Asymptotic Covariant Conservation Laws for Gravitational Radiation

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(Received March 1, 1962)

The usual conserved quantities of Lorentz-covariant theories are associated with the descriptors of coordinate transformations in Killing directions. By suitably defining the concept of asymptotic-Killing vector fields, it is possible to extend the definition of the usual conserved quantities to situations in which, properly speaking there are no Killing vector fields, but in which, nevertheless, it is still meaningful to speak of the energy, momentum and perhaps angular momentum radiated by the gravitational field. Among the results obtained in the course of the investigation are: (a) an understanding of the circumstance which singles out the energy as a positive-definite quantity; (b) a possible global consequence that vanishing energy implies that the space is flat; (c) an understanding of the position of the Fock harmonic coordinate conditions in Trautman's treatment of gravitational radiation; (d) an extension of the validity of the Møller pseudotensor to the Trautman radiative solutions. Also a brief indication is given of how this work might be extended to give a proper treatment of energy, momentum, and angular momentum densities for general metrics.

I. INTRODUCTION AND GENERAL CONSIDERATIONS

It is well known that the fundamental conservation theorems in physics are related to the invariance properties of the physical laws. Thus, in the theory of gravitation, which has as an invariance group the group of general curvilinear coordinate transformations, one can find innumerable conservation laws, no one finite set of which has any evident invariant distinction. In particular, if we describe an arbitrary infinitesimal curvilinear coordinate transformation by means of a vector field ξ^i as follows (Latin indices run from 1 to 4):

$$X^{i}=X^{i}+\xi^{i}, \tag{1}$$

then it has been shown[1] that the strong conservation law associated with this transofrmation may be written

$$E^{m}(\xi)_{;m}=0, \tag{2}$$

where the generalized energy flux vector $E^i(\xi)$ is

$$E^{i}(\xi)=2(\xi^{i;n}-\xi^{n;i})_{;n}. \tag{3}$$

In general, there appears to be no convincing way to single out a particular ξ^i for the purpose of identifying an analog of energy, momentum, or angular momentum. However, for particular highly specialized spaces, namely, those admitting Killing vector fields, the situation strikingly resembles that found in ordinary Lorentz covariant theories. For the generalized energy flux vector associated with the Killing vector field ξ^i is found from Eq. (3) to be

$$E^{i}(\xi)=-2\xi^{m}R_{m}{}^{i}, \tag{4}$$

where $R_j{}^i$ is the Ricci tensor of the manifold. (Use has been made of the commutation relations for covariant derivatives.)

We see from Eq. (4) that the generalized energy flux vector for the Killing vector ξ^i vanishes in regions where there is no matter. The total generalized energy,

$$E(\xi)\equiv\frac{1}{2K}\int E^{m}dS_{m}=\frac{1}{2K}\int E^{4}(-g)^{\frac{1}{2}}d_{3}x, \tag{5}$$

which, from Eq. (2), we can see is conserved, can be converted by use of Eq. (3) into a two-surface integral surrounding the matter

$$E(\xi)=\frac{1}{K}\oint(\xi^{m;n}-\xi^{n;m})dS_{mn}. \tag{6}$$

We see from Eq. (4) that this two-surface integral is surface-independent, providing it encloses all the regions where the Ricci tensor is nonvanishing or singular, that is, all the regions where there is matter. For the Schwarzchild solution, if ξ^i is taken to be the time-like Killing vector field, Eq. (6) provides a covariant, surface-independent characterization of the mass of the singularity. It is the strict analog of the Gauss flux integral for the total charge in the Maxwell theory.[2]

The fact that the vanishing of covariant derivatives of symmetric tensors lead, in the presence of Killing vectors to true integral conservation theorems, has already been noted by Trautman.[3] For, given T_{ij} such that

$$T_{ij}=T_{ji} \tag{7}$$

and

$$T_{i}{}^{m}{}_{;m}=0, \tag{8}$$

and given a Killing vector ξ^i, then

$$(\xi^{n}T_{n}{}^{m})_{;m}=\tfrac{1}{2}(\xi_{n;m}+\xi_{m;n})T^{nm}+\xi^{n}T_{n}{}^{m}{}_{;m}=0. \tag{9}$$

* Supported in part by the National Science Foundation.

[1] A. Komar, Phys. Rev. **113**, 934 (1959). (Note the slight change in notation; E^m of the present paper corresponds to P^m of this reference.)

[2] In order to see this analogy more sharply, compare Eq. (3) with the Maxwell equation: $J^i=F^{im}{}_{;m}=(A^{i;m}-A^{m;i})_{;m}$ and compare Eq. (6) with $e=\oint\mathbf{E}\cdot d\mathbf{S}=\oint F^{\mu\nu}dS_{\mu\nu}$.

[3] A. Trautman, "Lectures on Relativity," Kings College, London, 1958 (unpublished).

