THE

PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 127, NO. 4 AUGUST 15, 1962

Phonon-Phonon Interactions in Bose Systems*

MARK NELKIN

John Jay Hopkins Laboratory for Pure and Applied Science, General Atomic Division of General Dynamics Corporation, San Diego, California

(Received March 15, 1962)

The stability of phonons in Bose systems is considered. It is noted that the phonons in helium II are stable against decay into two or more phonons, in contrast to the low-density Bose gas where the lifetime for decay of one phonon into two is finite and has been calculated by several authors. The consequences of phonon stability for allowable mechanisms of phonon-phonon interaction are considered. It is concluded that virtual three-phonon processes may still occur although energy-conserving three-phonon processes cannot take place. The inclusion of such virtual processes makes it plausible that the interactions between excitations can be weak, even though the linewidths measured in neutron scattering are larger than the line shifts. The quantum hydrodynamics of Landau and Khalatnikov including nonlinear terms is considered as a calculational model. In this theory the stability of the phonons is indeterminate and must be legislated. If the phonons are assumed unstable against decay, the phonon lifetime in the low-density and long-wavelength limit can be calculated, and is in agreement with the result for the atomic theory of the low-density Bose gas. This lends considerable plausibility to the quantitative validity of the Landau-Khalatnikov calculation of phonon-phonon scattering in helium II via an almost energy-conserving intermediate one-phonon state. The paper concludes by comparing the results of an idealized neutron scattering experiment as calculated from the Feynman-Cohen point of view, and from the point of view of the theory of the low-density Bose gas.

I. INTRODUCTION

A QUANTITATIVE description of the thermodynamic functions of helium II at low temperatures can be given in terms of the energy vs momentum relationship for the low-lying excited states of the fluid. This relationship has been accurately determined from cold-neutron scattering experiments,[1] and the thermodynamic quantities calculated using the neutron scattering data are in good agreement with experiment.[2] The agreement between the directly measured and indirectly inferred $E(p)$ curve for the low-lying excited states has given striking confirmation to Landau's original picture[3] of helium II at low temperatures as a dilute gas of long-lived elementary excitations (normal fluid) moving in a background of fluid in the ground state (superfluid). The atomic basis for this theory as developed by Feynman[4] and by Feynman and Cohen[5] has also been given quantitative confirmation. The measured $E(p)$ curve[1] is shown in Fig. 1. For small p, $E=pc$ where c is the sound velocity. We see that the neutron scattering experiments are consistent with the measured sound velocity. The long-wavelength excitations corresponding to the linear portion of the $E(p)$ curve are known as phonons. These excitaions, which imply the characteristic T^3 behavior of the specific heat, are dominant at sufficiently low temperatures. At somewhat higher temperatures (between 0.5 and 1.0°K depending on the quantity being considered) the excitations near the minimum of the $E(p)$ curve, which are historically[3] called "rotons," become important.

When one considers effects dependent upon the interactions between the elementary excitations, the theoretical understanding is much less satisfactory. The most direct information is again obtainable from neu-

* Work performed, in part, at the Institute for Theoretical Physics, University of Utrecht, Netherlands.

[1] D. G. Henshaw and A. D. B. Woods, Phys. Rev. **121**, 1266 (1961). Reference to earlier work is given here.

[2] P. J. Bendt, R. D. Cowan, and J. L. Yarnell, Phys. Rev. **113**, 1386 (1959).

[3] L. Landau, J. Tech. Phys. (U.S.S.R.) **5**, 71 (1941).

[4] R. P. Feynman, Phys. Rev. **94**, 262 (1954); For a review, see article by R. P. Feynman in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1956), Vol. I.

[5] R. P. Feynman and M. Cohen, Phys. Rev. **102**, 1189 (1956).

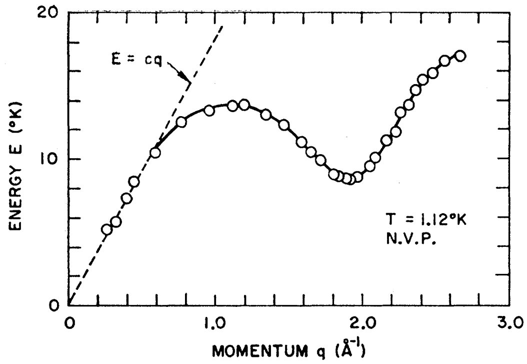

FIG. 1. The measured energy-momentum relationship (reference 1) for the elementary excitations of He II. The dashed line through the origin is drawn with a slope corresponding to the measured sound velocity of 237 m/sec.

tron scattering by observing the shift and the width of the line in the scattered neutron spectrum as a function of temperature. This shift and width are directly related to the asymptotic behavior of the propagator for the elementary excitation; this propagator being a basic concept of the theory.[6] Experimental information is available for the shift and width of the "roton" energy,[1] and Cohen has considered[7] the relationship between this information and the effects of the roton-roton interaction on the thermodynamic functions.

