Critical Velocities and Boundary Interactions in the Isothermal Flow of Superfluid Helium*

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The pressure gradient as a function of the superfluid velocity in a 1.1-mm-diam capillary has been measured at four temperatures between 1.26 and 1.57°K. In the flow tube the normal fluid was held at rest by ultrafine porosity filters, while the superfluid flowed isothermally. Superconducting microwave cavities were used as level indicators, making it possible to detect pressure gradients as small as 3×10^{-4} dyn/cm^3. A critical superfluid velocity, below which no pressure gradient could be detected, was observed directly. The critical velocity decreased linearly with temperature, being 1.25 mm/sec at 1.26°K, and 0.85 mm/sec at 1.57°K. For a superfluid velocity v_s greater than the critical velocity v_c, the variation of the pressure gradient with velocity could be described by $\alpha_2 v_s(v_s-v_c)$ or $\alpha_1(v_s-v_c)^{1.7}$, where α_1 and α_2 are constants that vary with temperature. It is believed that this represents an interaction between vorticity in the superfluid and the wall of the flow tube.

I. INTRODUCTION

A VERY sensitive technique has been used to measure the pressure gradient along a 1.1-mm capillary in which the superfluid component of helium II was flowing isothermally. Two microwave resonant cavities of superconducting tin were placed along the flow tube to act as standoff pipes. A klystron oscillator was electronically stabilized to each cavity, and the recorded beat frequency between the two klystrons was used to measure the relative level of liquid in the two cavities. The flow system was constructed so that the normal fluid was constrained by ultrafine porosity filters, while the superfluid was caused to flow by the thermomechanical (or "fountain") effect.

The apparatus was capable of detecting very small pressure gradients and a critical superfluid velocity, below which no pressure gradient could be detected, was observed directly. The measured critical velocities decreased with increasing temperature, while their magnitudes were in reasonable agreement with previous work.

While the primary objective was the direct detection of the critical velocities, the pressure gradients observed at velocities greater than the critical velocity are also of interest. This phenomenon was not investigated in detail and the number of experimental points is small. Nevertheless, it is believed that the most plausible explanation for the observed effect is an interaction between the superfluid and the wall.

The experimental data presented here are the same, except for a slight correction, as were reported previously.[1] The purpose of this paper is to give a more detailed description of the apparatus and the technique, and to enlarge upon the interpretation of the results by correlating them with some other work on the flow of helium II. The results of certain experiments would seem to preclude the existence of any superfluid-boundary interaction while other experiments, including the present one, are more easily interpreted with the assumption that such a phenomenon exists.

II. EXPERIMENTAL APPARATUS

A. Microwave Stabilizer

The microwave equipment used in this experiment was X band (3 cm) and the klystrons were operated at a frequency of approximately 10 kMc/sec. In order to stabilize a klystron to each cavity two modified versions of the electronic frequency stabilizer first introduced by Pound[2] were used. The block diagram of such a system is shown in Fig. 1. An essential characteristic of this particular circuit is that the sidebands and carrier both travel the same path. Even though the arm leading to the external cavity in a liquid helium cryostat must necessarily be quite long, phase shifts due to random changes of path length are reduced to second order.

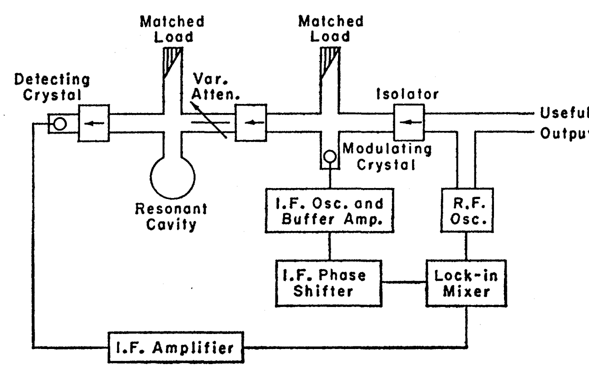

FIG. 1. Block diagram of the modified Pound stabilizer used to stabilize each klystron to a superconducting cavity.

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[1] The values of the critical velocities listed here are approximately 10% less than those previously published [J. N. Kidder and W. M. Fairbank, *Proceedings of the Seventh International Conference on Low-Temperature Physics* (University of Toronto Press, Toronto, 1961), p. 560]. This is because of the belated application of the velocity correction described in Sec. III B.

[2] R. V. Pound, Rev. Sci. Instr. 17, 490 (1946).