The conserved quantity thus obtained is

$$T=\int \xi^m T_m{}^n dS_n=\int \xi^m T_m{}^4(-g)^{\frac{1}{2}}d_3x. \tag{10}$$

From the above discussion, it would thus appear that the preferred conservation laws of Lorentz-covariant theories (energy, momentum, and angular momentum) stem from the fact that flat Minkowski space admits 10 independent Killing vector fields. These vector fields are the descriptors of the rigid translations and rotations of the flat Minkowski coordinate surfaces. Each family of flat coordinate surfaces in Minkowski space may be characterized in the following fashion: The equations for the coordinate surfaces may be written

$$\phi=\text{const};\quad \phi_{,m}\phi^{,m}\neq 0, \tag{11}$$

where

$$\xi_i\equiv(\phi_{,m}\phi^{,m})^{-1}\phi_{,i}, \tag{12a}$$

$$\xi_{i;j}+\xi_{j;i}=0. \tag{12b}$$

When we go to more general Riemannian manifolds, it would seem that when there are Killing vectors available they are the appropriate choice to take in Eq. (3) in order to have conservation laws which correspond naturally with those of Lorentz-covariant theories. The generalized energy-flux vector so obtained is everywhere locally defined, in view of the fact that Killing vectors are locally definable (that is, no reference to boundary conditions at infinity are usually required to specify a solution of Killing's equation).

For the most general and typical Riemannian manifolds it is no longer possible to find Killing vector fields. In this event it is no longer clear whether there is any meaningful, relevant choice of a preferred descriptor, ξ^i, which when used in Eq. (3) would give a correspondence with Lorentz-covariant energy or momentum. One thing we may insist upon is that any criteria or conditions placed upon a particular ξ^i in order to single it out should not exclude the hypersurface-orthogonal Killing vector field [i.e., ξ^i such that Eqs. (12) are satisfied] should one exist. If we further require that the descriptor ξ^i be locally defined, we are led rather immediately to the eigenvectors determined by the Weyl tensor.[4] Although this scheme does provide a formal generalization of the concept of energy and momentum, (in fact we have, in general, four such mutually orthogonal descriptors at each point in space-time), the expressions for "energy" and "momentum" so obtained do not seem at present to have any special properties which indicate their utility.

For Riemannian manifolds which are asymptotically flat at spatial infinity, several authors have suggested approaches for determining preferred descriptors which discard the requirements of locality. Fock[5] has observed that the flat coordinate surfaces of Minkowski space may be characterized by the property that they are harmonic, that is, they satisfy

$$\phi_{;m}{}^m=0. \tag{13}$$

Furthermore, if Eq. (13) is imposed together with an outgoing radiation condition, the solutions so obtained are unique up to a Lorentz transformation. One then determines the descriptor as in Eq. (12a) and the energy and momentum flux vectors according to Eq. (13). Fock then conjectures that a similar uniqueness theorem can be proven for general, asymptotically flat monifolds and proposes to carry through the program of determining the energy and momentum (presumed unique up to a Lorentz transformation) in the identical fashion. Although some doubt has recently been cast upon the uniqueness conjecture of Fock,[6] it is worth noting that the harmonic condition Eq. (13) is in complete accord with our requirement of not excluding hypersurface-orthogonal Killing vector fields Eqs. (12). For, differentiating Eq. (12a) and substituting the result into Eq. (12b), we find

$$\phi^{;m}\phi_{;m}(\phi_{;ij}+\phi_{;ji})=2(\phi_{;i}\phi_{;m}\phi^{;m}{}_j+\phi_{;j}\phi_{;m}\phi^{;m}{}_i). \tag{14}$$

If we multiply Eq. (14) by $\phi^{;i}\phi^{;j}$, we obtain

$$2\phi^{;m}\phi_{;m}\phi^{;p}\phi^{;q}\phi_{;pq}=0. \tag{15}$$

Thus, if we contract Eq. (14) on i and j and use Eq. (15) and the second condition of Eq. (11), we readily deduce Eq. (13). That is, ϕ is necessarily harmonic.

Dirac[7] and several other authors[8,9] suggest that the flat coordinate surfaces of Minkowski space may be characterized as being minimal (i.e., a solution of the Plateau problem). Mathematically, the precise requirement is that each family of surfaces satisfies

$$[(\phi^{;m}\phi_{;m})^{-\frac{1}{2}}\phi^{;n}]_{;n} \equiv(\phi^{;m}\phi_{;m})^{-\frac{3}{2}}(\phi^{;n}\phi_{;n}\phi^{;p}{}_p-\phi^{;p}\phi^{;q}\phi_{;pq})=0. \tag{16}$$

Equation (16) is then usually required by these authors in the general case (at times with and at times without supplementary boundary conditions at infinity) at least for the space-like hypersurfaces. Here again we should point out that the uniqueness (up to Lorentz transformation) conjectured by those authors who impose boundary conditions may be subject to some doubt.[6] Nevertheless, it is again worth noting that in view of the fact that a hypersurface-orthogonal Killing vector field satisfies Eqs. (13) and (14), it *a fortiori* satisfies Eq. (16). That is, the ϕ of Eqs. (12) is necessarily minimal as well as harmonic.

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[4] F. Pirani, Phys. Rev. **105**, 1089 (1957). Note particularly the proof that hypersurface-orthogonal Killing vectors are eigenvectors of the Weyl tensor.

[5] V. Fock, Revs. Modern Phys. **29**, 325 (1957).

[6] P. G. Bergmann, Phys. Rev. **124**, 274 (1961); P. G. Bergmann, I. Robinson, and E. Schucking, Phys. Rev. **126**, 1227 (1962).

