# Flow Instability in Liquid Helium II*

R. MESERVEY†
*Yale University, New Haven, Connecticut*

(Received December 11, 1961; revised manuscript received April 10, 1962)

The stability of steady, low-velocity laminar flow of the normal component of liquid helium II is shown to follow from the principle of minimum entropy production. It is pointed out that for two-fluid equations containing a nonlinear mutual friction term, this justification of stability is no longer valid when the energy dissipation by mutual friction is comparable to that by the viscosity of the normal-fluid component. An instability condition which is characterized by a dimensionless parameter $\mathcal{G}$ (Gorter number) formed from the ratio of these two dissipative terms, is shown to predict the magnitude and temperature dependence of the critical velocities observed in heat conduction, boundary movement, and isothermal flow experiments in channels wider than $10^{-3}$ cm. Instabilities observed in heat conduction at higher velocities are shown to correlate with a Reynolds number of the usual form. The onset of the mutual friction force is discussed in the light of this phenomenological theory.

## I. INTRODUCTION

THE superfluidity of liquid helium has been explained by Landau[1] on the basis of the available energy states in the liquid. This explanation also gives a critical velocity at which quantum excitations are formed and superfluidity must break down. Could we identify this critical velocity of Landau with the many observations of the onset of additional frictional forces, the main problem in the flow of liquid helium would be solved. However, when applied to the known energy states in helium II, this concept yields a minimum critical superfluid velocity of about 6000 cm/sec, which is much higher than those observed.

In fact, the observed critical velocities range from about 0.1 to 50 cm/sec, depending on the lateral dimension of the channel $d$.[2] In narrow channels ($10^{-6}\text{cm}<d<10^{-3}$ cm) the critical velocity is approximately proportional to $d^{-1/3}$; it has also been shown that the superfluid velocity is the critical parameter[3] and that there is little temperature dependence. In wide channels ($d>10^{-3}$ cm) the critical velocities can be approximated by $V_c=0.01d^{-1}$ (in cgs units) at $T=1.4°$K, but there is a temperature dependence which varies with the type of experiment and there is little evidence that the critical velocities observed are uniquely associated with the superfluid. These facts suggest, as was pointed out by Atkins,[2] that the critical velocities in these two cases are of a different character.

Feynman[4] showed that the Landau concept applied to Onsager's[5] quantized vortex motions should lead to critical velocities not much above those observed, and Atkins[2] has shown that this model fits much of the data in wide channels at 1.4°K. This theory is attractive, but open to the following objections: (1) The constants giving the correct magnitude are uncertain; (2) the temperature dependence is not explained; (3) the value chosen for $d$ in oscillating-boundary experiments is the penetration depth of the normal fluid, a choice which seems inconsistent with selecting the superfluid velocity as the critical parameter.

The present proposal is hydrodynamic in nature, starting from equations of motion which contain the mutual friction term introduced by Gorter and Mellink[6] to explain the measured heat conduction of helium II. From these equations, a thermodynamic argument concerning the mutual friction term suggests a criterion of flow instability, which is then compared with observed critical velocities in a wide variety of experiments. The high correlation which is found between theory and experiment finally leads us back to examine more closely the onset and nature of this mutual friction.

## II. STABILITY OF SLOW LAMINAR FLOW

To make clear the nature of the argument which justifies the stability of slow laminar flow in helium II, we first apply it to the familiar case of an incompressible viscous fluid. The motion of such a fluid is described by the Navier-Stokes equation, which, in the absence of external fields, is

$$\rho\partial\mathbf{V}/\partial t-\rho\mathbf{V}\cdot\nabla\mathbf{V}=-\nabla p-\eta\nabla\times\nabla\times\mathbf{V}, \qquad (1)$$

and the equation of continuity,

$$\nabla\cdot\mathbf{V}=0. \qquad (2)$$

The usual assumptions that the fluid velocity at a solid boundary equals the boundary velocity and that the viscosity $\eta$ is constant, are justified for present applications.

The dissipation of energy in a viscous incompressible fluid in a region of volume $\tau$ whose surface is $s$, can be

* Supported by the National Science Foundation and the Office of Ordnance Research.

† Present address: Lincoln Laboratory, Massachusetts Institute of Technology, Lexington, Massachusetts.

1 L. D. Landau, J. Phys. (U.S.S.R.) **5**, 71 (1941).

2 K. R. Atkins, *Liquid Helium* (Cambridge University Press, New York, 1959). Critical velocities are discussed on page 198.

3 P. Winkel, A. Broese van Groenou, and C. J. Gorter, Physica **21**, 345 (1955).

4 R. P. Feynman, *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1955), Vol. 1, p. 45.

5 L. Onsager, Nuovo cimento **6**, Suppl. 2, 249 (1949).

6 C. J. Gorter and J. H. Mellink, Physica **15**, 285 (1949).

written as[7,8]

$$\dot{E}=\eta\int_\tau |\nabla\times\mathbf{V}|^2 d\tau+\eta\int_s (d\mathbf{V}/dt)\cdot d\mathbf{s}. \tag{3}$$

For low velocities, where the accelerations are small, the surface integral can usually be neglected. The dissipation will then be

$$\dot{E}=\eta\int_\tau |\nabla\times\mathbf{V}|^2 d\tau. \tag{4}$$

When we can neglect the inertial term $\rho\mathbf{V}\cdot\nabla\mathbf{V}$ in Eq. (1), some general theorems hold.[8] Under these conditions Helmholtz[9] and Korteweg[10] showed that in a region whose boundary velocities are given and constant, the steady motion eventually reached is unique and stable, and dissipates less energy than any other kinematically possible motion with the same boundary conditions. This stable motion is the solution of Eq. (5) with appropriate boundary conditions.

$$\nabla p=-\eta\nabla\times\nabla\times\mathbf{V}. \tag{5}$$

Rayleigh[11,12] generalized this theorem to any dynamical system without potential energy in which the kinetic energy and dissipation function can be expressed as quadratic functions of the generalized velocities with constant coefficients.