Although the most important effects of interactions between the excitations on the thermodynamic functions will be due to the roton-roton interaction, we restrict our attention in this paper to the simpler problem of phonon-phonon interactions. The phonon excitations have many properties in common for all Bose fluids arising from their relationship to density fluctuations. This has been discussed by Feynman[4] in a paper which gives strong plausibility for Landau's[3] hydrodynamical approach. In concentrating on the long-wavelength excitations there is the additional advantage that a quantitative microscopic theory exists for the special case of the low-density Bose gas with repulsive interactions.[8]

In considering the mechanisms of phonon-phonon interaction in the low-density Bose gas as a guide to these mechanisms in liquid helium, we are immediately led to the significant conceptual difference that a phonon in helium cannot decay into two phonons with conservation of energy and momentum[9,10] while a phonon (as defined by perturbation theory) in the low-density Bose gas can so decay.[8] The consequences of this difference for allowable mechanisms of phonon-phonon interaction in helium are discussed in Sec. II.

[6] D. Pines, *The Many-Body Problem* (W. A. Benjamin, New York, 1961).
[7] Michael Cohen, Phys. Rev. **118**, 27 (1960).
[8] See, for example, T. D. Lee and C. N. Yang, Phys. Rev. **112**, 1419 (1958).
[9] L. P. Pitaevskii, Soviet Phys. JETP **9**, 830 (1959).
[10] L. D. Landau and I. M. Khalatnikov, Soviet Phys.–JETP **19**, 637, 709 (1949) [available in English as Atomic Energy Commission Reports, AEC tr-1255 and AEC tr-1256 (unpublished)].

The lifetime for a long-wavelength phonon at absolute zero temperature in the low-density Bose gas has been calculated[8] to be independent of the interaction between the particles. This suggests some sort of a hydrodynamical origin for the decay mechanism. In Sec. III we consider the mechanisms for phonon decay in Landau's quantum hydrodynamics. The lifetime at $T=0$ is found to be indeterminate without an arbitrary specification of the infinite phonon self-energy terms appearing in the theory. If these are specified to give an $E(p)$ curve which allows one phonon to decay into two, then the hydrodynamical theory leads to the correct phonon lifetime. This suggests that a correct hydrodynamical theory for the phonon-phonon interaction in helium can be constructed, and lends plausibility to the calculation of this interaction that was made along these lines by Landau and Khalatnikov.[10] Finally, in Sec. IV we summarize the consequences of the arguments presented here for neutron scattering experiments.

II. MECHANISMS OF PHONON-PHONON INTERACTION

Recent work on the quantum mechanical many-body problem has led to the description of the low-lying excited states of many-particle systems as long-lived elementary excitations or quasi-particles.[6] In most theoretical treatments these quasi-particles have a finite lifetime for decay even at absolute zero temperature when collisions between quasi-particles are of negligible frequency. The mechanisms for quasi-particle decay arise from terms which are neglected in the lowest order approximation of the theory, but the mechanisms are always present. For Bose systems the only low-lying states are the phonon states, and the mechanisms for phonon decay require the transformation of phonons initially present into other phonons with different energy and momentum.

The two cases of Bose systems which have been explicitly calculated lead to phonons which are unstable in the sense that they have a finite lifetime at $T=0$. The lattice vibrations of an anharmonic solid are described in terms of phonons which can decay into two phonons.[11] For the low-density hard-sphere Bose gas a phonon can decay into two phonons with conservation of energy and momentum.[8] In both of these cases the dominant mechanism of phonon-phonon interaction at low temperatures is a cubic term in the Hamiltonian which describes the annihilation of one phonon with the creation of two new phonons plus the Hermitian conjugate term in which two phonons are annihilated and one created.

A reasonable model Hamiltonian for describing a system of interacting phonons in a Bose fluid would

[11] R. E. Peierls, *Quantum Theory of Solids* (Oxford University Press, New York, 1954).

therefore appear to be

$$H=H_0+V_3, \quad (1)$$

where

$$H_0=\sum_{\mathbf{p}} E_0(p)a_{\mathbf{p}}^*a_{\mathbf{p}}, \quad (2)$$

and

$$V_3=\sum_{\mathbf{p},\mathbf{q}} M(\mathbf{p},\mathbf{q})a_{\mathbf{q}}^*a_{\mathbf{p}-\mathbf{q}}^*a_{\mathbf{p}}+\text{H.c.} \quad (3)$$

The $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^*$ are the annihilation and creation operators for the phonons, satisfying the Bose commutation rules

$$[a_{\mathbf{p}},a_{\mathbf{q}}]=0, \quad [a_{\mathbf{p}},a_{\mathbf{q}}^*]=\delta_{\mathbf{p},\mathbf{q}}. \quad (4)$$

If V_3 can be considered as a small perturbation, then the lifetime of a phonon for decay at $T=0$ can be calculated from first-order perturbation theory as

$$\frac{1}{\tau}=\lambda(p)=\frac{18\Omega}{(2\pi)^2}\int d^3\mathbf{q}\,|M(\mathbf{p},\mathbf{q})|^2 \times\delta[E_0(p)-E_0(q)-E_0(\mathbf{p}-\mathbf{q})]. \quad (5)$$

We use a system of units in which $\hbar=1$, and have made the substitution

$$\sum_q \rightarrow \frac{\Omega}{(2\pi)^3}\int d^3\mathbf{q},$$

where Ω is the volume of the system. The factor of eighteen in (5) is most easily seen if we consider λ as twice the imaginary part of the self-energy associated with the diagram of Fig. 2.