[7] P. A. M. Dirac, Phys. Rev. **114**, 924 (1959).

[8] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **116**, 1322 (1959).

[9] J. Rayski, Acta Phys. Polon. **9**, 33 (1961); **20**, 509 (1961).

At this point, one is tempted to conjecture that the preferred choice of coordinate surfaces in the general Riemannian manifold are familes of surfaces which are simultaneously minimal and harmonic. However, as we have already indicated earlier, all of the above attempts to generalize energy and momentum to arbitrarily curved manifolds are formal in character, and the "correct" choice, if indeed there is one, must be determined by the use to which we wish to put the resulting conservation laws. One is not particularly interested in a formal definition of energy if it teaches us nothing about the properties of the spaces under consideration.

Of particular physical interest are a class of manifolds which, due to the boundary conditions they satisfy at spatial infinity, appear to be radiating gravitational waves. In the treatment of these solutions by Trautman[10] an essential role was played by Fock's (or de Donder's) harmonic coordinate conditions. The harmonic coordinate conditions were used asymptotically by Trautman in order to show that the energy expressions were finite, and that the energy radiated was positive-definite. The purpose of the remainder of this paper is to investigate whether, at least for the radiative solutions, the harmonic coordinate conditions are truly fundamental or whether any of the other procedures mentioned above for generalizing the energy concept play some role in the understanding of the phenomena of gravitational radiation. In particular, it will be shown that the minimal coordinate system is required asymptotically in order to express the relevant conservation laws in covariant form [Eq. (2)]. Then paper will conclude with a discussion of some results obtained from a procedure for avoiding the specification of asymptotic coordinate conditions entirely. This will be accomplished by placing asymptotic boundary conditions on the descriptor vector fields, ξ^i, in order to assure that the energy-flux vectors obtained via Eq. (3) will be appropriate for the treatment of gravitational radiation.

The principal motivation for the program of dispensing with the harmonic coordinate conditions, and placing the emphasis instead on the descriptor vector fields, is to underline the fact that the preferred, relevant conservation laws for the treatment of gravitational radiation refer to the generators of rigid translations in directions which are asymptotically Killing. The preferred coordinate surfaces are *a fortiori* asymptotically harmonic (as well as minimal).

II. TRAUTMAN RADIATIVE SPACES

Building on a striking analogy with electromagnetic theory Trautman[10] has proposed that, for a space to be regarded as radiating gravitational energy, a coordinate system should exist in which the metric tensor has the asymptotic form

$$g_{ij}=\eta_{ij}+O(r^{-1}), \tag{17}$$

$$g_{ij,k}=i_{ij}k_k+O(r^{-2}), \tag{18}$$

where η_{ij} is the Minkowski metric, k_l is an outward drawn null vector, $i_{ij}=O(r^{-1})$, and $r^2=x^2+y^2+z^2$. In addition, in analogy with the Lorentz condition of electrodynamics, Trautman found it necessary to impose, asymptotically, the harmonic coordinate condition, Eq. (13), in order to obtain coordinate surfaces which are asymptotically flat. This has a consequence

$$(i_{ij}-\tfrac{1}{2}\eta_{ij}i_m{}^m)k^j=O(r^{-2}). \tag{19}$$

Having assumed the existence of one coordinate system in which Eqs. (17), (18), and (19) are satisfied, we can now obtain other such systems by performing coordinate transformations which preserve these conditions. Such transformations are given (by Trautman) as follows:

$$x'^i=x^i+a^i(x), \tag{20}$$

where

$$a^i=O(r^{-1});\quad a_{i,j}=b_ik_j+O(r^{-2}); \tag{21}$$

$$b_i=O(r^{-1});\quad b_{i,j}=c_ik_j+O(r^{-2});\quad c_i=O(r^{-1}). \tag{22}$$

The total energy and the flux of energy is most readily expressible in terms of the von Freud[11] expressions:

$$U_k{}^{[ij]}\equiv(-g)^{-\frac{1}{2}}g_{km}[-g(g^{im}g^{jn}-g^{jm}g^{in})]_{,n}. \tag{23}$$

With this definition the total energy-momentum four-vector may be written

$$P_m(\sigma)=\frac{1}{2K}\oint_S U_m{}^{[pq]}dS_{pq}, \tag{24}$$

where the integral is taken over the surface of a two-sphere at infinity S, which bounds all of a space-like hypersurface, σ. The total energy and momentum radiated between two space-like hypersurfaces σ and σ' is given by

$$p_m=\frac{1}{2K}\int_\Sigma U_m{}^{[pq]}{}_{,q}dS_p, \tag{25}$$

where the integration is taken over the surface of the time-like three cylinder, Σ, which, together with the hypersurfaces σ and σ', forms the boundary of the four-volume between the two space-like hypersurfaces.

If we substitute the asymptotic expressions for the metric tensor Eqs. (17) and (18) into Eq. (24), we find that the integrand falls off like r^{-1}, whereas the surface element increases as r^2. Thus, $P_m(\sigma)$ as defined in Eq. (24) would ordinarily diverge in the limit that the sphere, S, approaches infinite radius. It is at this point that we must impose the harmonic coordinate condition, Eq. (19), in order to assure that the integrand of Eq. (24) falls off as r^{-2}, and thus make the integral con-

[10] A. Trautman, Bull. Acad. Polon. Sci., Classe III, 5, 721 (1957).