Onsager[13] generalized Rayleigh's principle and showed that irreversible processes which can be described by linear phenomenological relations, and which obey reciprocal relations, can be described by a variational principle. He showed that, with time-independent constraints, the steady state to which such a system proceeds is one of minimum entropy production. There is presently controversy over how far away from equilibrium this principle is valid;[14] however, there is little question that for slow viscous flow it is completely justified. From this standpoint the theorems of Helmholtz and Korteweg follow as a special case of the principle of minimum entropy production for a liquid at uniform temperature.

### Liquid Helium II

To discuss the motion of helium II, the two-fluid equations essentially in the form given by Landau[1] and London[15] are assumed, to which has been added a mutual friction term $\mathbf{F}_{sn}$:

$$\rho_s\frac{\partial\mathbf{V}_s}{\partial t}+\rho_s\mathbf{V}_s\cdot\nabla\mathbf{V}_s=-\frac{\rho_s}{\rho}\nabla p+\rho_s S\nabla T-\mathbf{F}_{sn}+\frac{\rho_s\rho_n}{2\rho}\nabla|\mathbf{V}_n-\mathbf{V}_s|^2, \tag{6}$$

$$\rho_n\frac{\partial\mathbf{V}_n}{\partial t}+\rho_n\mathbf{V}_n\cdot\nabla\mathbf{V}_n=-\frac{\rho_n}{\rho}\nabla p-\rho_s S\nabla T+\mathbf{F}_{sn}-\frac{\rho_s\rho_n}{2\rho}\nabla|\mathbf{V}_n-\mathbf{V}_s|^2-\eta\nabla\times\nabla\times\mathbf{V}_n. \tag{7}$$

The mutual friction is written essentially in the form originally proposed by Gorter and Mellink[6]:

$$\mathbf{F}_{sn}=\rho_s\rho_n A|\mathbf{V}_s-\mathbf{V}_n|^2(\mathbf{V}_s-\mathbf{V}_n). \tag{8}$$

The empirical necessity for such a term has been demonstrated by many experiments, and Hall and Vinen[16] have suggested a plausible mechanism. The sufficiency and limitations of mutual friction in explaining effects observed in helium II will be discussed in Sec. IV.

It was assumed that the liquid was incompressible,

$$\nabla\cdot(\rho_n\mathbf{V}_n+\rho_s\mathbf{V}_s)=0, \tag{9}$$

and that the temperature differences were small enough so that variations of $\rho_n$ and the frictional term $\frac{4}{3}(\eta\nabla\nabla\cdot\mathbf{V}_n)$ could be neglected. The other frictional terms derived by Khalatnikov[17] should be extremely small for the present applications and were also neglected.

Equation (10) defines $\rho_s$:

$$\rho_s+\rho_n=\rho. \tag{10}$$

If we add Eqs. (6) and (7), we get the result

$$\rho_s\left(\frac{\partial\mathbf{V}_s}{\partial t}+\mathbf{V}_s\cdot\nabla\mathbf{V}_s\right)+\rho_n\left(\frac{\partial\mathbf{V}_n}{\partial t}+\mathbf{V}_n\cdot\nabla\mathbf{V}_n\right)=-\nabla p-\eta\nabla\times\nabla\times\mathbf{V}_n. \tag{11}$$

If we can neglect the accelerations, this equation reduces to

$$\nabla p=-\eta\nabla\times\nabla\times\mathbf{V}_n, \tag{12}$$

which is identical in form with Eq. (5) and shows that the pressure field is determined only by the normal-fluid velocity field.

Again, if we assume in Eq. (6) that the acceleration terms can be neglected, or more generally can be derived from a potential, we have by taking the curl of this equation and using Eq. (8),

$$\nabla\times[|\mathbf{V}_s-\mathbf{V}_n|^2(\mathbf{V}_s-\mathbf{V}_n)]=0. \tag{13}$$

---

[7] D. Bobyleff, Math. Ann. **6**, 72 (1873).

[8] J. Serrin, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. 8, Part 1, section on incompressible viscous fluids, pp. 246–262.

[9] H. Helmholtz, *Collected Works* (1869), Vol. 1, p. 223.

[10] D. J. Korteweg, Phil. Mag. (5) **16**, 112 (1883).

[11] Lord Rayleigh, Phil. Mag. (6) **26**, 776 (1913).

[12] H. Lamb, *Hydrodynamics* (Dover Publications, New York, 1945), 6th ed., p. 619.

[13] L. Onsager, Phys. Rev. **37**, 405 (1931); **38**, 2265 (1931).

[14] *Proceedings of the International Symposium on Transport Processes in Statistical Mechanics, Brussels, 1956* (Interscience Publishers, Inc., New York, 1958).

[15] F. London, *Superfluids* (John Wiley & Sons, Inc., New York, 1954), Vol. 2, pp. 130, 132, 141. A term representing the creation of normal fluid has been neglected in London's equations.

[16] H. E. Hall and W. F. Vinen, Proc. Roy. Soc. (London) **A238**, 215 (1956).

[17] I. M. Khalatnikov, Uspekhi Fiz. Nauk. **60**, 69 (1956).

In parallel flow Eq. (13) implies that $\nabla\times(\mathbf{V}_s-\mathbf{V}_n)=0$. Thus, for $\mathbf{F}_{sn}\neq 0$, the relative velocity across the channel is constant and

$$\langle\nabla\times\mathbf{V}_s\rangle_{\mathrm{av}}=\nabla\times\mathbf{V}_n. \tag{14}$$

Here the average vorticity designates the circulation around a small but macroscopic region divided by the area of the region and does not necessarily imply that the flow must be microscopically rotational. Actually, Eq. (14) follows for any mutual friction term which varies with greater than the first power of the relative velocity.