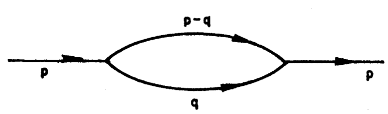

FIG. 2. The lowest order diagram contributing to the phonon self-energy.

There are three ways of choosing the creation operator for the final phonon, three ways of choosing the annihilation operator for the initial phonon, and two ways of pairing the intermediate state phonons.

We introduce a polar coordinate system with polar axis along q so that

$$(\mathbf{p}-\mathbf{q})^2=p^2+q^2-2pq\mu,$$

and transform (5) to the form

$$\lambda=\frac{9\Omega}{\pi}\int_0^p q^2dq\,|M(p,q,\mu_0)|^2 \times\left(\left|\frac{\partial E_0(\mathbf{p}-\mathbf{q})}{\partial\mu}\right|_{\mu_0}\right)^{-1}\int_{-1}^{1}\delta(\mu-\mu_0)d\mu, \quad (6)$$

where $\mu_0(p,q)$ is determined by

$$E_0(p)=E_0(q)+E_0((p^2+q^2-2pq\mu_0))^{1/2}. \quad (7)$$

In the special case that $E_0(p)=cp$, $\mu_0=1$ and the δ-function singularity occurs exactly at the limit of the range of integration so that the integral in (6) is not properly defined. This case corresponds to the decay of a phonon into two phonons moving parallel to the original direction of motion. The phase space available for this is vanishingly small so that a careful evaluation in this limiting case will give a vanishing decay constant. If $[E_0(p)/p]$ is an increasing function of p, then $\mu_0<1$ and one phonon can decay into two phonons moving a small angle to each other so that $\lambda(p)$ as defined by (2) is nonvanishing. If, however, $[E_0(p)/p]$ decreases as p increases, then $\mu_0>1$ and the decay of one phonon into two or more phonons is not energetically possible. The situation for the phonons in a solid is more complicated because of the separate longitudinal and transverse branches of the $E(p)$ vs p curve and because of the possibility of umklapp processes arising from the "quasi-momentum conservation" implied by the lattice structure.[11] For a fluid, however, we see that the stability of the long-wavelength phonons depends only on the way in which $E(p)/p$ varies with p. The low-density hard-sphere Bose gas has[8]

$$E_0(p)=\left[\frac{4\pi a\hbar^2\rho p^2}{m^2}+\left(\frac{p^2}{2m}\right)^2\right]^{1/2}, \quad (8)$$

where ρ is the density and a is the hard-sphere diameter. This leads to the possibility of one phonon decaying into two. The lifetime for this has been calculated by many authors[8,12–14] and is given in the long-wavelength limit by

$$\tau=(160\pi\rho\hbar^4/3p^5). \quad (9)$$

For a weak, repulsive interaction where the original Bogolyubov[15] method can be applied in a straightforward manner without the complications of replacing the potential by a proper two-particle scattering matrix, we have obtained the same answer[16] which has the notable property of being independent of the interaction between particles. The origin of this independence is indicated by the calculation in Sec. III.

There appears to be an important difference, however, between the phonons defined by perturbation theory in the low-density gas and the observed phonons in liquid helium II. The energy vs momentum curve for the elementary excitations as determined by slow-neutron scattering[1] indicates that one phonon cannot decay into two or more phonons with conservation of energy and momentum. If we assume that the phonons are the only low-lying excitations of the system, as made plausible by Feynman,[4] we must conclude that the

[12] S. Beliaev, Soviet Phys.—JETP **7**, 299 (1958).
[13] N. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959).
[14] F. Mohling and A. Sirlin, Phys. Rev. **118**, 370 (1960).
[15] N. Bogolyubov, J. Tech. Phys. (U.S.S.R.) **11**, 23 (1947).
[16] The calculation is similar to those reported in the literature and will not be given here. The result is most easily obtained by using several algebraic properties of the Bogolyubov transformation given in reference 14. In the limit of low density, the phonons are always unstable, independent of the range of the interaction, and the lifetime remains independent of the interaction.

phonons are stable elementary excitations of the system or, in other words, exact eigenstates.[17] This will be true at least up to some finite momentum above which decay into two phonons is energetically possible. This question of an instability threshold and its experimental consequences has been considered by Pitaevskii.[9] We will consider further the consequences of the stability of the phonons for small momentum.

The stability of the elementary excitations in helium II is also supported by the calculations by Feynman[4] and by Feynman and Cohen[5] of the energy vs momentum curve for the excitations in terms of the measured static structure. These calculations lead to predicted stability extending well beyond the "roton" minimum, in agreement with the experiments on inelastic neutron scattering.

