots,z_n)}{\alpha-\sum_{i=1}^{n} z_it_i}. \tag{3.1}$$

We assume that $\rho(\alpha,z_1,\cdots,z_n)$ vanishes unless

$$\alpha\geq\sum_{i=1}^{n} z_ia_i. \tag{3.2}$$

We denote by $D^s(n)$ the set of all points $(t_1,\cdots,t_n)$ such that

$$\sum_{i=1}^{n} z_i(t_i-a_i) \tag{3.3}$$

can become real and non-negative[4a] for some $(z_1,\cdots,z_n)$ satisfying $z_i\geq 0$ and $\sum_{i=1}^n z_i=1$. Then it is evident that $f(t_1,\cdots,t_n)$ is analytic in a domain $D^a(n)$, where $D^a(n)$ stands for the complement of $D^s(n)$. So our task is to find $D^s(n)$ explicitly.

Lemma. Let H_+, H_-, H_0 be three (disjoint) sets of natural numbers such that

$$H_++H_-+H_0=\{1,2,\cdots,n\}, \tag{3.4}$$

and consider a system of an equation and an inequality

$$\begin{aligned} \sum_{i\in H_+} b_iz_i-\sum_{j\in H_-} c_jz_j&=0, \\ \sum_{i=1}^{n} \lambda_iz_i&\geq 0, \end{aligned} \tag{3.5}$$

where b_i and c_j are positive numbers, λ_i being real. In order that (3.5) has a solution such that $z_i\geq 0$ $(i=1, 2, \cdots, n)$ and $\sum_{i=1}^n z_i>0$, it is necessary and sufficient that at least one of inequalities,

$$\begin{aligned} b_i\lambda_j+c_j\lambda_i&\geq 0, \quad (i\in H_+,\ j\in H_-) \\ \lambda_k&\geq 0, \quad (k\in H_0) \end{aligned} \tag{3.6}$$

holds.

Proof. Sufficiency. If $b_i\lambda_j+c_j\lambda_i\geq 0$, we set

$$\begin{aligned} z_j&=b_iz_i/c_j>0, \\ z_l&=0, \quad (l\neq i,\ j). \end{aligned} \tag{3.7}$$

Then,

$$\lambda_iz_i+\lambda_jz_j=(z_i/c_j)(c_j\lambda_i+b_i\lambda_j)\geq 0. \tag{3.8}$$

Thus, (3.5) is satisfied. If $\lambda_k\geq 0$, we set

$$\begin{aligned} z_k&>0, \\ z_l&=0, \quad (l\neq k). \end{aligned} \tag{3.9}$$

Then, (3.5) is naturally satisfied.

[4a] Hereafter we write simply "non-negative" instead of real and non-negative.

Necessity. Assume,

$$\begin{aligned} b_i\lambda_j+c_j\lambda_i<0, &\quad (i\in H_+,\ j\in H_-) \\ \lambda_k<0, &\quad (k\in H_0). \end{aligned} \tag{3.10}$$

From (3.5), we have

$$\frac{1}{b_l}\Big[\sum_{i\in H_+}(b_l\lambda_i-b_i\lambda_l)z_i+\sum_{j\in H_-}(b_l\lambda_j+c_j\lambda_l)z_j\Big] +\sum_{k\in H_0}\lambda_k z_k\geq 0,\quad (l\in H_+). \tag{3.11}$$

When $\sum_{i\in H_+}z_i>0$ (then of course $\sum_{j\in H_-}z_j>0$), from (3.11) and (3.10) we get

$$\sum_{i\in H_+}(b_l\lambda_i-b_i\lambda_l)z_i>0. \tag{3.12}$$

Therefore,

$$0=\sum_{l\in H_+}\sum_{i\in H_+}(b_l\lambda_i-b_i\lambda_l)z_iz_l>0. \tag{3.13}$$

This is self-inconsistent. When $\sum_{i\in H_+}z_i=0$ [then of course $z_i=0$ $(i\in H_++H_-)$], from (3.11) we get

$$\sum_{k\in H_0}\lambda_k z_k\geq 0. \tag{3.14}$$

This is inconsistent with (3.10) and $\sum_{k\in H_0}z_k>0$. Q.E.D.

Theorem II. The intersection of $D^s(n)$ and

$$\begin{aligned} \mathrm{Im}t_i>0, &\quad (j\in H_+) \\ \mathrm{Im}t_j<0, &\quad (j\in H_-) \\ \mathrm{Im}t_k=0, &\quad (k\in H_0) \end{aligned} \tag{3.15}$$

is a union of

$$\begin{aligned} \mathrm{Im}[(t_i-a_i)/(t_j-a_j)]\geq 0, &\quad (i\in H_+,\ j\in H_-) \\ t_k\geq a_k, &\quad (k\in H_0) \end{aligned} \tag{3.16}$$

within (3.15).