[11] P. von Freud, Ann. Math 40, 417 (1939).

vergent. The harmonic condition then has the additional virtue of assuring that the total energy radiated, p_4 of Eq. (25), is non-negative.

III. THE MØLLER PSEUDOTENSOR IN RADIATIVE SPACES

The expressions for energy and momentum used in Eqs. (24) and (25) were obtained from the consideration of the generators of translations along the coordinate axes. That is, the relevant descriptors for energy and the three components of momentum are $\delta_4{}^i$, $\delta_1{}^i$, $\delta_2{}^i$, $\delta_3{}^i$, respectively. However, if we placed one of these descriptors into Eq. (3) we would not, in general, obtain the expression of the corresponding conserved quantity used in Sec. II. The reason is that the expressions for the conserved quantities in Sec. II are obtained from the Einstein canonical pseudotensor, whereas the expressions of Eq. (3) were obtained from the Møller pseudotensor[12] and, in fact, for the case we are now considering of constant descriptor fields, coincide precisely with the Møller pseudotensor.

Møller introduced his expression for the purpose of rectifying a gross shortcoming of the Einstein pseudotensor, namely that it is not even covariant under purely spatial coordinate transformations. The Møller pseudotensor differs from the Einstein pseudotensor by a pure divergence, and therefore generates precisely the same coordinate transformations. In the integrated expressions for the total energy and momentum, the divergence term is convertible into a surface integral over the two-sphere at spatial infinity. Møller has so chosen the divergence that the surface integral vanishes for metrics which asymptotically approach the static spherically symmetric Schwarzschild metric. Thus, for the nonradiative, asymptotically static flat solutions of Einstein field equations the Møller and the Einstein pseudotensors agree as to total energy and momentum in those coordinate systems where the Einstein pseudotensor may be applied legitimately (i.e., asymptotically Minkowskian); and for those coordinate systems which preserve the preferred time direction, where the Einstein pseudotensor is inapplicable (e.g., polar coordinates) the Møller tensor remains valid and continues to give the correct values for the energy and momentum. (It should be noted, however, that under linear coordinate transformations which involve the time the Møller expression no longer transforms as a tensor, in contrast to the Einstein pseudotensor.)

The expression in Eq. (3) was constructed to generalize the Møller pseudotensor to a completely covariant quantity and to exhibit explicitly the descriptor of the coordinate transformation which is generated by the conserved quantity. The region of validity and applicability of Eq. (3) is therefore strongly delimited by the region of applicability of the Møller pseudotensor. In particular, we have no assurance that the Møller expression can be applied meaningfully to radiative spaces whose metrics are characterized by Eq. (17) and (18). For example, we have no knowledge of the significance of the disgarded surface integral which no longer vanishes in the radiative solutions. These questions we will now subject to closer scrutiny.

[12] C. Møller, Ann. Phys. (New York) 4, 347 (1958).

If we define

$$\bar{U}_k{}^{[ij]}\equiv 2U_k{}^{[ij]}-\delta_k{}^iU_m{}^{[mj]}+\delta_k{}^jU_m{}^{[mi]}, \qquad (26)$$

where $U_k{}^{[ij]}$ are the von Freud expressions, Eq. (23), the Møller expression for the total energy-momentum four-vector may now be written:

$$E_m(\sigma)=\frac{1}{2K}\oint_S \bar{U}_m{}^{[pq]}dS_{pq}, \qquad (27)$$

and the Møller expression for the total energy and momentum radiated between two space-like hypersurfaces, σ and σ', is given by

$$e_m=\frac{1}{2K}\int_\Sigma \bar{U}_m{}^{[pq]}{}_{,q}dS_p, \qquad (28)$$

where the regions of integration are defined precisely as in the corresponding expressions, Eqs. (24) and (25), respectively.

In analogy with our earlier discussion of the Einstein pseudotensor, we substitute the asymptotic expressions for the metric tensor, Eqs. (17) and (18), into Eq. (27) and we again find that the integrand falls off like r^{-1}, whereas the surface element increases as r^2. However, in our present case, the harmonic coordinate condition Eq. (19) no longer enables us to avoid the resulting divergence. The coordinate condition now required is readily found to be

$$i_{km}k^m=O(r^{-2}), \qquad (29)$$

which corresponds to requiring the coordinate surfaces asymptotically to satisfy the equation

$$(g^{mn}\phi_{,m})_{,n}=0. \qquad (30)$$

Since this equation is not covariant, the resulting family of surfaces cannot readily be geometrically characterized.