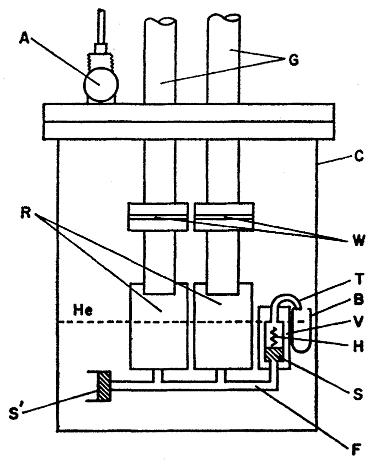

FIG. 2. Superfluid flow system and part of pressure gradient measuring apparatus, including: resonant cavities (R), coupling waveguide (G), vacuum windows (W), Hoke valve (A), superfluid filters (S,S′), flow tube (F), manganin wire heater (H), vacuum jacket around heater chamber (V), exhaust tube (T), velocity-calibrating bucket (B), and dashed line (He) indicating level of helium in can (C).

Since the path lengths in the circuit are fixed, the i.f. phase shifter between the i.f. oscillator and the lock-in-mixer was introduced to align the system. It consisted of a four pole phase shifting capacitor plus two stages of amplification. The ferrite isolators were introduced to permit continuous tuning of the klystron, and to avoid "pulling" by impedance mismatches in the circuit. The i.f. oscillators were controlled by a crystal rather than a tank circuit, but the rest of the i.f. circuit was essentially the same as that described by Pound. The variable attenuator reduced the power fed to the cavities to the minimum needed for good stability.

The useful output from each klystron was fed to a mixer crystal and the resultant beat frequency (usually about 10 Mc/sec) was measured by a Hewlett-Packard frequency counter. The beat frequency was printed out by a digital recorder and also plotted by a graphical recorder.

B. Microwave Cavities

The rectangular resonant cavities, made of high-purity tin, were of the same cross section as the coupling waveguide and electrically one-half wavelength long. They were fabricated by pressing half cavities with a highly polished steel die and then soldering the two halves together in an oven under high vacuum. Before soldering the inside cavity surfaces were polished in the water bath of an ultrasonic cleaner, with fine cesium oxide powder suspended in the bath. To remove surface strains the cavities were annealed at 200°C for a week in an evacuated oven. The unloaded Q of the best cavities was 700 000 at 1.3°K.

Before the cavity halves were soldered they were clamped together and a coupling iris was drilled with the diameter chosen to give a coupling factor as near as possible to unity. In the bottom of the cavities a small-diameter hole was drilled to attach the flow system.

C. Flow System

A drawing of the flow system, resonant cavities, and coupling waveguide is shown in Fig. 2. The entire system is enclosed in a copper can (C), which is immersed in a bath of liquid helium. For the cavities (R) to be most sensitive as level detectors it was necessary that the helium level in the can, indicated by the dashed line (He), be such that the cavities were approximately half full. The can was filled by opening a Hoke valve (A) at the top, and the level was set by comparing the wavelength of the superconducting cavities to that of a calibrated wavemeter.

The i.d. of the 7 cm circular cross-section flow tube (F) was 1.1 mm (0.044 in.). This stainless steel tube was joined to the cavities by a stainless steel tee junction. A hole of the same internal diameter as this tube was drilled across the top arm of the tee. Connecting into this hole at right angles was a hole 0.013 in. in diameter. The flow tube was in the form of a smooth semicircular bend to conserve space, and it was silver soldered to the tees, which in turn were soft soldered to the cavities.

The entrance to the flow tube was sealed with a superfluid filter (S′). The other end of the flow tube led through a second filter (S) to a small chamber containing a manganin wire heater (H). The filters were made by pressing barium oxide powder into a thin-walled stainless steel tube. It was found that for the filters to be most efficient the finest available powder had to be used and that it had to be pressed, a little at a time, at pressures up to 10 000 psi. To seal the powder to the wall of the tube and to prevent leakage around the filter the stainless steel was "tinned" with a layer of indium. The chamber containing the heater, and part of the flow tube in front of the second filter (S), were enclosed in a vacuum jacket (V). The top of the chamber was open to the bath through a small exhaust tube (T). Helium flowing out of the exhaust tube fell into a small glass bucket (B) placed under the exhaust spout. The purpose of this bucket will be described in the section on the velocity calibration technique.

To prevent excess heat flow from the top of the cryostat the coupling waveguides (G) were sealed by two electrically matched, vacuum tight windows (W). Above the windows the waveguide was evacuated, and just below the windows a small hole was drilled in the H-plane side of the waveguide so that the standoff pipes would be open at the top.

III. EXPERIMENTAL TECHNIQUE

A. Calibration of the Cavities

A necessary preliminary calibration was the determination of the resonant frequency of the cavities as a function of the helium level height therein. This calibration curve was obtained by observing the cavities visually when they were not enclosed in a copper can, then admitting helium to a series of known levels, and measuring the resonant frequency with a standardized wavemeter. The slope of this curve gave the change in resonant frequency with level at the chosen operating point. Since the total absolute level change during all

phases of the experiment was less than 5% of the cavity length, the slope over the whole operating range could be considered constant. In this experiment, a change in the helium level of 0.001 cm caused a change in the resonant frequency of 240 kc/sec.