Proof. By definition $D^s(n)$ consists of all points $(t_1,\cdots,t_n)$ such that

$$\begin{aligned} &\sum_{i=1}^n z_i\,\mathrm{Im}t_i=0, \\ &\sum_{i=1}^n z_i(\mathrm{Re}t_i-a_i)\geq 0, \end{aligned} \tag{3.17}$$

are satisfied by some $(z_1,\cdots,z_n)$ satisfying $z_i\geq 0$ and $\sum_{i=1}^n z_i>0$. Hence, according to lemma, we have

$$\begin{aligned} (\mathrm{Im}t_i)(\mathrm{Re}t_j-a_j)-(\mathrm{Im}t_j)(\mathrm{Re}t_i-a_i)\geq 0, &\\ (i\in H_+,\ j\in H_-), &\\ \mathrm{Re}t_k-a_k\geq 0,\quad (k\in H_0), & \end{aligned} \tag{3.18}$$

which can be rewritten as (3.16). Q.E.D.

Theorem III. The domain $D^a(n)$ is a domain of holomorphy.

Proof. It is sufficient to show the existence of a function which is analytic in $D^a(n)$ and actually singular at an arbitrary point in the interior of (3.16). For simplicity, we write $s=t_i-a_i$, $t=t_j-a_j$. Then a point

$$\begin{aligned} s^0=pe^{i\phi}, &\quad (p>0,\ 0\leq\phi<2\pi), \\ t^0=qe^{i\psi}, &\quad (q>0,\ 0\leq\psi<2\pi), \end{aligned} \tag{3.19}$$

belongs to $\{\mathrm{Im}s>0,\ \mathrm{Im}t<0,\ \mathrm{Im}(s/t)>0\}$ only when

$$\begin{aligned} &0<\phi<\pi, \\ &\pi<\psi<2\pi, \\ &-2\pi<\phi-\psi<-\pi. \end{aligned} \tag{3.20}$$

In order to construct a desired function, it is convenient to use a previous example[5]:

$$\begin{aligned} f(s,t)&=\int_0^1\frac{dz}{\{re^{i\theta}z+(1-z)\}\{z(-s)+(1-z)(-t)\}} \\ &=\frac{1}{s-re^{i\theta}t}\ln\left[\frac{(-t)}{(-s)}re^{i\theta}\right],\quad (0<\theta<\pi). \end{aligned} \tag{3.21}$$

This is a special case of (3.1), and hence analytic in $D^a(n)$. This function is actually singular at

$$\begin{aligned} &|s|=r|t|, \\ &\arg s=\arg t+\theta-2\pi. \end{aligned} \tag{3.22}$$

Therefore, setting $r=p/q$ and $\theta=2\pi+\phi-\psi$, we get a function which is singular at (s^0,t^0).

As for $t_k\geq a_k$ in (3.16), the existence of functions having such a cut is well known. Q.E.D.

IV. RECONSTRUCTION OF THE INTEGRAL REPRESENTATION

Theorem IV. If a function $f(t_1,\cdots,t_n)$ is analytic in $D^a(n)$, and if it is bounded by $|\sum_{i=1}^n z_it_i|^{-\delta}$, $(\delta>0)$ whenever $|\sum_{i=1}^n z_it_i|$ becomes larger than some large positive number M, where $z_1,\cdots,z_n$ are any non-negative numbers satisfying $\sum_{i=1}^n z_i=1$, then it can be represented as (3.1) with (3.2).

Proof. By setting $\beta=\alpha-\sum_{i=1}^n z_ia_i$ and $s_i=t_i-a_i$, the statement of the theorem is rewritten as follows. If $f(s_1,\cdots,s_n)$ is analytic in all points $(s_1,\cdots,s_n)$ such that $\sum_{i=1}^n z_is_i$ cannot become non-negative for any $(z_1,\cdots,z_n)$ satisfying $z_i\geq 0$ and $\sum_{i=1}^n z_i=1$, and if it is bounded by $|\sum_{i=1}^n z_is_i|^{-\delta}$ at infinity, then it can be represented as

$$f(s_1,\cdots,s_n)=\int_0^\infty d\beta\int_0^1 dz_1\cdots\int_0^1 dz_n\,\delta\Big(1-\sum_{i=1}^n z_i\Big)\times\frac{\rho(\beta,z_1,\cdots,z_n)}{\beta-\sum_{i=1}^n z_is_i}. \tag{4.1}$$

[5] N. Nakanishi, Prog. Theor. Phys. **24**, 1275 (1960).