However, if we wish to require the simultaneous validity of the Einstein pseudotensor and the Møller pseudotensor [and the consequently validity of the covariant expression, Eq. (3)], that is, if we require that Eqs. (19) and (29) be simultaneously valid, we can easily characterize the resulting surfaces geometrically; namely the coordinate surfaces asymptotically are simultaneously harmonic and minimal. That this can always be accomplished in a Trautman radiative space may be seen by performing a coordinate transformation of the type given by Eqs. (20), (21), and (22). Such a coordinate transformation will leave invariant Eqs. (17) and (18), as well as the harmonic coordinate condition,

Eq. (19). Under such a coordinate transformation i_{ij} transforms thus:

$$i'_{ij}=i_{ij}+k_ic_j+k_jc_i. \tag{31}$$

It is readily seen that if c_i is chosen such that

$$c_mk^m=-\tfrac{1}{2}(i_m{}^m), \tag{32}$$

then in the new coordinate system Eq. (29) will also be satisfied.

Our freedom to perform coordinate transformations which alter i_{ij} is by no means exhausted by imposing conditions (19) and (29). For, any c_i such that

$$c_mk^m=0 \tag{33}$$

will, when used in Eq. (31), continue to preserve Eqs. (19) and (29). If τ^i is taken to be any time-like vector field (e.g., $\tau^i=\delta^i{}_4$) and we define c_i thus

$$c_i=\tfrac{1}{2}(\tau^pk_p)^{-2}i_{mn}\tau^m\tau^nk_i-(\tau^pk_p)^{-1}i_{im}\xi^m, \tag{34}$$

we observe first that, as a consequence of Eq. (29), Eq. (33) remains valid. If we employ Eq. (34) in Eq. (31) we find that in addition to conditions (19) and (29) we may impose on i_{ij} the condition

$$i_{km}\tau^m=O(r^{-2}). \tag{35}$$

This exhausts the conditions which may simultaneously be imposed in i_{ij}. It is worth noting that as a consequence of Eqs. (19) and (29) we have

$$i_m{}^m=O(r^{-2}). \tag{36}$$

Thus, i_{ij} is explicitly exhibited to have but two independent nontrivial degrees of freedom, both of them spatial and transverse with respect to k_i as we should expect for gravitational radiation.

Having agreed to impose both conditions (19) and (29) on our choice of coordinate systems, both the Einstein pseudotensor and the Møller pseudotensor can be made finite. It does not follow from this that both expressions agree for the values of the total energy and momentum, Eqs. (24) vs (27), or for the energy and momentum radiated between two space-like hypersurfaces, Eqs. (25) vs (28). In order to obtain such an equality, additional conditions would have to be imposed on the r^{-2} terms in the metric tensor and its derivatives. This is not very surprising since, even in the static Schwarzschild solution, the Einstein and Møller expressions agree only in a particular class of coordinate systems, those which decompose the space-time into a three-space orthogonal to the time-like Killing direction and which are asymptotically rectangular at spatial infinity. The Møller expression which transforms as a tensor under all coordinate transformations which do not involve the time, is then presumed to apply when more general coordinate systems are used.

For the radiative spaces considered in this paper there is also, presumably, a class of coordinate systems in which the values given by the Einstein and Møller expressions coincide. Since, by means of descriptor fields, the Møller expression can be generalized more readily than the Einstein, we may then prefer to use it for other coordinate systems. In view of the fact that the Møller expression was contrived to agree with the Einstein expression for the static Schwarzschild solution, we may say with certainty that if a space is initially asymptotically static Schwarzschild, then radiates for a finite interval, and then returns to an asymptotically static Schwarzschild form, both the Møller and the Einstein expressions will agree for the total amount of energy and momentum radiated in that interval.

IV. ASYMPTOTIC KILLING FIELDS

It is evident that the Trautman-Fock device of employing harmonic coordinate systems is a means of selecting the conserved quantities which generate coordinate transformations whose descriptors are in some sense asymptotically Killing. In this section we wish to examine the extent to which the concept of asymptotically Killing can be made precise. Care must be taken to avoid definitions which, though they may be precise, are sterile.

The first indication in Sec. I that the preferred conservation laws are to be associated with Killing vector descriptors, was the observation that the Killing vectors enabled us to construct conserved quantities whose values could be determined by a Gauss-type flux integral over any surface surrounding the sources of the gravitational field. This might suggest that for asymptotic Killing fields one should be able to obtain expression for conserved quantities which are asymptotically surface independent. Duplicating the steps which lead to the derivation of Eq. (4), we are led to conclude that

$$\xi_{i;j}+\xi_{j;i}=O(r^{-(2+\epsilon)}) \tag{37}$$

is implied by the requirement of asymptotic surface-independence. However, Eq. (37) is much too strong a condition. For, in a coordinate system in which the vector field is a descriptor of a rigid translation of a coordinate surface, that is, in a coordinate system in which the descriptor has the form

$$\xi^i=\delta^i{}_a, \tag{38}$$

we find from Eq. (37)

$$g_{ij,a}=O(r^{-(2+\epsilon)}), \tag{39}$$

in contradiction to the requirement Eq. (18). This result should not be particularly surprising, since, if the value of the conserved quantity can be obtained asymptotically in a surface-independent fashion, it is evident that this quantity cannot be radiated to infinity.

It would be better to take the expression "asymptotically Killing" to mean that the vector field to which the expression is to be applied has the property that, in a neighborhood of infinity, it is hypersurface-orthogonal to a surface which, asymptotically (to order r^{-2}), is

simultaneously harmonic and minimal. That is, the surface ϕ satisfies the equations

$$\phi_{;m}{}^{m}=O(r^{-2}) \tag{40}$$

and

$$[(\phi^{;m}\phi_{;m})^{-\frac{1}{2}}\phi^{;n}]_{;n}=O(r^{-2}). \tag{41}$$

The asymptotic Killing field, ξ^i, is then obtained via Eq. (12a).