B. Calibration of Flow Velocity

It was first believed that the velocity of the superfluid could be calculated directly, assuming that the Joule heat dissipated in the manganin wire heater would all go into raising the entropy of the incoming superfluid from zero to the specific entropy of the bulk fluid. However, heat losses, principally caused by evaporation, necessitated calibrating the flow velocity as a function of heater current over the entire range of velocities studied.

When helium flowed out of the heater chamber into the small glass bucket under the exhaust spout the level in the can fell. This caused a net increase in the resonant frequency of each cavity. The velocity was calibrated by calculating the amount of helium that flowed in a given time, as measured by the increase in the resonant frequency of the cavities. One klystron was stabilized to a superconducting cavity in the cryostat, and the other to a fixed wavemeter. The heater current was switched on for short intervals, causing helium to flow into the bucket. By using the resultant change in beat frequency, the length of time the heater power was on, the frequency vs level calibration of the cavity, the relative density ρ_s of the superfluid, and the geometry of the can and flow system, the flow velocity was calculated. Each time the bucket became filled, it was emptied by evaporation with another small manganin wire heater placed in it. The hole in the top of the glass bucket was small enough to make the film flow in and out of the bucket negligible.

Not all the superfluid flowing into the heater chamber came from the bath inside the can, as there was a net change in the level in the cavities also. The velocity in

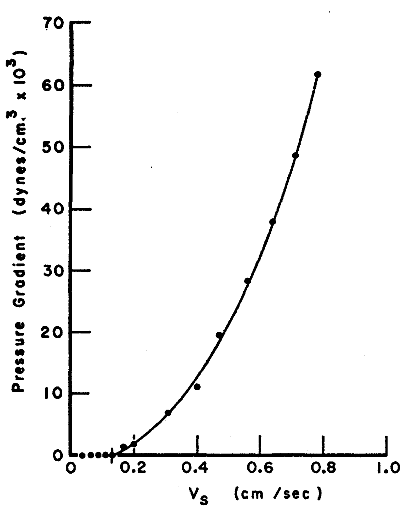

FIG. 3. Pressure gradient vs superfluid velocity at 1.3°K.

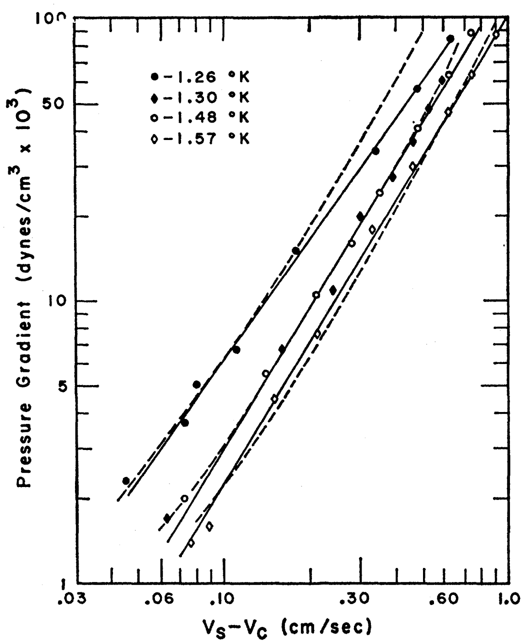

FIG. 4. The measured pressure gradients are plotted logarithmically against the difference (v_s-v_c) at each of the four temperatures studied. The solid lines are plots of Eq. (1), except in the case of the curve at 1.26°K, where the exponent is 1.4. The dashed lines are plots of the pressure gradients as calculated from Eq. (2). The data taken at 1.30 and 1.48°K lie on the same curve. The proportionality constants for Eqs. (1) and (2) are given in Table I.

the flow tube between the standoff pipes was due to helium flowing from the bath and from the first cavity. It was assumed that the ratio of the helium flowing from the bath to that flowing from the cavities was equal to the ratio of the surface area of helium in the can to the inside cross-sectional area of the cavities. The velocities measured had to be corrected by this factor. The Bernoulli pressure drop caused by this velocity change along the flow tube was calculated and it made necessary a small (approximately 2%) correction to the pressure gradient data.

C. Pressure Gradient Measurements

The pressure gradient measurements were made under the same conditions as those under which the velocity was calibrated. However, both klystrons were stabilized to a superconducting cavity, so that a change in the beat frequency represented a change in the relative level of helium in the two cavities. The drop in level due to the filling of the bucket was the same for both cavities, so the resulting error in the absolute resonant frequency was of second order.

With the system in equilibrium the flow heater was switched on. The beat frequency went through a transient, and then changed to a new steady value. The heater current was then switched off and the beat frequency returned to its initial equilibrium value. From the values of the change in beat frequency for

TABLE I. Critical superfluid velocities v_c and the proportionality constants α_1 and α_2 at various temperatures.