Now, since $f(s_1,\cdots,s_n)$ is analytic in $\{\mathrm{Im}s_1>0,\cdots,\mathrm{Im}s_n>0\}$, Cauchy's theorem leads

$$f(s_1,\cdots,s_n)=\frac{1}{(2\pi i)^n}\oint\frac{ds_1'}{s_1'-s_1}\cdots\oint\frac{ds_n'}{s_n'-s_n}\times f(s_1',\cdots,s_n') \quad (4.2)$$

for such $(s_1,\cdots,s_n)$, where contours are large semicircles in upper half planes. Using the generalized Feynman identity, we have

$$f(s_1,\cdots,s_n)=\int_0^1 dz_1\cdots\int_0^1 dz_n\,\delta(1-\sum_{i=1}^n z_i)\times\psi(\sum_{i=1}^n z_is_i,z_1,\cdots,z_n), \quad (4.3)$$

with

$$\psi(\sum_{i=1}^n z_is_i,z_1,\cdots,z_n)\equiv\frac{(n-1)!}{(2\pi i)^n}\oint ds_1'\cdots\oint ds_n'\times\frac{f(s_1',\cdots,s_n')}{[\sum_{i=1}^n z_is_i'-\sum_{i=1}^n z_is_i]^n}. \quad (4.4)$$

Since at least one of $z_1,\cdots,z_n$ does not vanish, consider the case $z_n\neq 0$, for instance. We transform the integration variable s_n' into $w\equiv\sum_{i=1}^n z_is_n'$ in (4.4). Deforming the contours, we can analytically continue ψ as far as w and $\sum_{i=1}^{n-1} z_is_i'$ do not become nonnegative. Setting

$$g(w,z_1,\cdots,z_n)\equiv\frac{1}{(2\pi i)^n}\frac{\partial^{n-1}}{\partial w^{n-1}}\oint ds_1'\cdots\oint ds_{n-1}'\times f(s_1',\cdots,s_{n-1}',[w-\sum_{i=1}^{n-1}z_is_i']/z_n), \quad (4.5)$$

we obtain (after $n-1$ partial integrations)

$$\psi(\sum_{i=1}^n z_is_i,z_1,\cdots,z_n)=\int_0^\infty\frac{d\beta}{z_n}\times\frac{g(\beta+i\epsilon,z_1,\cdots,z_n)-g(\beta-i\epsilon,z_1,\cdots,z_n)}{\beta-\sum_{i=1}^n z_is_i},\quad (\epsilon\to 0+), \quad (4.6)$$

because the contributions from infinity vanish on account of the boundedness condition. (4.3) with (4.6) immediately yields (4.1). Q.E.D.

When one tries to evaluate the integral (4.5), it is convenient to transform the integration variables into

$$s_1',\quad z_1s_1'+z_2s_2',\quad\cdots,\quad \textstyle\sum_{i=1}^{n-1}z_is_i',$$

because then the integral in each stage reduces essentially to a real integral from 0 to $+\infty$. In particular, in the case $n=2$ we have

$$g(w,z_1,z_2)=\frac{1}{(2\pi i)^2}\times\int_0^\infty ds'\frac{\partial}{\partial w}\{f(s'+i\epsilon',[w-z_1s']/z_2)-f(s'-i\epsilon',[w-z_1s']/z_2)\},\quad(\epsilon'\to 0+) \quad (4.7)$$

for any complex value of w. For instance, a previous example (3.21),

$$f(s,t)=[\ln(-t)-\ln(-s)+\ln(r\,e^{i\theta})]/(s-re^{i\theta}t), \quad (4.8)$$

gives

$$g(w,z_1,z_2)=\frac{-1}{2\pi i}\int_0^\infty ds'\frac{\partial}{\partial w}\frac{1}{s'-re^{i\theta}(w-z_1s')/z_2}=\frac{-z_2}{2\pi i}\frac{1}{re^{i\theta}z_1+z_2}\frac{1}{w}, \quad (4.9)$$

that is,

$$\rho(\beta,z_1,z_2)=\delta(\beta)/(re^{i\theta}z_1+z_2). \quad (4.10)$$

If $f(t_1,\cdots,t_n)$ is bounded by $|\sum_{i=1}^n z_it_i|^{N-\delta}$ at infinity, subtraction procedure becomes necessary. The subtracted from at $(t_i=b_i<a_i)$ is