It has been pointed out by Cohen[7] that the stability of the elementary excitations can be experimentally determined, in principle, from neutron scattering experiments. If the width of the line in the scattered neutron spectrum approaches zero as the temperature approaches zero, then the excitations are exact eigenstates of the system. It would appear that this is likely to be the case in helium II.

Although a one-phonon state is an exact eigenstate, a state with more than one phonon is not. This situation can be described in terms of phonon-phonon scattering processes. Perhaps the simplest way to describe these processes would be to consider the physical phonons as the eigenstates of some H_0 of the form (2), with $E_0(p)$ the energy momentum relationship that would be measured by a neutron scattering experiment. Since a one-phonon state is an eigenstate of the total Hamiltonian and of H_0, and interaction of the form V_3 as given by (3) must be excluded and the simplest interaction that one could consider is a direct-scattering term of the form

$$V_s=\sum_{\mathrm{l},\mathrm{m},\mathrm{p},\mathrm{q}}(lm|v|pq)a_{\mathrm{l}}^*a_{\mathrm{m}}^*a_{\mathrm{p}}a_{\mathrm{q}}. \qquad (10)$$

This is the point of view that has been taken by Cohen,[7] but it is not necessary, as he asserts, to consider only models in which terms like V_3 are excluded. Such terms are present both in the dynamical theory of the low-density Bose gas[8,15] and in the quantum hydrodynamic theory of Landau and Khalatnikov.[10] They may also be present in a correct theory of phonon-phonon interactions in helium.

It is of some interest to examine the case in which the creation operators appearing in the dynamical theory create approximate eigenstates, but the neutrons couple with finite probability to the exact one-phonon eigenstate. In such a case, (1) is an acceptable Hamiltonian, but the eigenstates of H cannot be separately eigenstates of H_0 and of V_3. The phonon energy as measured by a neutron scattering experiment at $T=0$ will not be $E_0(p)$, but will be $E_0(p)$ plus the self-energy due to the virtual emission and reabsorption of phonons through the perturbation V_3. The lowest order contribution to the self-energy will be given in second-order perturbation theory as

$$\Delta E_{(2)}(p)=18\sum_q \frac{|M(\mathbf{p},\mathbf{q})|^2}{E_0(p)-E_0(q)-E_0(\mathbf{p}-\mathbf{q})}. \qquad (11)$$

As long as the energy denominator in (11) cannot vanish, the self-energy will be real, and the one phonon state will remain an exact eigenstate. The renormalized phonon energy

$$E(p)=E_0(p)+\Delta E_{(2)}(p)+\cdots \qquad (12)$$

will be the observable quantity. If the self-energy is real it is possible to transform the problem so that the physical phonon states with energy $E(p)$ are eigenstates of the transformed H_0 and the transformed V contains an infinite series of terms, of which the simplest is of the form (10). All these terms will contain at least two annihilation and two creation operators and will give zero when operating on a one-phonon state.

An example of a system with the qualitative features of the preceding paragraph is given by quantum electrodynamics. A state containing a single physical electron is an exact eigenstate of the total Hamiltonian, but not of H_0 or V separately. The observable electron energy is the sum of two separately unobservable portions; the bare electron energy, and the self-energy due to the virtual emission and reabsorption of photons. There is the significant difference that the quantum which exchanges the force between the electrons is distinct from the particles themselves, while in our problem the phonon is both particle and quantum of the force field.

Both points of view described above lead to a temperature-dependent self-energy which has a real and an imaginary part. The difference between the finite-temperature and zero-temperature self-energies is observable as the shift and width of the line in the scattered neutron spectrum. An exact solution for a given Hamiltonian will give the same observable line shifts and widths in either representation. In practice, however, one truncates the interaction Hamiltonian after a finite number of terms such as (3) and (10). If one considers the two representations with truncated Hamiltonians, there will be observable differences between the two points of view.

The above considerations allow for a possible qualitative explanation of a difficulty pointed out by Cohen.[7] Working with a direct-scattering interaction of the type (10), he proved that a weak interaction between the excitations implies line shifts which are large compared to linewidths. The quantitative success of calculations based on a weakly interacting gas of excitations[2] in

[17] It has been pointed out to the author that the apparent difference between helium II and the low-density Bose gas may arise from the use of perturbation theory in the theory of the low-density gas. It seems reasonable that the existence of a stable one-phonon state is common to helium II and the low-density hard-sphere gas.

reproducing the thermodynamic functions of helium II from the neutron scattering data implies that the interactions are weak. On the other hand, the observed linewidths in neutron scattering are larger than the line shifts.[1] One possible conclusion from this apparent paradox is that the actual interaction between the excitations is not well described by a direct scattering of the form (10). We know from the work of Landau and Khalatnikov[10] that an almost energy-conserving three-phonon term plays an important role in the interaction between the bare phonons. It does not seem likely that this interaction is well represented by a direct scattering interaction between the "clothed" phonons.