Temp. (°K)	v_c (cm/sec)	α_1 (cgs)	α_2 (cgs)
1.26	0.125	0.16	0.28
1.30	0.115	0.15	0.15
1.48	0.098	0.15	0.15
1.57	0.085	0.11	0.11

various heater currents a pressure gradient vs velocity curve was constructed.

IV. EXPERIMENTAL RESULTS

Figure 3 shows a typical plot of the experimental results for one of the four temperatures at which data were taken. The apparatus was able to detect pressure gradients as small as 3×10^{-4} dyn/cm^3 (corresponding to a relative level difference of 1.5×10^{-5} cm), and at velocities below the critical velocity no gradient was observed. The error in the measurement of the superfluid velocity was estimated to be about ±0.005 cm/sec.

It must be noted that one of the four curves is inconsistent with the other three. This can be seen in Fig. 4 where, for each temperature, the pressure gradient is plotted logarithmically against (v_s-v_c), the difference between the superfluid velocity and the critical velocity. At the three higher temperatures the points fall near or on the line described by

$$\nabla p=\alpha_1(v_s-v_c)^{1.7}. \tag{1}$$

At 1.26°K the equation has the same form, but the exponent is 1.4.

The data can also be approximated by the relation

$$\nabla p=\alpha_2 v_s(v_s-v_c). \tag{2}$$

In Fig. 4 it can be seen that, for three temperatures, most of the points fall near or on the dashed lines which are plots of the pressure gradients as calculated from Eq. (2). At 1.26°K, the marked deviation at higher velocities and the relatively large value of the constant α_2 are not in good agreement with the other data. The values of the proportionality constants α_1 and α_2, and the critical velocities at each temperature are given in Table I.

Besides the measurements already described, an investigation was made of the transient effects at the onset of resistance to superfluid flow, and of the possibility of a hysteresis loop in the pressure gradient vs velocity curve. The transient effects observed were believed to be due to a second sound pulse generated by the sudden switching on (and switching off) of the heater current. This effect was quite large, and it turned out to be unavoidable, so no measurements could be made of transients in the mechanism of resistance to flow. There was no measureable (i.e., longer than 10 sec, the duration of the aforementioned transient) time delay in the formation of resistance to superfluid flow, even at velocities near the critical velocity. Finally, there was no observable hysteresis associated with the pressure gradient.

V. DISCUSSION

A. Critical Velocities

Atkins[3] has pointed out, from an analysis of previous experiments, that for channels wider than 10^{-3} cm, the critical superfluid velocity v_c at 1.4°K is related to the characteristic channel width d by the equation

$$v_c=(4\hbar/md)\ln(d/4a). \tag{3}$$

The constant a is approximately 2×10^{-8} cm and m is the mass of the helium atom.

For a channel of diameter 1.1 mm this would give a critical velocity of 0.081 cm/sec. If the critical velocities measured in this experiment are plotted against temperature, they fall near or on a straight line that passes through the point 0.105 cm/sec at 1.4°K. This can be considered good agreement with previous data.

It is also of interest to note that the critical velocity decreases with increasing temperature. One might expect in the case of pure superfluid flow in wide channels to observe the same temperature dependence of the critical velocity as in the case of flow through helium films. The comparison is not easy to make, because experiments with the mobile film measure the critical mass transfer rate, which is a function of the relative superfluid density and the film thickness as well as the superfluid velocity. The mass transfer rate is found[4] to decrease as $[1-(T/T_\lambda)^7]$, while the relative superfluid density varies as $[1-(T/T_\lambda)^{5.6}]$. Measurements of the variation of the film thickness with temperature have not been made with sufficient accuracy or reliability to calculate the temperature variation of the superfluid critical velocity in the film, though in general it is found[5] that the film becomes thicker as the temperature increases.

In evaluating the critical velocities derived from critical heat currents in helium II there are three relevant velocities to be considered: that of the superfluid, the normal fluid, and the relative velocity between the two fluids. According to Vinen's[6] measurements of critical heat currents, as the temperature increased so did the critical superfluid velocity, while the critical normal fluid velocity and the critical relative velocity decreased. Chase[7] has suggested that in thermal conduction below (approx) 1.6°K the onset of resistance to heat flow occurs when the normal fluid flows sufficiently fast to

[3] K. R. Atkins, *Liquid Helium* (Cambridge University Press, New York, 1959), pp. 198–201.

[4] B. Smith and H. A. Boorse, Phys. Rev. **99**, 2, 367 (1955).

[5] K. R. Atkins, *Progress in Low-Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1955), Vol. II, p. 110.

[6] W. F. Vinen, Proc. Roy. Soc. **A243**, 400 (1958).