$$f(t_1,\cdots,t_n)=P(t_1,\cdots,t_n)+\int d\alpha\int dz_1\cdots\int dz_n\,\delta(1-\sum_{i=1}^n z_i)\times\left[\frac{\sum_{i=1}^n z_i(t_i-b_i)}{\alpha-\sum_{i=1}^n z_ib_i}\right]^N\frac{\rho(\alpha,z_1,\cdots,z_n)}{\alpha-\sum_{i=1}^n z_it_i}, \quad (4.11)$$

where a polynomial $P(t_1,\cdots,t_n)$ is formally defined by

$$P(t_1,\cdots,t_n)\equiv\left[\sum_{l=0}^{N-1}\frac{1}{l!}\left\{\sum_{i=1}^n(t_i-b_i)\frac{\partial}{\partial t_i'}\right\}^l\times f(t_1',\cdots,t_n')\right]_{t_i'=b_i}$$
$$=\sum_{l=1}^{N-1}\int d\alpha\int dz_1\cdots\int dz_n\,\delta(1-\sum z_i)\times\frac{[\sum z_i(t_i-b_i)]^l}{[\alpha-\sum z_ib_i]^{l+1}}\rho(\alpha,z_1,\cdots,z_n). \quad (4.12)$$

The integral in (4.11) converges if $\rho(\alpha,z_1,\cdots,z_n)$ is bounded by $|\alpha|^{N-\delta}$ at infinity.

V. UNIQUENESS THEOREM A

For mathematical convenience, we hereafter rewrite (3.1) as follows:

$$f(t_1,\cdots,t_n)=\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}dz_1\cdots\int_{-\infty}^{\infty}dz_n \times\frac{\sigma(\alpha,z_1,\cdots,z_n)}{\alpha-\sum_{i=1}^{n}z_it_i}, \quad (5.1)$$

with

$$\sigma(\alpha,z_1,\cdots,z_n)\equiv\delta(1-\sum_{i=1}^{n}z_i)\rho(\alpha,z_1,\cdots,z_n) \quad (5.2)$$

and $\rho(\alpha,z_1,\cdots,z_n)$ vanishes unless $z_1\geq0$, $\cdots$, $z_n\geq0$.

Theorem V. If $f(t_1,\cdots,t_n)$ is an entire function of t_1 when t_2, $\cdots$, t_n are real and $t_2<a_2$, $\cdots$, $t_n<a_n$ where a_2, $\cdots$, a_n are real constants, and if

$$\sigma(-\infty,z_1,\cdots,z_n)=0, \quad (5.3)$$

then $\sigma(\alpha,z_1,\cdots,z_n)$ vanishes unless $z_1=0$, namely,

$$\sigma(\alpha,z_1,\cdots,z_n)=\sum_{j=1}^{N}\delta^{(j)}(z_1)\sigma_j(\alpha,z_2,\cdots,z_n) \quad (5.4)$$

and hence,

$$f(t_1,\cdots,t_n)=\sum_j(-t_1)^j\int d\alpha\int dz_2\cdots\int dz_n \times\frac{(\partial/\partial\alpha)^j\sigma_j(\alpha,z_2,\cdots,z_n)}{\alpha-\sum_{i=2}^{n}z_it_i}. \quad (5.5)$$

Proof. From the assumption of the theorem two limits from both sides of the real axis on the t_1 plane must coincide, namely,

$$\int dz_1\cdots\int dz_n\,\sigma(\sum_{i=1}^{n}z_it_i,z_1,\cdots,z_n)=0 \quad (5.6)$$

for any real value of t_1 as far as $t_2<a_2$, $\cdots$, $t_n<a_n$. Using (5.2), we can rewrite (5.6) as

$$\int dz_2\cdots\int dz_n \times\rho(t_1+\sum_{i=2}^{n}z_iu_i,\,1-\sum_{i=2}^{n}z_i,\,z_2,\,\cdots,\,z_n)=0, \quad (5.7)$$

with $u_i\equiv t_i-t_1<a_i-t_1$. We define $n-1$ operators,

$$L_i\equiv\int_{-\infty}^{t_1}dt_1\frac{\partial}{\partial u_i},\quad (i=2,\cdots,n). \quad (5.8)$$