In all approximate dynamical treatments of Bose systems,[10,15] the interaction between the bare phonons is complicated and contains several terms. Since different contributions to the line shift can have different signs while all contributions to the width are intrinsically positive, it seems likely that a more complicated interaction would not restrict the ratio of width to shift even if the effective interaction were in some sense weak. An intuitively reasonable criterion for using quasi-particles with temperature-dependent excitation energies as in reference 2 is that both the width and shift be small compared to the zero-temperature excitation energy. This criterion is reasonably well satisfied in helium II up to 1.8°K, where the ratio of roton line width to roton energy is $\frac{1}{4}$.

In the preceding we have used the concepts of linewidth and shift rather loosely. We have implicitly assumed that these are associated with an asymptotic behavior of the form $\exp(iEt-\Gamma t)$ for the one-phonon propagator. We are, therfore, assuming a Lorentzian line shape. To the extent that the line shape is to be described in more detail, it is necessary to go beyond the one-phonon propagator and explicitly include multiphonon processes, i.e., to calculate line shapes, it is necessary to know the interaction of the probe with the system.

III. QUANTUM HYDRODYNAMICS CONSIDERATIONS

Since the original work of Landau[18] it has been known that many of the properties of superfluid helium can be understood in terms of a quantized hydrodynamics which does not specifically refer to the interacting particles. In this section we examine some of the properties of the nonlinear terms in this theory in order to understand how considerations of phonon stability enter in the hydrodynamic approach. We also present a simple calculation indicating a hydrodynamic origin for the lifetime of a long-wavelength phonon in the low-density Bose gas.

The description in terms of a continuum fluid has been given considerable plausibility by Feynman's[4] arguments concerning the dominance of long-wavelength density fluctuations in determining the low-lying excitations of the fluid. Feynman proposed that a phonon of momentum $\mathbf{p}$ have a wave function given by $\rho_{\mathbf{p}}\phi_0$, where ϕ_0 is the ground state wave function and

$$\rho_{\mathbf{p}}=N^{-1/2}\sum_{j=1}^{N}e^{-i\mathbf{p}\cdot\mathbf{x}_j} \tag{13}$$

is the operator representing a density fluctuation. The sum in (13) runs over all the atoms in the fluid. This wave function implies the familiar relationship

$$E(p)=p^2/2mS_0(p), \tag{14}$$

where $S_0(p)$ is the x-ray structure factor at zero temperature, and is given by

$$S_0(p)=(\phi_0|\rho_{-\mathbf{p}}\rho_{\mathbf{p}}|\phi_o). \tag{15}$$

$S_0(p)$ is the mean square density fluctuation of momentum p in the ground state.

It has been shown by Pitaevskii[19] that (14) can be derived from Landau's continuum fluid picture. On the other hand, it is necessary in Feynman's theory in order to get the correct phonon spectrum from (14) to argue that $S_0(p)$ for small p is given by

$$S_0(p)=p/2mc. \tag{16}$$

This form for $S_0(p)$ has not been derived on an atomic basis, nor has it been measured. The quantitative description of the long-wavelength excitations remains dependent on the introduction of the sound velocity c as it appears in the Landau theory. This point has been discussed by Feynman and Cohen (Appendix B of reference 5) where the behavior of $S(p)$ for small p is discussed both for finite and zero temperatures. The Landau and Feynman theories therefore fit together to give a single highly plausible picture of the long-wavelength excitations of a Bose fluid.

For quantitative calculation of phonon lifetimes we will work with the formulation of Landau's theory by Kronig and Thellung[20] in which a Hamiltonian for irrotational flow is constructed in terms of the velocity potential and density fluctuation as canonically conjugate variables. Quantization and expansion for small density fluctuations yield the expression

$$H=H_0+V, \tag{17}$$

where

$$H_0=\sum\nolimits_{\mathbf{q}} c_0 q(a_{\mathbf{q}}{}^*a_{\mathbf{q}}+\tfrac{1}{2}). \tag{18}$$

In (18) $c^2=dp/d\rho$, and the subscript zero indicates evaluation at the equilibrium density ρ_0. The a_q and $a_q{}^*$ obey the commutation relations (4) for Boson annihilation and creation operators.

One term appearing in V arises from the cubic term $\frac{1}{2}v(\rho-\rho_0)v$ in the kinetic energy of the fluid. The remaining terms arise from the dependence of the potential

[18] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, New York, 1959), Chap. XVI.

[19] L. P. Pitaevskii, Soviet Phys.—JETP **4**, 439 (1956).

[20] R. Kronig and A. Thellung, Physica **18**, 749 (1952).

energy on the density fluctuations beyond the quadratic term $\frac{1}{2}(c_0^2/\rho_0)(\rho-\rho_0)^2$. These terms depend on various derivatives of c^2/ρ with respect to ρ, and vanish for a low-density system where c^2/ρ is constant. At liquid helium densities, however, they play an important role, and have been considered by Landau and Khalatnikov.[10]

In the low-density limit we need only the term

$$V'=\frac{i}{4}\left(\frac{c_0}{2\rho_0\Omega}\right)^{1/2}\sum_{\mathbf{p},\mathbf{q}}\left(\frac{(\mathbf{p}\cdot\mathbf{q})^2|\mathbf{p}-\mathbf{q}|}{pq}\right)^{1/2}\times a_{\mathbf{p}-\mathbf{q}}{}^*a_{\mathbf{q}}{}^*a_{\mathbf{p}}+\text{H.c.} \quad (19)$$

arising from $\frac{1}{2}v(\rho-\rho_0)v$.