At low velocities where we can neglect accelerations, the energy dissipation of the normal fluid in a volume $\tau$ is

$$\dot{E}_n=\eta\int_\tau |\nabla\times\mathbf{V}_n|^2 d\tau, \tag{15}$$

being of the same form as Eq. (4). At higher velocities the energy dissipation implied by the mutual friction must be considered, and is

$$\dot{E}_{sn}=\rho_s\rho_n A\int_\tau |\mathbf{V}_s-\mathbf{V}_n|^4 d\tau, \tag{16}$$

To measure the relative importance of these dissipative terms we define a dimensionless parameter $\mathcal{G}$, which might be called the Gorter number, as follows:

$$\mathcal{G}^2\equiv\frac{\dot{E}_{sn}}{\dot{E}_n}=\frac{\rho_s\rho_n A\int|\mathbf{V}_s-\mathbf{V}_n|^4 d\tau}{\eta\int|\nabla\times\mathbf{V}_n|^2 d\tau}. \tag{17}$$

It is clear that for low velocities $\mathcal{G}\ll 1$ and the stability of the laminar solutions of Eq. (12) can be justified, just as those of Eq. (5), by the principle of minimum entropy production.[18] However, for $\mathcal{G}\approx 1$, the thermodynamic justification of the stability of this steady laminar flow is no longer valid. This last conclusion may seem obvious, but is mentioned because, in the past, it has often been assumed that the normal fluid must be stable to much higher velocities. Furthermore, the above discussion provides a background for the hypothesis which will now be made.

This hypothesis was suggested by the criterion of instability for an ordinary viscous liquid, where we know that for values of the Reynolds number $\mathcal{R}\ll 1$ the principle of minimum entropy production assures stability. [The Reynolds number is essentially the ratio $|\rho\mathbf{V}_n\cdot\nabla\mathbf{V}_n|/|\eta\nabla\times\nabla\times\mathbf{V}_n|$ in Eq. (1).] We also know that for some higher value of this same parameter $\mathcal{R}=\mathcal{R}_c$ the flow becomes unstable. For the normal component of helium II it was shown that the additional requirement $\mathcal{G}\ll 1$ was necessary before we could justify that the flow was stable by the principle of minimum entropy production. Thus, it seems natural to assume that at some higher value of this same parameter $\mathcal{G}=\mathcal{G}_c$, the flow will become unstable. Incidentally, the change at such an instability will not necessarily be to turbulence, and the subsequent state might be another steady regime of flow which again minimizes the entropy production subject to the existing constraints by reducing the average relative velocity. If this last conjecture is correct, the instability criterion might have a rigorous thermodynamic basis, but here it is introduced as a hypothesis.

Now it is reasonable to ask why we chose an instability criterion which is the ratio of two entropy production rates rather than the more traditional ratio of two forces, as is done in defining the Reynolds number. This choice is possible because both terms are dissipative and seems natural in the context of the thermodynamic argument. Furthermore, it has the advantage of being the ratio of two scalars. On the other hand, the Reynolds number, for instance, is the ratio of the absolute value of two vectors, whose directions, although evidently important, are neglected. For this reason it is hardly surprising to find that $\mathcal{R}_c$ varies greatly with the geometry of the flow. On the other hand, $\mathcal{G}$ has for every fluid element a physical meaning which depends only on the relative magnitude of the two entropy productions. If an instability can be described by such a parameter, the most obvious assumption is that this instability should take place when the two entropy production terms are equal. Thus, we are led to make the hypothesis that the flow of helium II becomes unstable when $\mathcal{G}=\mathcal{G}_c$ where $\mathcal{G}_c$ is approximately constant for different experiments and of the order of unity. The subsequent sections are mainly devoted to comparing the instabilities predicted by this hypothesis with the flow transitions experimentally observed.

When terms of the form $\mathbf{V}\cdot\nabla\mathbf{V}$ in Eq. (11) can no longer be neglected, we expect a type of turbulence or secondary flow analogous to that found in a normal viscous fluid. For oscillating boundary experiments, Donnelly and Hallett[19] have shown that usually two critical velocites can be identified. The lower one, which they identify with the entrainment of the superfluid, will be shown in the next section to satisfy the condition $\mathcal{G}=\mathcal{G}_c$. The higher one they showed to be correlated with a critical value of a Reynolds number of the form

$$\mathcal{R}=\rho Vd/\eta, \tag{18}$$

where $\rho$ is the total density, $V$ is the boundary velocity (which equals $V_n$ at the boundary), and $\eta$ is approximately the viscosity of the normal fluid. There is evidence[19] of an increase of effective viscosity for $\mathcal{G}>\mathcal{G}_c$, but near the critical velocity this is not large and will be ignored at present.

[18] In He II the temperature gradient furnishes an entropy production term, but at low heat current densities this is proportional to that of viscous dissipation and so does not alter the functional dependence of $\mathcal{G}$.

[19] R. J. Donnelly and A. C. Hollis Hallett, Ann. Phys. (New York) 3, 320 (1960).

For $\mathcal{G}<\mathcal{G}_c$, we might expect a turbulence in the normal fluid corresponding to a critical value of a Reynolds number $\mathcal{R}_n=\rho_n V_n d/\eta$, but it would be difficult to fulfill both of these requirements except near $T_\lambda$, when the transition would probably be indistinguishable from that given by Eq. (18).

In thermal counterflow the two fluids are constrained to have a relative linear velocity, and perhaps some more complex expression is necessary; but at low temperatures where $|\mathbf{V}_n|\gg|\mathbf{V}_s|$ we might expect Eq. (18) with $V=V_n$ to be appropriate.