When L_i operates on (5.7), we get

$$\int dz_2\cdots\int dz_n \times z_i\rho(t_1+\sum_{i=2}^{n}z_iu_i,\,1-\sum_{i=2}^{n}z_i,\,z_2,\,\cdots,\,z_n)=0, \quad (5.9)$$

because of (5.3). In general, an operation of a polynomial of L_i yields

$$\int dz_2\cdots\int dz_n\,P(z_2,\cdots,z_n) \times\rho(t_1+\sum_{i=2}^{n}z_iu_i,\,1-\sum_{i=2}^{n}z_i,\,z_2,\,\cdots,\,z_n)=0, \quad (5.10)$$

where $P(z_2,\cdots,z_n)$ is an arbitrary polynomial of z_2, $\cdots$, z_n. By assumption, ρ vanishes outside a region $\{z_2\geq0,\cdots,z_n\geq0,\sum_{i=2}^{n}z_i\leq1\}$. Since this region is compact, according to Weierstrass' approximation theorem, for any infinitely differentiable function $\phi(z_2,\cdots,z_n)$ there exists a polynomial $P(z_2,\cdots,z_n)$ such that

$$\phi(z_2,\cdots,z_n)=P(z_2,\cdots,z_n)+\epsilon\psi(z_2,\cdots,z_n) \quad (5.11)$$

in this region, where ϵ is an infinitesimal constant and $|\psi|<1$. Because of (5.10), therefore, we have

$$\int dz_2\cdots\int dz_n\,\phi(z_2,\cdots,z_n)\rho(\alpha,\,1-\sum_{i=2}^{n}z_i,\,z_2,\,\cdots,\,z_n)$$
$$=\epsilon\int dz_2\cdots\int dz_n\,\psi(z_2,\cdots,z_n) \times\rho(\alpha,\,1-\sum_{i=2}^{n}z_i,\,z_2,\,\cdots,\,z_n)\rightarrow0, \quad (5.12)$$

where

$$\alpha\equiv t_1+\sum_{i=2}^{n}z_iu_i=(1-\sum_{i=2}^{n}z_i)t_1+\sum_{i=2}^{n}z_it_i. \quad (5.13)$$

This means

$$\rho(\alpha,\,1-\sum_{i=2}^{n}z_i,\,z_2,\,\cdots,\,z_n)=0 \quad (5.14)$$

as a distribution as far as $1-\sum_{i=2}^{n}z_i\neq0$. Q.E.D.

Theorem VI. If $f(t_1,\cdots,t_n)$ is an entire function of t_1 for any real values of t_2, $\cdots$, t_n, and if (5.3) is satisfied, then $\sigma(\alpha,z_1,\cdots,z_n)$ vanishes identically.

Proof. In the last step of the proof of Theorem V, α can take any real value even for

$$1-\sum_{i=2}^{n}z_i=0$$

in the present case. Q.E.D.

Theorem VII. In the representation (5.1) with (5.2) $\sigma(\alpha,z_1,\cdots,z_n)$ is *uniquely* determined by $f(t_1,\cdots,t_n)$ as far as (5.3) is satisfied.

Proof. If $f(t_1,\cdots,t_n)$ is represented in terms of two weight functions σ_1 and σ_2, zero must be represented in terms of $\sigma_1-\sigma_2$. Since zero is of course an entire function, $\sigma_1-\sigma_2$ must identically vanish on account of Theorem VI. Q.E.D.

This theorem assures the uniqueness of the integral representation for vertex function[6] and particularly of the Deser-Gilbert-Sudarshan-Ida integral representation.[7]

VI. UNIQUENESS THEOREM B

Theorem VIII. When an identity

$$\sum_{i=1}^{n+1} t_i=c \tag{6.1}$$

holds between $n+1$ variables t_i, if a function $f(t_1,\cdots,t_n)$ is represented in the following two ways (we use the notation mentioned at the beginning of the last section):

$$f(t_1,\cdots,t_n)=\int d\alpha\int dz_1\cdots\int dz_n\frac{\sigma(\alpha,z_1,\cdots,z_n)}{\alpha-\sum_{i=1}^{n} z_it_i} \tag{6.2}$$

where $\sigma(\alpha,z_1,\cdots,z_n)$ vanishes unless

$$\alpha\geq\sum_{i=1}^{n} z_ia_i, \tag{6.3}$$

and

$$f(t_1,\cdots,t_n)=\int d\beta\int dz_2\cdots\int dz_{n+1} \times\frac{\sigma'(\beta,z_2,\cdots,z_{n+1})}{\beta-\sum_{i=2}^{n+1} z_it_i} \tag{6.4}$$

where $\sigma'(\beta,z_2,\cdots,z_{n+1})$ vanishes unless

$$\beta\geq\sum_{i=2}^{n+1} z_ia_i, \tag{6.5}$$

and if

$$\sum_{i=1}^{n+1} a_i>c, \tag{6.6}$$

then $f(t_1,\cdots,t_n)$ is written as

$$\sum_k t_1^k\int d\alpha\int dz_2\cdots\int dz_n\frac{\sigma_k(\alpha,z_2,\cdots,z_n)}{\alpha-\sum_{i=2}^{n} z_it_i}, \tag{6.7}$$