[7] C. E. Chase, private communication (to be published).

become turbulent, and so induces turbulence (or vorticity) in the superfluid.

Recently Staas and Taconis[8] have published measurements of critical superfluid velocities observed in a capillary in which the normal fluid was at rest. The critical velocity vs temperature plot had a pronounced maximum at 1.5°K. Their measurements were made with a flow system similar to ours, and the difference in the temperature variation of the critical velocity is surprising.

B. Hydrodynamical Equations

A generally accepted form[9] of the steady-state hydrodynamical equations of liquid helium II is

$$-(\rho_s/\rho)\nabla p+\rho_s S\nabla T-\mathbf{F}_{sn}-\mathbf{F}_s=0, \quad (4)$$

$$-(\rho_n/\rho)\nabla p-\rho_s S\nabla T+\mathbf{F}_{sn}-\mathbf{F}_n+\eta_n\nabla^2 v_n=0. \quad (5)$$

These equations give the net force per unit volume on the superfluid and normal fluid, respectively. The thermomechanical force is expressed in the term $\rho_s S\nabla T$, where ρ_s is the density of the superfluid and S is the specific entropy of the whole fluid. The last term in Eq. (5) is the normal fluid viscous term, and for laminar flow in a circular cross-section tube of radius r it is written $-8\eta_n v_n/r^2$.

From a series of careful measurements of heat conduction and second sound attenuation in helium II Vinen[10,11] has established that $\mathbf{F}_{sn}$, a nonlinear mutual friction force between the two fluids, is of the form $A\rho_s\rho_n(|v_s-v_n|-v_c)^3$, for $|v_s-v_n|>v_c$. The constant A has the value 40 cm sec/g at 1.4°K and increases with temperature. The general terms $\mathbf{F}_s$ and $\mathbf{F}_n$ are included to represent nonlinear interactions of the superfluid and the normal fluid, respectively, with the boundaries of the flow system. In many discussions it has been assumed that these terms could be ignored. Recently Staas, Taconis, and Van Alphen[12] have shown that a nonlinear term in the normal fluid velocity must be included, and that it is the solution to the Navier-Stokes viscous term when the flow is turbulent.

C. Superfluid Force Term

It is believed that in the present experiment the superfluid force $\mathbf{F}_s(\mathbf{v}_s)$ was measured directly. Assuming that the normal fluid was not moving, Eqs. (4) and (5) may be added to give

$$\nabla p+\mathbf{F}_s(\mathbf{v}_s)=0. \quad (6)$$

A superfluid-boundary interaction is the most reasonable explanation for the observed pressure gradients. The normal fluid was constrained by the filters, and even if they were not completely effective, the direction of the normal fluid flow would have been away from the heater. This would have given a pressure gradient opposite in direction to that observed. The mutual friction interaction exerts a force that is equal and opposite in both directions, and cannot be observed directly as a pressure gradient.

There are spurious effects that could have caused a pressure gradient without a superfluid-wall interaction. The possibility of a complicated convection flow in the tube with the normal fluid being carried down along the wall and flowing back up the center has been considered. To describe such an effect completely and quantitatively would be difficult, if not impossible. Other investigators[13,14] working with flow systems similar to the one used in this experiment have found no indication of any such effects. The most reasonable assumption is that the normal fluid was not moving at all, and certainly not in the same direction as the superfluid.

Finally one must consider the possibility of a temperature gradient existing along the flow tube. It has been stated that in this experiment the superfluid was flowing isothermally. Actually this cannot be the case for velocities greater than the critical velocity, as can be seen by considering Eqs. (4) and (5). For the case $v_s<v_c$, $v_n=0$, these equations become

$$-(\rho_s/\rho)\ \nabla p+\rho_s S\ \nabla T=0, \quad (4')$$

$$-(\rho_n/\rho)\ \nabla p-\rho_s S\ \nabla T=0, \quad (5')$$

so that the temperature and pressure gradients are both zero.

Now, when $v_s>v_c$, and $v_n=0$, these equations are written

$$-(\rho_s/\rho)\ \nabla p+\rho_s S\ \nabla T =\rho_s\rho_n A\,(v_s-v_c)^3-\alpha_2 v_s(v_s-v_c)=0, \quad (4'')$$

$$-(\rho_n/\rho)\ \nabla p-\rho_s S\ \nabla T+\rho_s\rho_n A\,(v_s-v_c)^3=0. \quad (5'')$$

Here we have used Vinen's[10] form for the mutual friction term and the superfluid force term as given by Eq. (2).