## III. APPLICATION TO EXPERIMENTAL RESULTS

### Thermal Counterflow

Assume that a long, straight cylindrical tube of negligible thermal conductivity is filled with liquid helium and has a heat source at one end and a heat sink at the other. The normal fluid carries all the entropy so that we have

$$\mathbf{w}=\rho ST\mathbf{V}_n, \tag{19}$$

where $\mathbf{w}$ is the heat current density and $S$ is the entropy per unit mass. Since there is no net mass transfer, we have from Eq. (9) and the assumption of nearly uniform temperature

$$\rho_s\langle\mathbf{V}_s\rangle_{\text{av}}+\rho_n\langle\mathbf{V}_n\rangle_{\text{av}}=0. \tag{20}$$

At low heat current densities, $V_n$ and $V_s$ are both small, and we may assume that Eq. (12) gives the steady, low-velocity laminar solution. If the axis of the tube (radius $=a$) coincides with the $x$ axis and the radial distance is $r$, the boundary conditions

$$\mathbf{V}_n=0 \quad \text{at} \quad r=a, \tag{21a}$$

$$\mathbf{V}_n\neq\infty \quad \text{at} \quad r=0, \tag{21b}$$

lead to the usual Poiseuille solution. The normal fluid velocity not too close to the ends is only in the $x$ direction and is given by

$$V_n=-(1/4\eta)(dp/dx)(r^2-a^2)=2\bar{V}_n(1-r^2/a^2). \tag{22}$$

Equations (10) and (20) imply

$$\langle\mathbf{V}_n\rangle_{\text{av}}-\langle\mathbf{V}_s\rangle_{\text{av}}=(\rho/\rho_s)\langle\mathbf{V}_n\rangle_{\text{av}}, \tag{23}$$

and from (22),

$$\langle|\nabla\times\mathbf{V}_n|^2\rangle_{\text{av}}=8\bar{V}_n/a^2. \tag{24}$$

Rather than this exact expression, we chose one of greater generality using a characteristic lateral channel dimension $d$. This is useful since we know the exact steady-state solution only in very simple cases, and even with these the experimental boundary conditions are geometrically imperfect. Small-scale surface roughness has little effect on viscous flow because the velocities near the wall are small, but no such statement can be made for the superfluid whose maximum velocities will presumably be near convex wall irregularities, so the apparent accuracy of $\mathcal{G}$ would usually not be real.

Thus, for thermal counterflow we write Eq. (17) in the form

$$\mathcal{G}=f_1(T)V_n d, \tag{25}$$

where

$$f_1(T)=(\rho_s\rho_n A/\eta)^{1/2}(\rho/\rho_s)^2$$

and $V_n$ is the average normal-fluid velocity.

Figure 1 (solid lines) shows a family of theoretical curves for the product $V_n d$ as a function of temperature according to Eq. (25) for various values of the parameter $\mathcal{G}$. The velocity symbol $V$ is used in this figure so that theoretical curves for other types of flow, to be described later, can be represented on this same figure (dashed lines) with their appropriate characteristic velocity. The values of the constants used in the temperature-dependent factor are shown in Table I. The values of $\rho_n$ are from Bendt, Cowan, and Yarnell[20]; the values of $\rho$ from Kerr.[21] The normal fluid viscosity $\eta$ is from measurements by Heikkila and Hallett[22] and by Brewer and Edwards.[23] The values of the Gorter-Mellink constant $A$ were calculated from the same data which these authors used. The magnitude of $A$ increases with the relative velocity apparently to some limiting value at high velocities; it is this limiting value which is used, and it agrees rather well with the values given by Vinen.[24] The values below 1.2°K were extrapolations, and above 2.0°K the scatter in the data is large.

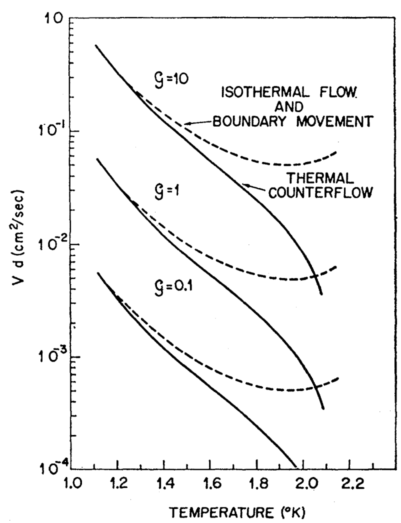


FIG. 1. Theoretical curves for the product of the critical velocity and the characteristic channel dimension for various values of the Gorter number, $\mathcal{G}$. For thermal counter flow $V=V_n$; for isothermal flow $V=V_s$, and for boundary motion $V=V_{\text{boundary}}$.

[20] P. J. Bendt, R. D. Cowan, and J. L. Yarnell, Phys. Rev. **113**, 1386 (1959).
[21] E. C. Kerr, J. Chem. Phys. **26**, 511 (1957).
[22] W. J. Heikkila, A. C. Hollis Hallett, Can. J. Phys. **33**, 420 (1955).
[23] D. F. Brewer and D. O. Edwards, Proc. Roy. Soc. (London) **A251**, 247 (1959).
[24] W. F. Vinen, Proc. Roy. Soc. (London) **A240**, 114 (1957).

Figure 2 shows the experimental data on the observed critical velocities in thermal counterflow. The product of the critical value of $V_n$ and $d$ is plotted as a function of temperature. The theoretical curve from Eq. (25) with $\mathcal{G}=4$ is shown for comparison. Since the channels were different in cross-sectional shape, $d$ was taken to be the hydraulic diameter.[25] Mellink[26] used a glass slit $1.05\times10^{-3}$ cm wide ($d=2.10\times10^{-3}$cm) and the data plotted were recalculated by Lifshitz and Andronikashvili,[27] who corrected for the effect of vapor pressure. Winkel, Broese van Groenou, and Gorter[3] used a similar glass slit $2.4\times10^{-4}$ cm wide ($d=4.8\times10^{-4}$ cm). Meservey[28] measured the slope of the free surface of a horizontal layer of liquid helium about $10^{-2}$ cm thick as a function of heat current density and $d$ is taken to be 4 times the liquid depth; details of this experiment will be published soon.