where $\sigma_k(\alpha,z_2,\cdots,z_n)$ vanishes unless

$$\alpha\geq\sum_{i=2}^{n} z_ia_i. \tag{6.8}$$

Proof. From (6.2)–(6.5) we see that when $t_2<a_2$, $\cdots$, $t_n<a_n$, $f(t_1,\cdots,t_n)$ is analytic with respect to t_1 *either* unless

$$t_1 \text{ is real and } t_1\geq a_1 \tag{6.9}$$

or unless

$$t_1 \text{ is real and } t_1\leq c-a_{n+1}-\sum_{i=2}^{n} t_i. \tag{6.10}$$

On account of (6.6) there exists a nonempty domain

$$D\equiv\{t_2<a_2,\ \cdots,\ t_n<a_n,\ \sum_{i=2}^{n} t_i>c-a_1-a_{n+1}\}. \tag{6.11}$$

From (6.9) and (6.10) we see that $f(t_1,t_2,\cdots,t_n)$ is an entire function of t_1 as far as $(t_2,\cdots,t_n)$ belongs to D. When $(t_2,\cdots,t_n)$ does not belong to D, $f(t_1,t_2,\cdots,t_n)$ may have a finite cut [the intersection of (6.9) and (6.10)]. But using Cauchy's formula

$$f(t_1,t_2,\cdots,t_n)=\frac{1}{2\pi i}\oint dt_1'\frac{f(t_1',t_2,\cdots,t_n)}{t_1'-t_1}, \tag{6.12}$$

we can make analytic completion, since $f(t_1,t_2,\cdots,t_n)$ is "real analytic" with respect to the auxiliary variables t_2, $\cdots$, t_n.[8] Thus, we see that $f(t_1,t_2,\cdots,t_n)$ is an entire function of t_1 as far as $t_2<a_2$, $\cdots$, $t_n<a_n$. We can, therefore, make use of Theorem V [the condition (5.3) is automatically satisfied because of (6.3)]. Thus, we get (6.7) with (6.8). Q.E.D.

Now, we conjecture that if a function $F(t_1,\cdots,t_n)$ is represented as

$$F(t_1,\cdots,t_n)=\int d\alpha\int dz_1\cdots\int dz_{n+1}[\sum_{i=1}^{n+1}\delta(z_i)] \times\frac{\sigma(\alpha,z_1,\cdots,z_{n+1})}{\alpha-\sum_{i=1}^{n+1} z_it_i} \tag{6.13}$$

with (6.1), if $\sigma(\alpha,z_1,\cdots,z_{n+1})$ vanishes unless

$$\alpha\geq\sum_{i=1}^{n+1} z_ia_i, \tag{6.14}$$

and if (6.6) is satisfied, then the weight functions $\sigma(\alpha,z_1,\cdots,z_{i-1},0,z_{i+1},\cdots,z_{n+1})$ are uniquely determined except for the parts of $z_j=0$ $(j\neq i)$. Namely, we expect that if $F=0$ then $\sigma(\alpha,z_1,\cdots,z_{i-1},0,z_{i+1},\cdots,z_{n+1})$ must vanish unless $z_j=0$ $(j\neq i)$.

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[6] N. Nakanishi, Suppl. Prog. Theor. Phys. **18**, 1 (1961).

[7] S. Deser, W. Gilbert and E. C. T. Sudarshan, Phys. Rev. **115**, 731 (1959); M. Ida, Prog. Theor. Phys. **23**, 1151 (1960). Their proofs are incorrect, but their result is correct in every order of perturbation theory under stability conditions (see reference 6).

[8] S. Bochner and W. T. Martin, *Several Complex Variables* (Princeton University Press, Princeton, New Jersey, 1948), Chap. IV.

Let D_i^a be the analyticity domain of

$$f_i(t_1,\cdots,t_n)\equiv\int d\alpha\int dz_1\cdots\int dz_{n+1}\,\delta(z_i)\times\frac{\sigma(\alpha,z_1,\cdots,z_{n+1})}{\alpha-\sum_{i=1}^{n+1}z_it_i}. \quad (6.15)$$

The assumption $F=0$ means

$$\sum_{i=1}^{n+1}f_i=0. \quad (6.16)$$

Then if these functions f_i can be decomposed into functions f_{ij} analytic in $D_i^a\bigcup D_j^a$, such that

$$f_i=\sum_{j=1}^{n+1}f_{ij}, \quad (6.17)$$

$$f_{ij}=-f_{ji}\quad (f_{ii}=0), \quad (6.18)$$

then our conjecture is valid, because Theorem VIII together with Theorems IV and VII assures that f_{ij} is just the contribution from $z_j=0$ in (6.15).