Since the phonons of the linearized theory have $E_0(q)=c_0q$, their stability against decay into two or more phonons is indeterminate. It is first necessary to calculate the energy shifts due to the nonlinear terms to determine whether $E(q)/q$ is an increasing or decreasing function of q. The phonon self-energy as calculated from these terms is strongly divergent due to the contributions of virtual intermediate states containing high-momentum phonons. This is not surprising since the natural high-momentum cutoff associated with the atomic structure is not present in a theory which treats the fluid as a continuum. It does not seem reasonable, therefore, to consider seriously a detailed derivation of the self-energy terms from a hydrodynamic basis. The stability of the long-wavelength excitations must be determined by a theory in which the interatomic forces are explicitly considered. Alternately, the experimental $E(p)$ vs p can be used.

For the low-density Bose gas we know that an atomic calculation leads to unstable phonons. In calculating the phonon lifetime from the hydrodynamic theory, we have the matrix element from (19) as

$$|M(\mathbf{p},\mathbf{q})|^2=\left(\frac{c_0}{32\rho_0\Omega}\right)\frac{(\mathbf{p}\cdot\mathbf{q})^2|\mathbf{p}-\mathbf{q}|}{pq}. \quad (20)$$

To evaluate (6) for small p we note that μ_0 approaches 1 from below as p approaches zero. All the terms in (6) with the matrix element (20) are smoothly behaved in this limit. The final result for the phonon lifetime is, therefore, independent of the quantitative deviation of $E(p)$ from pc, and is given by

$$1/\tau=3p^5/160\pi\rho_0\hbar^4 \quad (21)$$

in exact agreement with the result (9) derived from the atomic theory of the low-density Bose gas. The absence of the interatomic potential, which appeared fortuitous in the atomic calculation, is natural in the hydrodynamic calculation. It therefore appears that the nonlinear terms in the quantum hydrodynamics theory have a definite physical basis.

In the light of the preceding considerations, it is interesting to reexamine the calculation of phonon-phonon interactions in helium II made by Landau and Khalatnikov.[10] They assumed an $E(p)$ curve which implied stability of phonons against decay. The lowest-order process of interest then becomes phonon-phonon scattering. The terms in the Hamiltonian which they included were

$$V_3=V'+\frac{1}{3!}\left(\frac{d}{d\rho}\frac{c^2}{\rho}\right)_{\rho_0}\Omega^{-1/2}\sum_{\mathbf{p},\mathbf{q}}A_{-\mathbf{p}}A_{-\mathbf{q}}A_{\mathbf{p}+\mathbf{q}} \quad (22)$$

and

$$V_4=\frac{1}{4!}\Omega^{-1}\left(\frac{d^2}{d\rho^2}\frac{c^2}{\rho}\right)_{\rho_0}\sum_{\mathbf{p},\mathbf{q},\mathbf{r}}A_{-\mathbf{p}}A_{-\mathbf{q}}A_{-\mathbf{r}}A_{\mathbf{p}+\mathbf{q}+\mathbf{r}}, \quad (23)$$

where

$$A_q=\left(\frac{\rho_0 q}{2c_0}\right)^{1/2}i(a_{\mathbf{q}}{}^*-a_{-\mathbf{q}}), \quad (24)$$

and V' is given by (19). The lowest order contributions to the scattering matrix element come from V_4 in first-order perturbation theory and V_3 in second-order perturbation theory. The second-order term in which the intermediate state contains one phonon diverges if one assumes $E(p)=cp$ in the energy denominator. This arises from a "kinematical resonance" in which two phonons moving parallel to each other form an intermediate one-phonon state with the same energy and momentum. The treatment of a similar resonance for unstable phonons in solids has been discussed by Carruthers.[21] If we assume that the phonons are stable, then this energy denominator cannot vanish, but can become very small. Unlike the decay constant for an unstable phonon discussed previously, the scattering of two stable phonons depends quantitatively on the departure of $E(p)$ from pc. Assuming that $E^2(p)$ has a power series expansion in p^2, one obtains

$$E(p)=cp-\gamma p^3+\cdots \quad (25)$$

for small p. The dominant contribution to the cross section arises from the second-order term in V_3 with a one-phonon intermediate state and yields a cross section proportional to $1/\gamma$. At sufficiently low temperatures the phonon-phonon scattering is the dominant relaxation mechanism in the liquid and should determine, for example, the attenuation of first sound.

Since only low-momentum intermediate states are involved, and we have demonstrated the quantitative accuracy of the matrix element of V' for such states, it seems reasonable that the Landau-Khalatnikov calculation[10] is quantitatively applicable to very low temperature relaxation phenomena in helium II.