In the data of Kramers, Wiarda, and Broese van Groenou,[29] the tubes were so short that the normal fluid boundary layer could not have propagated across the channel, so $d$ was taken as twice the dimensional expression for the boundary layer thickness[30] of the normal fluid,

$$\delta=(\eta L/\rho_n\bar{V}_n)^{1/2}, \tag{26}$$

where $L$ is the length of the tube. The definition of the hydraulic diameter does not apply to this case so that the factor of 2 has little significance and may be regarded as a normalizing factor to bring out the similarity of the temperature dependence with other measurements.

Table I. Values of the temperature-dependent quantities used in the calculations.

| $T$ (°K) | $\rho_n$ (g/cm³) | $\rho$ (g/cm³) | $\eta\times10^5$ (poise) | $A$ (cm sec/g) |
|---|---|---|---|---|
| 2.15 | 0.126 | 0.1463 | 2.33 | 320[a] |
| 2.1 | 0.120 | 0.1462 | 1.96 | 115[a] |
| 2.0 | 0.0874 | 0.1460 | 1.48 | 110 |
| 1.9 | 0.0647 | 0.1458 | 1.36 | 99 |
| 1.8 | 0.0473 | 0.1457 | 1.32 | 92 |
| 1.7 | 0.0342 | 0.1456 | 1.32 | 82 |
| 1.6 | 0.0241 | 0.1456 | 1.34 | 71 |
| 1.5 | 0.0165 | 0.1455 | 1.36 | 60 |
| 1.4 | 0.0108 | 0.1455 | 1.44 | 48 |
| 1.3 | 0.00672 | 0.1455 | 1.60 | 35 |
| 1.2 | 0.00391 | 0.1455 | 1.90 | 27 |
| 1.1 | 0.00207 | 0.1455 | 2.44 | 19[b] |

[a] Estimated from scattered data of reference 6.
[b] Extrapolated from data of reference 6.

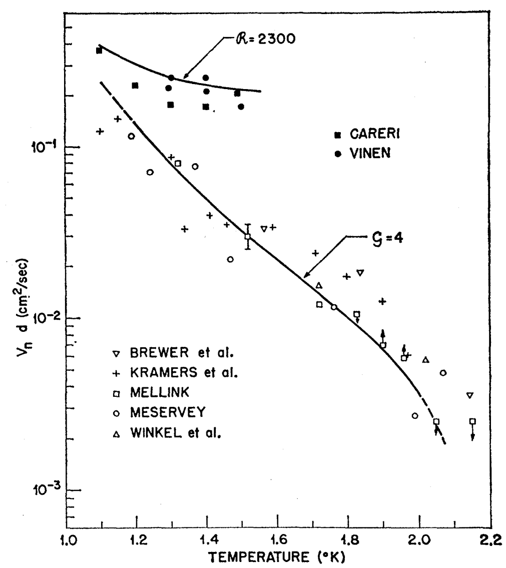


Fig. 2. Product of the critical normal-fluid velocity and the characteristic lateral channel dimension in thermal counterflow as a function of temperature. The measured points are from references, 3, 26, 28, 23, 29, 31, 32, 33; the solid lines are calculated from equations (18) and (25) for the given values of the Reynolds number $\mathcal{R}$ and the Gorter number $\mathcal{G}$.

The experiments of Brewer, Edwards, and Mendelssohn[31] on heat flow in a circular glass tube (diam$=d$ $=5.2\times10^{-3}$ cm) are interesting because, with precautions against vibrations and initial turbulence, they sometimes observed a metastable laminar flow at a heat current considerably larger than the usual critical value. This may account for the somewhat higher value of $\mathcal{G}_c$, although, if we carry out the exact calculation for this case, we find that the entropy production in the viscous liquid equals that due to the mutual friction force for $\mathcal{G}\approx7$, which is very close to the $\mathcal{G}$ value for the transitions actually observed.

The solid circles in Fig. 2 show the critical velocities given by Vinen[32] for rectangular channels using an ingenious second-sound detection technique. His channels were $0.240\times0.645$ cm and $0.400\times0.783$ cm in cross section and were both 10 cm long. The data of Careri, Scaramuzzi, and McCormick[33] are also shown by the solid squares. They used a rectangular tube of cross section similar to Vinen's smaller tube, but 24 cm long. Here the critical velocity was detected by measuring the mobility of ions in the liquid. The upper line of Fig. 1 is a plot of $V_nd$ from Eq. (18) with $V=V_n$, with $d$ again taken as the hydraulic diameter, and with a Reynolds number of 2300, which is often taken as the

[25] The hydraulic diameter $d=4$ area/perimeter is commonly used as the characteristic dimension for turbulence in noncircular pipes. See, for instance, H. Schlichting, Boundary Layer Theory (Pergamon Press, London, 1955). p. 32.
[26] J. H. Mellink, Physica **13**, 180 (1947).
[27] E. M. Lifshitz and E. L. Andronikashvili, A Supplement to *Helium* (Consultants Bureau Inc., New York, 1959).
[28] R. Meservey, Ph.D. thesis, Yale University, 1960 (unpublished).
[29] H. C. Kramers, T. M. Wiarda, and A. Broese Van Groenou, *Proceedings of the Seventh International Conference on Low-Temperature Physics, Toronto, 1960* (University of Toronto Press, Toronto, 1961), p. 562.
[30] H. Schlichting, *Boundary Layer Theory* (Pergamon Press, New York, 1955).
[31] D. F. Brewer, D. O. Edwards, and K. Mendelssohn, Phil. Mag. **1**, 1130 (1956).
[32] W. F. Vinen, Proc. Roy. Soc. (London) **A243**, 400 (1957).
[33] G. Careri, F. Scaramuzzi, and W. D. McCormick, *Proceedings of the Seventh International Conference on Low-Temperature Physics, Toronto, 1960* (University of Toronto Press, Toronto, 1961), p.502.