The above decomposition is not always possible for arbitrary domains of holomorphy. This problem is closely related to the cohomology group of a domain with coefficients in the sheaf of holomorphic functions.[9]

Now, our conjecture is trivially correct for $n=1$, because (6.13) then reduces to a usual dispersion relation whose two cuts are separated. We can prove that our conjecture is valid also for $n=2$.

Theorem IX. If a function $F(t_1,t_2)$ is represented as

$$F=f_1+f_2+f_3 \quad (6.19)$$

where

$$f_1\equiv\int_{-\infty}^{\infty}d\alpha_1\int_0^1dz\frac{\rho_{23}(\alpha_1,z)}{\alpha_1-zt_2-(1-z)t_3},$$
$$f_2\equiv\int_{-\infty}^{\infty}d\alpha_2\int_0^1dz\frac{\rho_{31}(\alpha_2,z)}{\alpha_2-zt_3-(1-z)t_1}, \quad (6.20)$$
$$f_3\equiv\int_{-\infty}^{\infty}d\alpha_3\int_0^1dz\frac{\rho_{12}(\alpha_3,z)}{\alpha_3-zt_1-(1-z)t_2},$$

with

$$t_1+t_2+t_3=c, \quad (6.21)$$

if the weight functions ρ_{23}, ρ_{31}, ρ_{12} vanish unless

$$\begin{aligned}\alpha_1&\geq za_2+(1-z)a_3,\\ \alpha_2&\geq za_3+(1-z)a_1,\\ \alpha_3&\geq za_1+(1-z)a_2,\end{aligned} \quad (6.22)$$

respectively, and if

$$a_1+a_2+a_3>c, \quad (6.23)$$

then the weight functions are uniquely determined for $0<z<1$. In other words, if $F=0$ then the supports of ρ_{23}, ρ_{31}, ρ_{12} are $z=0$ and $z=1$ only.

Proof. Let D_i^a be the analyticity domain of the function f_i defined in (6.20) and D_i^s be the complement of D_i^a ($i=1$, 2, 3). If $F=0$, f_1 is analytic in the envelope of holomorphy of

$$D_1^a\bigcup(D_2^a\bigcap D_3^a)=(D_1^a\bigcup D_2^a)\bigcap(D_1^a\bigcup D_3^a)$$

because $f_1=-(f_2+f_3)$. We will first compute the complement of $D_1^a\bigcup D_2^a$, i.e., $D_1^s\bigcap D_2^s$.

From Theorem II we see that when $\mathrm{Im}t_3<0$, D_1^s consists of all points such that

$$\mathrm{Im}t_2>0\quad\text{and}\quad\mathrm{Im}[(t_2-a_2)(t_3^*-a_3)]\geq0, \quad (6.24)$$

while D_2^s consists of all points such that

$$\mathrm{Im}t_1>0\quad\text{and}\quad\mathrm{Im}[(t_1-a_1)(t_3^*-a_3)]\geq0. \quad (6.25)$$

Since $t_2=c-t_1-t_3$, the second inequality of (6.24) is rewritten as

$$\mathrm{Im}[(t_1-a_1)(t_3^*-a_3)]+|t_3-a_3|^2+(a_1+a_2+a_3-c)\,\mathrm{Im}t_3^*\leq0. \quad (6.26)$$

Because of (6.23) this is clearly incompatible with (6.25). Namely, (6.24) and (6.25) are disjoint. The same is true also for $\mathrm{Im}t_3>0$. When $\mathrm{Im}t_3=0$, D_1^s and D_2^s overlap in a cut

$$t_3\geq a_3 \quad (6.26)$$

and some exceptional points such that

$$\mathrm{Im}t_1=0\quad\text{and}\quad a_1\leq t_1\leq c-a_2-t_3. \quad (6.27)$$

Similar consideration applies also to $D_1^s\bigcap D_3^s$. Thus, we find that f_1 is analytic except for

$$\mathrm{Im}t_2=0\quad\text{and}\quad t_2\geq a_2$$
or (6.28)
$$\mathrm{Im}t_3=0\quad\text{and}\quad t_3\geq a_3.$$

Namely, f_1 is represented as

$$(t_2-b_2)^N(t_3-b_3)^N\int_{a_2}^{\infty}dt_2'\int_{a_3}^{\infty}dt_3'\frac{\phi_{12}(t_2',t_3')}{(t_2'-t_2)(t_3'-t_3)} \quad (6.29)$$

plus single dispersion terms for t_2 and for t_3 with a polynomial. The other two functions f_2 and f_3 are likewise represented. Since $f_1+f_2+f_3=0$, (6.29) must vanish because of the uniqueness property of the Mandelstam representation when (6.23) is satisfied