Combining these two equations gives

$$\rho_s S\ \nabla T=\rho_s\rho_n A\,(v_s-v_c)^3+(\rho_n/\rho)\alpha_2 v_s(v_s-v_c). \quad (7)$$

Clearly, there must be a temperature gradient, for without it the superfluid would not flow. The magnitude of ∇T as given by Eq. (7) was calculated for each set of data taken, and the maximum was of the order of 10^{-7} °K/cm. A positive temperature gradient could produce a negative pressure gradient because of the difference in the vapor pressure in the two stand-off pipes. But, the above maximum value of ∇T is too small by an order of magnitude to account for the results given here. Furthermore, the effect would be minimized

[8] F. A. Staas and K. W. Taconis, Physica **27**, 924 (1961).
[9] Reference 3, p. 171.
[10] W. F. Vinen, Proc. Roy. Soc. (London) **A240**, 114 (1957).
[11] W. F. Vinen, Proc. Roy. Soc. (London) **A242**, 493 (1957). See footnote to Eq. (2), p. 495.
[12] F. A. Staas, K. W. Taconis, and W. M. Van Alphen, Physica **27**, 893 (1961).
[13] T. R. Koehler and J. R. Pellam, Phys. Rev. **125**, 3, 791 (1962).
[14] P. P. Craig and J. R. Pellam, Phys. Rev. **108**, 1109 (1957).

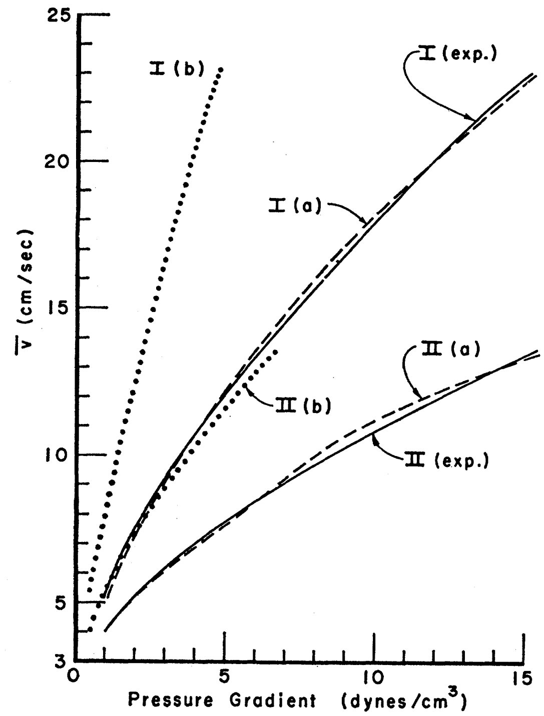

FIG. 5. Atkins' experimental data for capillaries I and II at 1.22°K [curves I(exp) and II(exp)], and theoretical curves with [I(a) and II(a)] and without [I(b) and II(b)] the superfluid force term included.

because both standoff pipes were open to the bath through a small hole drilled in the waveguide.

While a superfluid force term of either

$$\mathbf{F}_s = \alpha_1 (v_s - v_c)^{1.7} \tag{8}$$

or

$$\mathbf{F}_s = \alpha_2 v_s (v_s - v_c) \tag{9}$$

might appear unusual, the two forms are not without precedent. In 1957, Craig and Pellam[14] published the results of a flow experiment performed with an apparatus much like the one described here. Rather than measuring the pressure gradient they measured the lift on a delicate airfoil suspended on a sensitive torsion fiber. The torque was found to vary as $v_s(v_s - v_c)$, while the critical velocity was 6 mm/sec at 1.3°K.

Vinen[15] has shown that the onset of mutual friction is related to the development of vorticity in the superfluid. It is plausible that the superfluid force would also be related to vorticity, or turbulence, in the superfluid. For turbulent flow of classical fluids in circular cross section pipes it has been determined empirically[16] that, when the Reynolds number is less than 100 000, the pressure gradient varies as the mean velocity raised to the power 7/4. The proportionality constant includes the density and viscosity of the fluid and the diameter of the pipe. Since there is little meaning to the concept of a classical superfluid viscosity, the classical proportionality constant cannot be related to these results.

D. Comparison with Atkins' Data

If there is an interaction between the superfluid and the walls of a flow tube, then evidence for such an effect should appear in any flow experiment involving supercritical velocities. However, in all experiments involving heat conduction in helium II, and especially at temperatures where the normal fluid constitutes a small percentage of the total liquid, a relatively small superfluid force would be obscured by the mutual friction and normal fluid viscous forces. If the superfluid force is to be observed it will probably have to be with an isothermal flow scheme where the normal fluid is at rest.

The data from Atkins' experiment[17] measuring the pressure gradient caused by the isothermal flow of both the superfluid and the normal fluid have been analyzed, and it is found that some of the data, at least, can be better explained by the two-fluid equations if the superfluid force term is included.