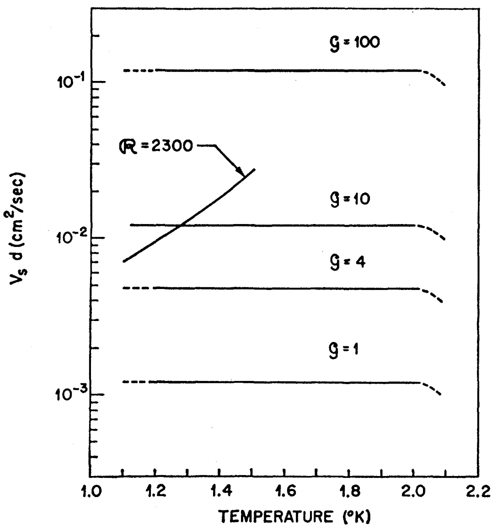


FIG. 3. Transformation of the curves for thermal counterflow shown in Fig. 2 to a plot of the product of the critical superfluid velocity and the characteristic lateral channel dimension as a function of temperature.

critical Reynolds number for a viscous liquid in a similar geometry.[34] Although such a Reynolds number is a plausible explanation for the quite different magnitude and temperature dependence of these critical velocities, inlet and end effects in such short tubes are considerable and probably cannot be ignored. For instance, such effects probably account for the 25% increase in the critical Reynolds number in Vinen's wider channel. Recent measurements by Chase[35] using both the delay-time technique and a measurement of thermal resistance confirm and extend Vinen's results and when completed should throw considerable light on these complex effects.

Although $V_n d$ appeared to be the most natural parameter to use in Fig. 2 because of the origin of $\mathcal{G}$, the fact that $\rho_s\langle \mathbf{V}_s\rangle_{\text{av}}+\rho_n\langle \mathbf{V}_n\rangle_{\text{av}}=0$ in thermal counterflow assures just as orderly a result if we use the parameter $V_s d$. In terms of the superfluid velocity, Eq. (25) becomes

$$\mathcal{G}=(\rho_s/\rho_n)f_1(T)V_s d. \tag{25a}$$

The transformation of $\mathcal{G}=4$ and $\mathcal{R}=2300$ to this type of plot is shown in Fig. 3. Within the limit of error of the numerical factors in Eq. (25a), the values of $V_s d$ corresponding to a given value of $\mathcal{G}$ are independent of temperature (at least up to $T=2.0°$K) in agreement with the relation suggested by Atkins,[2]

$$(V_s d)_c=(4\hbar/m)\ln(d/4a). \tag{27}$$

The magnitude of this expression also approximately agrees with Eq. (25a) for $\mathcal{G}=4$, but the uncertainty in the choice of both $a$ and $d$ does not allow us to verify or exclude Eq. (27) on the basis of the rather scattered experimental results.

### Moving Boundary

When a solid of revolution, which is immersed in helium II, executes torsional oscillations about its axis, Andronikashvili[36] showed that for small amplitude oscillations $V_s=0$. Furthermore, since the normal-fluid velocity at the boundary equals the boundary velocity $V$, we can write Eq. (17) as

$$\mathcal{G}=f_2(T)Vd, \tag{28}$$

where $f_2(T)=(\rho_s\rho_n A/\eta)^{1/2}$. In this case, $d$ is taken to be the penetration depth in the normal fluid,[30] which is a function of the angular frequency, $\omega$.

$$d=(2\eta/\rho_n\omega)^{1/2}. \tag{29}$$

A closely related case concerns flow past a plane solid boundary parallel to the flow; in this case $d$ is the boundary-layer thickness at a distance $L$ downstream from the leading edge and is the same as $\delta$ in Eq. (26). Theoretical values of $Vd$ as a function of temperature for various values of $\mathcal{G}$ in Eq. (28) are shown as the dashed lines in Fig. 1.

Figure 4 shows the data of Anglin and Benson[37] on oscillating cylinders, one with diameter 3.19 cm and periods 4.5 and 36 sec, the other with diameter 2.545 cm and period 5.25 sec. These data are the latest and perhaps the best of the oscillating-boundary measurements and are shown to agree rather well with Eq. (28)

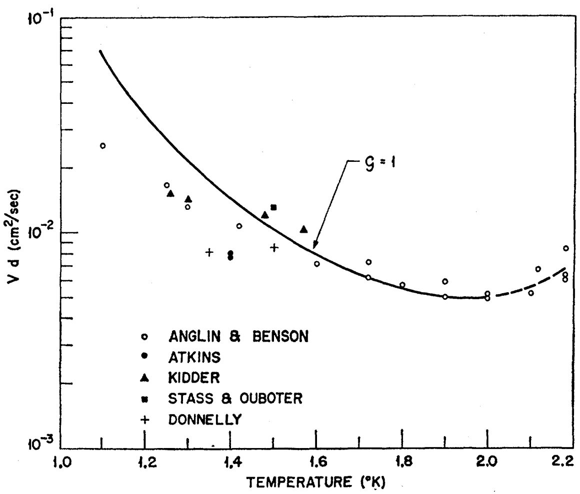


FIG. 4. Product of the critical velocity and the characteristic lateral channel dimension in boundary motion and isothermal flow. The measured points are from references 2, 37, 38, 39, and 40; the solid line is calculated from equation (28) or (31) for the given value of the Gorter number $\mathcal{G}$.

[34] See, for instance, reference 30, p. 32.