IV. SLOW NEUTRON SCATTERING

The physical content of the preceding sections is best summarized by considering an idealized slow neutron scattering experiment in the limit of absolute zero temperature. The discussion of the limiting behavior of

[21] P. Carruthers, Phys. Rev. 125, 123 (1962).

linewidths is not intended to be directly applicable to experiment, but to illustrate the difference between two theoretical approaches in terms of an idealized experiment which is sensitive to the microscopic theory. We will compare the theory of Cohen and Feynman[22] with the scattering from a low-density Bose gas as calculated from the Bogolyubov transformation.[15] A related comparison has been given by Pines (reference 6, p. 81), but with somewhat different conclusions.

The scattering of slow neutrons by a fluid can be accurately calculated in the Fermi pseudopotential approximation. In the limit of absolute zero temperature, the differential energy transfer cross section is given by[23]

$$d^2\sigma/d\Omega dE = a^2(E/E_0)^{1/2}S_0(q,\omega), \qquad (26)$$

where a is the neutron-nucleus scattering length, E_0 and E are the initial and final neutron energies, and $\hbar\omega$ and $\hbar q$ are the energy and momentum transfers in the collision. The function $S_0(q,\omega)$ is given by[6,23]

$$S_0(q,\omega) = (2\pi)^{-1}\int dt\, e^{-i\omega t}(\phi_0\rho_{\mathbf{q}}e^{i(H-E_0)t}\rho_{-\mathbf{q}}\phi_0), \qquad (27)$$

where ϕ_0 is the ground-state wave function of the fluid, $\rho_{\mathbf{q}}$ is the density fluctuation operator as given by (13), H is the Hamiltonian operator for the fluid, and E_0 is the ground-state energy.

For velocity-independent forces, we have the very useful sum rule,[6,22,24]

$$\int \omega S(q,\omega)d\omega = q^2/2m, \qquad (28)$$

which holds at finite temperature as well as for $T=0$.

In the Feynman theory the wave function of an elementary excitation approaches $\rho_{\mathbf{q}}\phi_0$ in the limit of small q. For this wave function it is easily seen that (27) reduces to

$$S_0(q,\omega) = S_0(q)\delta[\omega - E_f(q)], \qquad (29)$$

with $S_0(q)$ and $E_f(q)$ connected by (14). In the limit of small momentum transfer, the neutron scattering goes entirely into the excitation of a single phonon and exhausts the sum rule (28). The physical reason for this is that the neutron interacts directly with the density fluctuations of the fluid, and the phonon excitations have the character of density fluctuations.[25]

As the momentum of the excitation increases the wave function deviates from $\rho_{\mathbf{q}}\phi_0$, but the excitation remains stable against decay. In the calculations of Cohen and Feynman, the approximate wave function employed continues to have a finite overlap with $\rho_{\mathbf{q}}\phi_0$ so that a finite fraction of the neutron scattering goes into the production of a single elementary excitation. The neutron scattering will contain a delta-function contribution for $\omega = E(q)$, plus a continuum for $\omega > E(q)$. There will be no scattering with $\omega < E(q)$ since $E(q)$ is defined as the lowest energy state of the fluid of momentum q.

To discuss the scattering from the point of view of the Bogolyubov transformation, it is convenient to use the second quantized representation[6] of $\rho_{\mathbf{q}}$ in terms of the particle creation and annihilation operators $c_{\mathbf{p}}^*$ and $c_{\mathbf{p}}$:

$$\rho_{\mathbf{q}} = N^{-1/2}\sum\nolimits_{\mathbf{p}} c_{\mathbf{p}-\mathbf{q}}^* c_{\mathbf{p}}. \qquad (30)$$

The Bogolyubov transformation[15] is a linear transformation from the particle operators $c_{\mathbf{p}}$ to the phonon operators $a_{\mathbf{p}}$; both of which obey the Bose commutation rules (4).

$$c_{\mathbf{p}} = (1-\alpha_p^2)^{-1/2}(a_{\mathbf{p}} - \alpha_p a_{-\mathbf{p}}^*), \quad p\neq 0. \qquad (31)$$

The state of zero momentum requires special treatment since it is occupied by a macroscopically large fraction of the particles. For the zero-momentum state, the approximate expression

$$c_0^* \approx c_0 \approx N_0^{1/2} \qquad (32)$$

is employed. Substitution of (29–31) into (27) yields

$$S_0(q,\omega) = (2\pi)^{-1}\int dt\, e^{-i\omega t}\left[F_q(t) + \frac{1}{N}\sum_{\mathbf{p}}{}' G_{\mathbf{p},\mathbf{q}}(t)\right]. \qquad (33)$$

We note that $S_0(q,\omega)$ consists entirely of one-phonon and two-phonon production processes, and can be expressed in terms of the one-phonon propagator

$$\frac{N}{N_0}\left(\frac{1+\alpha_q}{1-\alpha_q}\right)F_q(t) = (\phi_0 a_{\mathbf{q}} e^{i(H-E_0)t} a_{\mathbf{q}}^*\phi_0), \qquad (34)$$

and the two-phonon propagator

$$\left[\frac{(1-\alpha_p^2)(1-\alpha_{q-p}^2)}{\alpha_p(\alpha_p+\alpha_{q-p})}\right]G_{\mathbf{p},\mathbf{q}}(t) = (\phi_0 a_{\mathbf{q}-\mathbf{p}} a_{\mathbf{p}} e^{i(H-E_0)t} a_{\mathbf{q}-\mathbf{p}}^* a_{\mathbf{p}}^*\phi_0). \qquad (35)$$

The primed sum in (33) indicates that terms in which either $p=0$ or $\mathbf{p}-\mathbf{q}=0$ are to be excluded.