Given such asymptotic Killing fields it does not as yet follow that the conserved quantities obtained via Eq. (6) will prove to be finite. For this, we need that in the preferred coordinate systems determined by Eqs. (40) and (41), the metric tensor satisfy the Trautman conditions Eqs. (17) and (18). It is possible to characterize the Trautman conditions covariantly by placing equivalent conditions on a vierbein of vector fields, although such a procedure is of questionable practical value. We will very briefly sketch at this point how such a characterization may be accomplished, since a few of the resulting expressions have interesting consequences.

In a coordinate system which satisfies Eqs. (17), (18), (29), (35), and (36), consider the four-vector fields $\xi_{a|}{}^{i}$ given by Eq. (38). It is readily verified that each of these four vector fields is asymptotically Killing (in the above precisely defined sense of the words), and in addition satisfy the following relations:

$$\xi_{a|i;j}+\xi_{a|j;i}=i_{ij}k_a+O(r^{-2})=O(r^{-1}), \tag{42}$$

$$k^a k_a=O(r^{-1}), \tag{43}$$

$$i_{ij}=O(r^{-1}), \tag{44}$$

$$\xi_{a|}{}^{m}{}_{;m}=O(r^{-2}), \tag{45}$$

$$k^m(\xi_{a|m;i}+\xi_{a|i;m})=O(r^{-2}), \tag{46}$$

$$\tau^m(\xi_{a|m;i}+\xi_{a|i;m})\equiv\xi_{4|}{}^{m}(\xi_{a|m;i}+\xi_{a|i;m})=O(r^{-2}), \tag{47}$$

$$\xi_{a|}{}^{m}\xi_{b|m}=\eta_{ab}+O(r^{-1}), \tag{48}$$

$$\xi_{a|}{}^{m}\xi_{b|}{}^{i}{}_{;m}-\xi_{b|}{}^{m}\xi_{a|}{}^{i}{}_{;m}=O(r^{-2}). \tag{49}$$

If we define what we mean by r via

$$r^{-1}\equiv k^m{}_{;m}/2, \tag{50}$$

it is possible to turn the above conditions about and deduce the Trautman metric in the preferred coordinate system. In this regard, Eq. (49) is particularly important for it is the condition which assures us of the possibility of finding a coordinate system in which, at least asymptotically (to the correct order), $\xi_{a|}{}^{i}=\delta_a{}^i$ for all a. It is also the condition which says that the coordinate transformations, generated by the four conserved quantities obtained from the four descriptors $\xi_{a|}{}^{i}$, commute.

One other relation, which is readily deducible, deserves particular mention

$$\xi_{4|}{}^{m}k^n(\xi_{a|m;n}-\xi_{a|n;m})=O(r^{-2}). \tag{51}$$

It is this condition which assures us that the surface integral, Eq. (6), yields a finite results, since the two surface over which the integration is to be performed has as surface element

$$dS_{ij}\sim k_i\xi_{4|j}-k_j\xi_{4|i}. \tag{52}$$

V. ENERGY

The condition expressed in Eq. (35), or equivalently in Eq. (47), yields a preferred status for the time-like descriptor $\xi_{4|}{}^{i}=\tau^i$. This has an important consequence for the resulting conserved quantity, which we wish to identify with the energy. In view of the asymptotic relations Eqs. (45) and (47), let us consider in general the following conditions on a time-like vector field τ^i:

$$\tau^m{}_{;m}=0, \tag{53}$$

$$\tau^m(\tau_{m;i}+\tau_{i;m})=0. \tag{54}$$

We note that Eqs. (53) and (54) form precisely half of the Killing equations. We may therefore call such a vector field semi-Killing. In view of the fact that these are five differential equations for the four components of τ^i, they are somewhat overdetermined. It is therefore not clear whether the assumption of the existence of a semi-Killing vector field is a limitation on the generality of the Riemann space. However, we found it no limitation to impose such five relations asymptotically to $O(r^{-2})$. The assumption of the existence of a time-like semi-Killing vector field is equivalent to the assumption that we can find a coordinate system in which the metric tensor takes the form:

$$g_{i4,4}=0, \tag{55a}$$

$$(\det g_{ij})_{,4}=0. \tag{55b}$$

It is well known that there is no difficulty to imposing any four of the above five conditions locally.

Let us assume that we can find a time-like hypersurface-orthogonal semi-Killing vector field τ^i. The conserved quantity determined by it (i.e., the energy) is [via Eqs. (3) and (5)]

$$E=\frac{1}{K}\int(\tau^{i;m}-\tau^{m;i})_{;m}dS_i. \tag{56}$$

Since it is understood that the integration is to be performed over a surface of constant time, we have [note Eq. (12a)]

$$dS_i\sim\phi_{,i}=(\tau^p\tau_p)^{-1}\tau_i=-|\tau^p\tau_p|^{-1}\tau_i. \tag{57}$$

[The last step in Eq. (57) results from the tacit assumption throughout this paper that the signature of the metric is (1, 1, 1, −1).]