Atkins' apparatus consisted of a small glass reservoir with a long capillary attached to the base. With the reservoir partially filled a pressure gradient was introduced by raising or lowering the reservoir in the bath. The pressure gradient was measured by observing the difference in the helium level between the reservoir and the bath, while the mean velocity $\bar{v}$, given by

$$\bar{v} = (\rho_s/\rho) v_s + (\rho_n/\rho) v_n, \tag{10}$$

was calculated from the measured rate of change of the level difference. It is not possible in this type of experiment to measure v_s and v_n separately.

In the analysis of Atkins' data it was assumed that the two-fluid equations could be written in the form of Eqs. (4) and (5), with ∇T and $\mathbf{F}_n(\mathbf{v}_n)$ taken equal to zero. One can combine Eq. (5) with Eq. (10) to get a cubic equation with v_n as the only unknown. It is thus possible to calculate a value of v_s and v_n for every point on Atkins' experimental curves. Equations (4) and (5) can be added to give an expression for ∇p in terms of v_s and v_n:

$$\nabla p = -\mathbf{F}_s(v_s) - (8\eta_n/r^2) v_n. \tag{11}$$

It should be possible to fit Atkins' data to this equation, provided the calculated values of v_n and v_s are correct, and that the introduction of the superfluid force is justified.

Such a fit can be made with some of Atkins' data, at least. In his experiment he used four different capillaries (labeled I through IV) of radii 0.22, 0.10, 0.041, and 0.013 mm. Measurements were made at 1.22 and 1.52°K. In Fig. 5 we have plotted the experimental data for capillaries I and II at 1.22°K, as well as two

[15] W. F. Vinen, Proc. Roy. Soc. (London) **A240**, 128 (1957).
[16] H. Schlichting, *Boundary Layer Theory* (McGraw-Hill Book Company, Inc., New York, 1955), p. 401.
[17] Reference 3, p. 188; Proc. Phys. Soc. (London) **A64**, 833.

theoretical curves for each capillary. The theoretical curves were calculated from Eq. (11), with and without the superfluid force term included. Curves I(a) and II(a) were calculated with the superfluid force written as in Eq. (9). The value of α, picked to give the best fit, was 0.020 for curve I(a) and 0.046 for II(a). The two fluid velocities were calculated from Eqs. (5) and (10) as explained above, using the values of ρ, ρ_s, ρ_n, and η_n as compiled by Benson and Hollis-Hallett.[18] As an indication that Poiseuille flow of the viscous normal fluid is not a sufficient explanation for the observed pressure gradient, Eq. (11) is plotted with the term $\mathbf{F}_s$ set equal to zero in curves I(b) and II(b).

Similar plots can be made from the data taken at 1.22°K in capillaries III and IV. The calculations of v_s and v_n are more difficult because v_n is small compared to v_s and it is not possible to get as accurate a solution to the cubic equation. Yet in all cases it is found that the pressure gradients are too large to be explained by laminar flow of the normal fluid. At 1.52°K it appears that the temperature difference between the reservoir and the bath was appreciable, as Eqs. (4), (5), and (11) do not have a single, consistent solution for grad p if grad T is set equal to zero. This temperature difference would arise because the superfluid was flowing out of the reservoir faster than the normal fluid, and the resulting increase in the specific entropy of the remaining liquid was not entirely overcome by cooling through evaporation.

Staas *et al.*[12] have made a similar analysis of Atkins' data by adding not the superfluid force, but rather by considering turbulence in the normal fluid. They considered only the largest capillary at 1.22°K, where it is probable that the normal fluid was turbulent over most of the measured range of velocities. However, from the data taken with the other capillaries we have calculated that the Reynolds number would not be large enough to suggest turbulent flow of the normal fluid, except possibly in capillary II at the highest observed velocities. It would appear that a complete and detailed interpretation of Atkins' data is still to be found.

E. Recent Work at Leiden

In discussing an interaction between the superfluid and the walls of a flow system, one must certainly consider the recently published experiments of Staas, Taconis, and Van Alphen,[12] where no such effect was observed. Their method employed a modified thermal conduction technique, whereby the superfluid flowed through a filter into an insulated chamber containing a heater, while the normal fluid flowed out of the chamber through a small, precisely measured capillary. The normal fluid velocity was calculated thermodynamically from the heat dissipated in the chamber, and the temperature and pressure gradients along the capillary were measured simultaneously. The reader is referred to the original paper for experimental details.

Since the temperature and pressure gradients were found to be related by London's equation $\nabla p = \rho S \nabla T$, it could be assumed that the superfluid and the normal fluid were moving with the same velocity in the capillary. The pressure gradient measurements could be interpreted by assuming that over a certain region of velocities Poiseuille flow existed, and that above that region the flow was turbulent. The authors introduced the two dimensionless numbers

$$Re_v = 2r\rho v_n/\eta_n, \quad Re_p = \rho r^3/4\eta_n^2,$$

and their data could be described by $Re_v = Re_p$ in the laminar region, and $Re_p = 4.94\times10^{-3} Re_v^{1.75}$ in the turbulent region. The transition occurred when $Re_v = Re_p \approx 1200$. Except that the critical Reynolds number is smaller than expected by about one half, and contains the total fluid density, these results are in accord with measurements on classical fluids.