[35] C. E. Chase, *Proceedings of the Seventh International Conference on Low-Temperature Physics, Toronto, 1960* (University of Toronto Press, Toronto, 1961), p. 438; and (private communication).

[36] E. L. Andronikashvili, J. Phys. (U.S.S.R.) **10**, 210 (1946).

[37] F. Anglin and C. B. Benson, *Proceedings of the Seventh International Conference on Low-Temperature Physics, Toronto, 1960* (University of Toronto Press, Toronto, 1961), p. 558.

for $\mathcal{G}=1$, using for $d$ the penetration depth as given in Eq. (29).

For the sake of clarity, only one oscillating-boundary experiment is shown in Fig. 4. However, experiments with oscillating spheres, disks, and oscillations in U-tubes were reviewed by Donnelly and Hallett,[19] and they proposed an empirical Reynolds number

$$\mathcal{R}_{ns}\equiv(Vd/\nu)[(\rho_n\rho_s)^{1/2}/\rho], \qquad (30)$$

where $\nu\approx\eta/\rho$, although measured values of this kinematic viscosity were used which are slightly higher than $\eta/\rho$.

The functional form of $\mathcal{R}_{ns}$ is the same as Eq. (28) for $\mathcal{G}$ in this case, and the temperature-dependent factor is numerically similar. These authors showed that all low-velocity transitions correlated with such an empirical Reynolds number. The correlation with a value of $\mathcal{G}_c$ for each geometry is thus assured to a fair degree of accuracy.

Below 1.6°K, the empirical temperature dependence of $\mathcal{R}_{ns}$ fits the data in Fig. 4 within the limit of error, whereas $\mathcal{G}$ is too high. Actually this low-temperature effect is perhaps present in the thermal counterflow results shown in Fig. 2. Whether this discrepancy is fundamental to the theory, or owing to a neglected correction, is not clear.

### Isothermal Flow

When $V_s\gg V_n$ and we assume a mutual friction force, the normal fluid velocity is determined by the balance of the mutual friction force and the normal fluid resistance on the channel walls so that $\rho_s\rho_nAV_s^3\approx\eta V_n/d^2$ and

$$1/\mathcal{G}=f_2(T)V_sd, \qquad (31)$$

where $f_2(T)=(\rho_s\rho_nA/\eta)^{1/2}$. Theoretical values of $V_sd$ as a function of temperature for various values of $\mathcal{G}$ in Eq. (31) are again given by the dashed lines of Fig. 1, assuming in this case $V=V_s$.

Data for isothermal flow are plotted in Fig. 4, and $V$ is to be interpreted as $V_s$. Atkins' data[2] are for gravity flow in glass capillaries with diameters $2.6\times10^{-3}$ cm and $8.1\times10^{-3}$ cm. Kidder and Fairbank[38] measured the superfluid velocity at which a pressure gradient appeared in a 0.11-cm-diam stainless steel tube whose ends were closed with packed powder filters which allowed only the superfluid to flow into and out of the tube. The measurement of Stass and Ouboter[39] is for gravity flow in a glass capillary of diameter $2.60\times10^{-2}$ cm with temperature regulation at each end.

[38] J. N. Kidder and W. M. Fairbank, *Proceedings of the Seventh International Conference on Low-Temperature Physics, Toronto, 1960* (University of Toronto Press, Toronto, 1961), p. 560.

[39] F. A. Stass and R. De Bruyn Ouboter, Physica Suppl. **24**, 143 (1958).

### Steady Rotation

Experiments involving steady rotation can perhaps be analyzed like other moving-boundary experiments. Donnelly,[40] using concentric cylinders with the inner one rotating, has observed two critical velocities, the lower of which at $T=1.35$ and 1.5°K is shown on Fig. 4 to agree fairly well with other measurements. Few other data giving definite critical velocities in steady rotation have been published, and these are difficult to analyze because of transient effects or centrifugal forces, so further comparison will be deferred. In fact, Donnelly's lower critical velocities just cited were only about half the higher critical velocities, which correspond rather closely to ordinary Taylor instability for a liquid with a kinematic viscosity of $\eta/\rho$.[41] This implies that neglect of the centrifugal forces even for the lower critical velocities is perhaps not justified.

When centrifugal forces cannot be neglected we expect the two-fluid analog of Taylor instability. Chandrasekhar and Donnelly[42] have derived such an instability criterion for liquid helium II between rotating coaxial cylinders with the assumption that the coupling between the two fluids is proportional to both the relative velocity and the constant vorticity of the steady state in this geometry. Presumably, both branches of such an instability, which is based on the assumption that $\mathbf{V}_s=\mathbf{V}_n$ initially, would be observable only above the low-velocity transitions with which we are presently concerned. In this connection it should be noted that the thermodynamic instability discussed previously is different from the dynamic instability calculation carried out by Chandrasekhar and Donnelly. In the latter, the stability of a system is investigated in the vicinity of a steady state where the relative velocity of the two fluids vanishes; in the former, the initial state for $\mathcal{G}\approx\mathcal{G}_c$ usually involves finite or even high relative velocity, and linearity of the interaction is not essential to the argument.

## IV. DISCUSSION

The above analysis is successful enough that we should examine more closely the validity of the mutual friction term on which it is based. That such a term is necessary to explain the heat conduction in liquid helium was demonstrated by Gorter and Mellink[6] and by Atkins.[43]

It has often been suggested that other frictional forces besides normal fluid viscosity and mutual friction are

[40] R. J. Donnelly, Phys. Rev. Letters **3**, 507 (1959).

[41] The point given by Donnelly at $T=2.1$°K is actually at a slightly larger velocity than that calculated for the onset of Taylor instability [G. I. Taylor, Phil. Trans. Roy. Soc. London **A223**, 280 (1923)] in the normal fluid. Possibly a transition at lower velocity was not observed because of the very small change in effective kinematic viscosity to be expected from the entrainment of the superfluid at this temperature.