Comparing Eqs. (56) and (57) we are led to consider

$$\begin{aligned}-|\tau^p\tau_p|^{-1}\tau_i(\tau^{i;m}-\tau^{m;i})_{;m}&\\ =-|\tau^p\tau_p|^{-1}\tau_i(\tau^{i;m}+\tau^{m;i}-2\tau^{m;i})_{;m}&\\ =-|\tau^p\tau_p|^{-1}\{-\tau_{i;m}(\tau^{i;m}+\tau^{m;i})-2\tau^i\tau^jR_{ij}&\\ +[\tau_i(\tau^{i;m}+\tau^{m;i})]_{;m}-2\tau^m{}_{;m}{}^{i}\tau_i\}.&\end{aligned} \tag{58}$$

We have made use of the commutation relations for covariant derivatives as well as an integration by parts. The last two terms in Eq. (58) vanish as a consequence of Eqs. (53) and (54). We conclude from Eq. (58)

$$(\tau^{i;m}-\tau^{m;i})_{;m}dS_i\sim|\tau^p\tau_p|^{-1}\times[\tfrac{1}{2}(\tau_{i;j}+\tau_{j;i})(\tau^{i;j}+\tau^{j;i})+2\tau^i\tau^jR_{ij}]\geq 0. \quad (59)$$

The positive-definiteness of the first term in the brackets is a consequence of Eq. (54). The positive-definiteness of the second term in the brackets is a consequence of the property of all known energy-momentum tensors for matter distributions. Thus, the energy density, as determined by a semi-Killing descriptor, is necessarily positive definite. We have, in addition:

$$E=0\leftrightarrow\tau_{i;j}+\tau_{j;i}=R_{ij}=0. \quad (60)$$

We can therefore paraphrase a well-known global theorem[13] and say that under suitable boundary conditions at infinity and with the exclusion of singularities, $E=0$ implies that the space is locally flat.

This result, which may be regarded as a version of Mach's principle, depends on the existence of a hypersurface-orthogonal time-like semi-Killing vector field globally. If we now return to the case of asymptotic semi-Killing vector field, we find that in place of the zeros on the right-hand side of Eqs. (53) and (54), we really have $O(r^{-2})$. This does not appear to suffice for a proof of the positive-definite character of the energy. What we apparently require to carry through a proof of the positive character of the energy is that the right-hand side of Eqs. (53) and (54) should behave as $O(r^{-(2+\epsilon)})$. It does not appear particularly difficult to accomplish this by means of a coordinate transformation, however such an investigation would be beyond the scope of the present paper, since it would require examining the terms in the metric tensor which go as $O(r^{-2})$. It is fair to say that the results of Eqs. (59) and (60) encourage us in the belief that we are at least on the right track.

VI. ANGULAR MOMENTUM

The vector fields which we have considered, thus far, described coordinate transformations which were rigid translations of the coordinate surfaces. The conserved quantities associated with them are therefore suitable generalizations of the linear momentum and energy. This is particularly evident in view of the commutation relations Eq. (49). We shall now present a brief discussion of the angular momentum radiated by an asymptotically Trautman gravitational field.

In flat Minkowski space the angular momentum is associated with the rigid rotations of the coordinate surfaces described by the Killing vector fields of the form:

$$\xi_i=\epsilon_{ipqr}x^pa^{qr}, \quad (61)$$

[13] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Electromagnetisme* (Masson et Cie, Paris, 1955).

where a^{qr} is an arbitrary constant antisymmetric tensor. The six possible independent choices for a^{qr} lead to six independent Killing vector fields, and therefore six components of angular momentum.

If we require that the vector fields of Eq. (61) be hypersurface-orthogonal, the tensor a^{qr} has to satisfy the additional (necessary and sufficient) condition:

$$\det a^{pq}=0. \quad (62)$$

Thus, a^{pq} is a simple bivector and can always be written in the form

$$a^{pq}=\lambda^p\mu^q-\lambda^q\mu^p, \quad (63)$$

where λ^p and μ^q are constant vector fields. In this event the vector field ξ_i can be written in the form Eq. (12a), where the hypersurface, $\phi=\text{const}$, is harmonic and minimal.[14]

If the six possible independent choices for the simple bivector field, a^{qr}, are obtained by forming the six possible antisymmetric combinations of pairs of the vierbein, $\xi_{a|}{}^i$, of section IV, we obtain the usual components of angular momentum. The three descriptors, ξ_i, formed via Eqs. (61) and (63) from those three pairs of vectors of the vierbein which contain $\xi_{4|}{}^i$, yield, via Eq. (3), the three components of what one ordinarily understands by the expression "the angular momentum." The three remaining components refer really to the motion of the center-of-mass. Properly speaking, the reason we are able to recognize that the above descriptors are associated with the corresponding integrals of motion stems from the fact that the descriptors satisfy the usual commutation relations for the Lorentz group. Thus, for example, if we label the descriptors associated with the x, y, and z components of the angular momentum by $\lambda_{1|}{}^i$, $\lambda_{2|}{}^i$, and $\lambda_{3|}{}^i$, respectively, it is easy to see that we obtain

$$\lambda_{1|}{}^m\lambda_{2|}{}^i{}_{;m}-\lambda_{2|}{}^m\lambda_{1|}{}^i{}_{;m}=\lambda_{3|}{}^i, \quad (64)$$

as well as the two other similar relations found by cycling the digits 123.