Similar measurements were then made with the apparatus modified by closing off the superfluid filter, so that "pure" thermal conduction counterflow occurred. In this case the temperature gradients became quite large because of the presence of mutual friction, and a large correction to the measured pressure gradients had to be made because of the difference in vapor pressure. Nevertheless, the data could again be explained by considering only the normal fluid viscosity and turbulence. These results would seem to leave little room for any effect due to a superfluid interaction with the wall of the capillary.

Yet, it would appear in the first case, where the superfluid and the normal fluid are flowing together with the same velocity, that the superfluid was taking part in the turbulence. A slight mutual friction force would have to be exerted on the superfluid to make it flow out of the heater chamber. If the two fluids are intimately locked together, then it is reasonable that the appropriate density to be used in the Reynolds number is that of the total fluid. It is quite possible that a superfluid force, observed when the superfluid is turbulent and the normal fluid is at rest, would be obscured when both fluids are moving.

For the second case, that of "pure" heat conduction, it is doubtful that the superfluid force would be observed at all. At 1.7°K, the only temperature at which data were given for this case, ρ_s/ρ_n is 0.74, so that v_s/v_n is 0.35. We have estimated the pressure gradient that might be caused by a superfluid force by calculating grad $p = 0.1 v_s^{7/4}$. This would give a level difference (the pressure gradients were measured with liquid helium manometer tubes) that is only 0.07 times the level difference caused by the normal fluid turbulence, and less than 0.01 times the "gross" level difference (before the vapor pressure correction).

The data from a second experiment, taken a few years ago, were published by Staas and Taconis[8] simul-

[18] C. B. Benson and A. C. Hollis-Hallett, Can. J. Phys. **38**, 1376 (1960).

taneously with the results just discussed. This earlier experiment was an attempt to study mutual friction when the superfluid was moving and the normal fluid was held at rest by a superfluid filter. The superfluid flowed up a capillary through a filter into a reservoir containing a heater. The pressure gradients along the capillary were measured with a liquid helium manometer. Again, the reader is referred to the original paper for details.

It was assumed that the entire level difference Δh was due to a difference in vapor pressure, caused by a temperature gradient along the capillary, so that

$$\rho g \Delta h = (\partial p_v / \partial T) \Delta T,$$

where p_v is the vapor pressure. Assuming that F_s and v_n are zero, Eqs. (4) and (5) show that $\mathbf{F}_{sn}$ and ∇T are related by

$$\mathbf{F}_{sn} = A \rho_s \rho_n (v_s - v_c)^3 = \rho_s S \, \nabla T.$$

For the case $v_s > v_c$ this would give

$$A = S g \Delta h / L (\rho_n / \rho) (\partial p_v / \partial T) v_s^3,$$

where L is the length of the capillary. One would then expect to have Δh vary as v_s^3, and to be able to calculate the mutual friction constant A from the above formula.

Actually, as noted by the authors, the plot of Δh vs v_s is less than cube law, and approaches square law at the lower temperatures. The calculated values of the mutual friction constant are greater than other previous measurements[10,19] by a factor of 2 or 3, and contrary to these previous results, A is not a monotonically increasing function of temperature, but has a maximum. These discrepancies might be resolved in the following way: Assume a superfluid force of the form $\alpha v_s^{7/4}$, and then the level difference Δh will be given by

$$\Delta h = \frac{A L (\rho_n / \rho)(\partial p_v / \partial T)}{g S} v_s^3 + \frac{L \alpha}{\rho g} v_s^{7/4}.$$

The calculated value of A will be higher if one ignores the superfluid force. Because of the factor ρ_n/ρ, the contribution of the cubic term will be less at low temperatures than near the λ point, which is what was observed. Finally, if one substitutes the values of A as measured by Vinen[10] into the above expression, along with all the other constants except the hypothetical α, then the calculated value of α is approximately 0.2 (cgs units). This is in qualitative agreement with our results.

F. Conclusion

One of us (JNK) is planning further experiments to investigate the ideas discussed here, and it is hoped that they will be more conclusive. A sensitive technique for measuring pressure and temperature gradients and a flow system which is effectively isothermal are needed to measure critical velocities and to investigate the superfluid-boundary interaction. It would also be desirable to work below 1°K, where the percentage of normal fluid is negligible and corrections for differences in vapor pressure are small.

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[19] H. C. Kramers, T. M. Wiarda, and A. Broese Van Groenou, *Proceedings of the Seventh International Conference on Low-Temperature Physics* (University of Toronto Press, Toronto, 1961), p. 562.