[9] M. Sato derived the following theorem from Theorem B of H. Cartan and J. P. Serre [Seminaire Henri Cartan (1951–52)] and J. Leray's lemma on the Čech cohomology [Seminaire Bourbaki (1953–54)]. If domains Δ_i ($i=0$, 1, $\cdots$, n) and their union $\bigcup_{i=0}^n\Delta_i$ are domains of holomorphy, if functions f_i are holomorphic in $\bigcap_{k\neq i}\Delta_k$, and if $\Sigma_{i=0}^n f_i=0$, then we can always find functions f_{ij} holomorphic in $\bigcap_{k\neq i,j}\Delta_k$ such that $f_i=\Sigma_{j=0}^n f_{ij}$ and $f_{ij}+f_{ji}=0$. This theorem can be applied to our problem only in the case $n=2$.

(this property is proved by applying two times the uniqueness of the usual single dispersion relation whose two cuts are separated).

Since single dispersion terms can be represented as (6.20), f_1 minus these terms must be an entire function. Therefore, Theorem VI shows that $\rho_{23}(\alpha,z)$ vanishes except for $z=0$ and $z=1$. The same is true for the other two weight functions. Q.E.D.

In the above theorem the condition (6.23) is very important. Without it even the Mandelstam representation loses its uniqueness property. Our previous results[2] show that this condition is satisfied in almost all practical cases (e.g., equal-mass, nucleon-nucleon, pion-nucleon, and kaon-nucleon scatterings) in every order of perturbation theory.

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Factorization of the Residues of Regge Poles*

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A proof of the factorizability of the residues of Regge poles, valid for a many-channel potential scattering problem, is given. Unitarity and certain other plausible assumptions about the S matrix allow the proof to be extended to the relativistic S-matrix theory.

RECENTLY, Gell-Mann (private communication) has postulated that the residues of Regge poles of the S matrix for a many-channel problem are factorizable, viz., for

$$\beta_{ij}(E)=\lim_{J\to\alpha(E)} [J-\alpha(E)]S_{ij}(E), \qquad (1)$$

where i, j label the channels and $\alpha(E)$ is the position of a pole, then

$$\beta_{ij}(E)=\gamma_i(E)\gamma_j(E). \qquad (2)$$

Gell-Mann has given a proof of this equation based on the nonrelativistic Schrödinger equation.[1]

In the course of a general study of analyticity in J for the nonrelativistic potential scattering problem we had also obtained a simple proof of this result, which is worth reporting, since the method, being based directly on the S matrix, can immediately be generalized to enable us to say something about the relativistic problem.[2]

For the potential case (and for a wide class of potentials), it can be shown that the S matrix can be written in the form

$$S(J,E)=F_1(J,E)[F_2(J,E)]^{-1}, \qquad (3)$$

where F_1 and F_2 are n-by-n matrices,[3] n being the number of channels, with F_1 and F_2 analytic functions of J. Thus, poles of $S(J,E)$ occur for

$$\det[F_2(J,E)]=0. \qquad (4)$$

Except for accidental degeneracies, the zeros in

$$\det[F_2(J,E)]$$

are simple zeros. Since the elements of F_2 are analytic in J, it follows that the rank of F_2 is $n-1$.

If we write

$$[F_2]^{-1}=G/\det F_2, \qquad (5)$$

where G is the matrix of cofactors, then Sylvester's law of nullity tells us that

$$r(G)+r(F_2)-n\leqslant 0, \qquad (6)$$

where $r(A)$ means the rank of the matrix A. Thus

$$r(G)\leqslant 1, \qquad (7)$$

i.e., all 2-by-2 cofactors of G are zero. Simple calculation then shows that the residues satisfy

$$\beta_{ii}\beta_{jj}=\beta_{ij}\beta_{ji}. \qquad (8)$$

Apart from an irrelevant sign, Eq. (2) follows from Eq. (8) and the fact that S is symmetrical.

If, as seems plausible, the relativistic S matrix can also be written in the form of Eq. (3), with F_1 and F_2

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[1] Murray Gell-Mann, Phys. Rev. Letters 8, 263 (1962).

[2] Since writing this paper, we have seen an article by V. N. Gribov and I. Ya. Pomeranchuk [Phys. Rev. Letters 8, 343 (1962)] in which the result is proven for a two-channel problem, with the same assumptions that we have made.

[3] The matrices F_1 and F_2 are generalizations of Jost functions; see, for example, R. G. Newton, J. Math. Phys. 1, 319 (1960).