If we now return to the case of a curved space satisfying the Trautman boundary conditions, we must determine what is the correct generalization of Eq. (61) for the purpose of obtaining a useful and meaningful definition of angular momentum. The criteria which we must employ are: (a) Eq. (51) must remain valid so that the resulting integral of motion [Eq. (6)] will not diverge; (b) Eq. (64) (as well as the five other commutation relations) must remain valid to $O(r^{-2})$ in order to assure that the commutator of each two such transformations is the descriptor for the correct integral of motion, [when used in Eq. (6)], in accordance with homo-

[14] For example, for the z component of angular momentum where only a^{34} is taken unequal to zero and where $\xi_i=(y,\ -x,\ 0,\ 0)$, take $\phi=\arctan(y^{-1}x)$. We readily confirm the validity of Eqs. (12a), (12b), (13), and (16). We see from this example that ξ_i generates a rigid rotation in the sense that it describes a rigid motion of the (flat) surfaces of constant longitude into one another.

geneous Lorentz group; (c) the vector fields should be determined in a covariant fashion. Condition (c) is no limitation at all since our coordinate frame is already uniquely determined by the vierbein $\xi_{a|}{}^{i}$ constructed in Sec. III.

Possible candidates for the correct generalization of Eq. (61) for the descriptor which yields the z component of angular momentum, for example could be

$$\lambda_{3|}{}^{i}=(y,\,-x,\,0,\,0) \tag{65}$$

or

$$\lambda_{3|i}=(y,\,-x,\,0,\,0) \tag{66}$$

or perhaps

$$\lambda_{3|i}=2(-g)^{1/2}\epsilon_{imnp}k^{m}\xi_{3|}{}^{n}\xi_{4|}{}^{p}(k^{q}{}_{;q})^{-1}. \tag{67}$$

Although the three vector fields defined by Eqs. (65), (66), and (67) coincide in Minkowski space with the correct descriptor for the z component of angular momentum, all three vector fields are different to $O(r^{0})$ in radiative solutions. The generalization represented by Eq. (65) satisfies condition (b) precisely, however it does not satisfy condition (a). The generalization represented by Eq. (66) satisfies condition (a), but not condition (b). The generalization represented by condition (67) satisfies neither (a) or (b). It is mentioned since in appearance it seems a most likely generalization of Eq. (61).

We have as yet been unable to construct a set of vector fields consistant with (a) and (b), above. We conjecture that the correct choice should be vectors which are asymptotic Killing in the precise sense of Sec. III. [None of the above generalizations of Eq. (61) appear to have this property.] However, we have as yet been unable to determine whether the asymptotic Killing fields are in fact consistant with both conditions (a) and (b).

It may well be that in the radiative solutions it is not possible to define an expression for angular momentum in conformity with (a) and (b). This possibility may be a consequence of the nonintegrability of the affine connection even asymptotically, a property of radiative solutions recently found by Bergmann, Robinson, and Schücking.[6] Thus, the commutation relations for the Lorentz group [condition (b)] may have to modified, and the concept of angular momentum thereby altered. However, it is as yet premature to discuss such possibilities.[15]

If we are prepared, at least for the time being, to relinquish the possibility of writing an expression for the total angular momentum, and ask only for an expression which yields the angular momentum per unit time radiated by the gravitating source, then it easily confirmed that the descriptor fields of the type given in Eq. (65) will give a convergent result, in addition to preserving, at least formally, condition (b).

VII. CONCLUSION

We observed that the preferred conservation laws of physical theories are properly associated with the generators of transformations in the Killing directions. By properly defining what is meant by asymptotic Killing vector fields we were able to extend the definition of preferred conservation laws to situations where, in the strict sense of the word there are no Killing fields, but where it still becomes meaningful to speak of the energy, momentum, and possibly angular momentum radiated by the gravitational field. We were able to understand the circumstance which singles out the energy as a positive-definite quantity, with the possible global consequence that vanishing energy implies that the space is flat.

The particular definition of the term "asymptotic Killing" involved using surfaces which are simultaneously minimal and harmonic. We showed that in Trautman radiative solutions it was no restriction to assume the existence (asymptotically) both of families of such surfaces, and of a time-like semi-Killing vector field. It is natural to conjecture that such geometric structures can always be found locally and would thereby provide a local, meaningful scheme for constructing energy, momentum, and angular-momentum densities. It might also be possible to trace through a relationship between the vector fields here conjectured and the vierbein determined by the eigen-directions of the Weyl tensor. But this possibility seems more remote.

[15] The customary treatment of angular momentum [e.g., V. Fock, *Theory of Space, Time, and Gravitation* (Pergamon Press, New York, 1959)] requires an analysis of the form of the metric tensor accurate to $O(r^{-3})$ which is beyond the intended scope of this paper. Furthermore, a consideration of the extent of the validity of the usual commutation relations is invariably ignored.

First Reprinting, March 1963