The coefficients α_q are chosen to diagonalize the portion of the Hamiltonian containing two phonon creation or annihilation operators. For a repulsive interaction with Fourier transform v_q, they are given by

$$\alpha = 1 + y^2 - y(y^2+2)^{1/2},$$

where

$$y = (q^2/2m\rho_0 v_q)^{1/2} \qquad (36)$$

[22] M. Cohen and R. P. Feynman, Phys. Rev. **107**, 13 (1957).
[23] L. Van Hove, Phys. Rev. **95**, 249 (1954).
[24] G. Placzek, Phys. Rev. **86**, 377 (1952).
[25] The derivation of (14) from the wave function $\rho_{\mathbf{q}}\phi_0$ and the derivation of the sum rule (28) are identical. This explains why the assertion that one-phonon excitation exhausts the sum rule implies the relationship (14) between the phonon energy and the structure factor.

and ρ_0 is the density of atoms in the zero-momentum state. In the low-density limit the terms containing three or four phonon creation operators can be neglected, and the phonons defined by (31) and (36) become eigenstates with excitation energies given by

$$\omega_0(q)=\left[\frac{q^2v_q}{m}+\left(\frac{q^2}{2m}\right)^2\right]^{1/2} \tag{37}$$

A useful algebraic property of the α_q is[14]

$$[(1+\alpha_q)/(1-\alpha_q)]=2m\omega_0(q)/q^2. \tag{38}$$

In the low-density limit, the one-phonon contribution to $S_0(q,\omega)$ can be calculated from (34) and (37) to be

$$S_0^{(1)}(q,\omega)=\frac{N_0}{N}\frac{q^2}{2m\omega_0(q)}\delta[\omega-\omega_0(q)]. \tag{39}$$

This gives a contribution to the sum rule (28) of (N_0/N) times the total value. To lowest order where (N_0/N) can be set equal to one, the neutron scattering goes entirely into single-phonon excitation. In the limit of long wavelength this agrees with the Feynman result. In the limit of large momentum transfer where the quasi-particle energy (37) approaches the free particle value, one obtains the reasonable result that the scattering corresponds to the recoil of a free particle initially at rest.

When we consider first-order departures from the low-density limit, the phonon energies contain an imaginary part due to the finite lifetime of a phonon against decay into two photons. This leads to a broadening of the one-phonon peak in the scattering. The two-phonon term giving a smooth background in the scattering also plays a role. It should be noted that the two-phonon term as given by (33) and (35) does not obey the sum rule (28). Its contribution to the integral in (28) is independent of q in the limit of small q. We also note that the one-phonon term no longer exhausts the sum rule even in the limit of small momentum transfer. For a more complete discussion of the neutron scattering using the Bogolyubov transformation, the reader is referred to the paper by Parry and Turner.[26] In that paper temperature dependent Green's functions are employed to calculate expressions for the line shift and width at finite temperatures. In the limit of zero temperature, their results reduce to those given in the above discussion.

To summarize, the theory of Feynman and Cohen predicts that the neutron scattering $S_0(q,\omega)$ in the limit of zero temperature will consist of a delta-function peak plus a continuous background. In the limit of small momentum transfer the peak corresponding to one-phonon excitation will exhaust the sum rule. Calculations using the Bogolyubov transformation predict an $S_0(q,\omega)$ containing only one- and two-phonon terms. The one-phonon peak is broadened due to phonon decay even at $T=0$, and exhausts only a fraction N_0/N of the sum rule. The two-phonon contribution does not obey the sum rule.

It is the author's opinion that the Landau-Feynman-Cohen point of view is the appropriate one for helium II, and gives several clues for a more complete microscopic theory of interacting bosons. It is probably an exact result that the phonon wave function in the limit of long wavelength is of the Feynman form $\rho_q\phi_0$. It also seems likely that the properly defined phonon excitations will be stable against decay. The proof of these two results from many particle quantum mechanics is, in the author's opinion, an important and challenging objective.

Note added in proof. A more quantitative discussion of the relation between the Feynman-Cohen theory and a correct microscopic theory has recently been given by A. Miller, P. Nozieres, and D. Pines in "Elementary Excitations in Liquid Helium" (unpublished).

ACKNOWLEDGMENT

The author would like to thank Professor Leon Van Hove for the hospitality of the Institute for Theoretical Physics in Utrecht during 1960–61. He would also like to thank Professor Van Hove for several stimulating discussions during the early part of this work.

[26] W. E. Parry and R. E. Turner, Ann. Phys. **17**, 301 (1962).