[42] S. Chandrasekhar and R. J. Donnelly, Proc. Roy. Soc. (London) **A241**, 9 (1957).

[43] See reference 2, p. 190.

necessary to explain certain experimental results. Heikkila and Hallett's[22] experiment with a rotating cylinder viscometer apparently demonstrated that an additional frictional force was necessary to explain the observed drag, but actually the effect was later observed[44] using carbon disulfide at room temperature and was presumably the onset of a secondary flow near the ends of the cylinder. The damping of an oscillating disk also measured by Hallett[45] was, at low temperatures, much greater than that calculated by Zwanniken,[46] assuming mutual friction and that the superfluid velocity is zero at all times. In a later analysis, Donnelly and Hallett[19] concluded that in the velocity range in question the superfluid in the boundary layer must move with the normal fluid. If we assume that the thickness of this boundary layer is approximately equal to the penetration depth of the normal fluid, the temperature dependence of the extra damping is approximately correct, but an exact calculation would necessitate an exact knowledge of the flow, which at present we do not have. Kidder and Fairbank,[38] in their experiment with flow in a 1.1-mm tube between superfluid filters, found that the pressure increased with superfluid velocity more rapidly than that calculated from the Gorter-Mellink term, assuming the normal-fluid velocity is everywhere zero. However, Newton's third law precludes the normal fluid remaining stationary if there is a mutual friction force, and we would expect some kind of circulating flow of the normal fluid in the space between the filters. Again, until the exact nature of the flow is known we cannot make an exact calculation of the pressure gradient, but we also cannot conclude that a direct interaction between the superfluid and the wall is required. Thus, these experiments which have been cited to show the inadequacy of mutual friction currently seem to be compatible with it.

However, there is a group of experiments which show that the Gorter-Mellink equations do not always apply. Experiments by Craig and Pellam,[47] Reppy and Lane,[48] Brewer, Edwards, and Mendelssohn,[23,31] and Kidder and Fairbank[38] all demonstrate that, below a certain critical velocity, $\mathbf{F}_{sn}=0$ or at least has a value much less than that given by Gorter and Mellink. Above the critical velocity, the mutual friction rapidly rises to its expected value. The last two experiments just mentioned are of particular interest because they seem to fall into the present pattern of results without apparently having initially present both of the competing mechanisms of dissipation.

One way to resolve this paradox is to assume that the instability criterion initially applies only to a thin boundary region whose effect on the overall pressure and temperature gradients is negligible. It was shown by Hall[49] that the surface roughness of a boundary is important in the development of additional friction. And the metastability observed by Brewer, Edwards, and Mendelssohn and by Reppy and Lane is perhaps connected with the smoothness of the drawn glass walls used, since there is no clear evidence for such metastability with metal containers, mica disks, or optical flats, all of which are microscopically rougher. These facts lead us to believe that the additional frictional forces develop close to the boundary and then propagate into the body of the fluid. Apparently, the steady motion of a front of this kind has been observed by Pellam[50] in a cylindrical container brought into steady rotation.

A second way to explain the results, which does not necessarily conflict with the first, is that the flow instability arises from a neglected inertial term of the hydrodynamic equations. If we assume that the equations given by Landau and London are correct [Eqs. (6) and (7) without $\mathbf{F}_{sn}$], the only apparent source of such an instability which has not yet been discussed is in the terms in the square of the relative velocity,

$$(\rho_s\rho_n/2\rho)\nabla|\mathbf{V}_n-\mathbf{V}_s|^2.$$

It would be interesting to see if a dynamic stability calculation including such a term would lead to the observed critical velocities.[51] From this point of view, mutual friction would be attributable to the new flow pattern above the critical velocity rather than to the appearance of an additional term in the equations. An analogous conclusion is generally accepted concerning turbulence in a viscous liquid, namely, that the Navier-Stokes equation is still applicable, but above a certain velocity other solutions besides the laminar solution are possible. Thus, all dissipation is attributable to viscosity, and the nonlinear frictional forces are not caused by a new dissipative mechanism, but rather by the higher shear rates in the new flow pattern. A similar view can be taken with helium II, since there seems to be no evidence that the mutual friction and the friction of normal-fluid viscosity are additive, either above or below the critical velocity. This leads us to the possibility that practically all dissipation, even above the critical velocity, may be attributable to the viscosity of the normal fluid in the new flow pattern in which the rate of shear of the normal fluid is much greater.

---

[44] A. D. B. Woods and A. C. Hollis Hallett, *Proceedings of the Fifth International Conference on Low-Temperature Physics and Chemistry, Madison, Wisconsin, 1957* (University of Wisconsin Press, Madison, 1958), p. 16.

[45] A. C. Hollis Hallett, Proc. Roy. Soc. (London) **A210**, 404 (1952).

[46] G. C. J. Zwanniken, Physica **16**, 805 (1950).

[47] P. P. Craig and J. R. Pellam, Phys. Rev. **108**, 1109 (1957).

[48] J. D. Reppy and C. T. Lane, *Proceedings of the Seventh International Conference on Low-Temperature Physics, Toronto, 1960* (University of Toronto Press, Toronto, 1961), p. 443.

[49] H. E. Hall, Phil. Trans. Roy. Soc. (London) **A250**, 369 (1957).

[50] J. R. Pellam, *Proceedings of the Seventh International Conference on Low-Temperature Physics, Toronto, 1960* (University of Toronto Press, Toronto, 1961), p. 446.

[51] The instability criterion $\mathcal{G}=\mathcal{G}_c$, based on thermodynamic reasoning cannot yield details of the process of instability; a dynamical stability calculation from the true hydrodynamic equations could yield such details and naturally the result would not conflict with the thermodynamic hypothesis if the latter were correct.

The view that the buildup of mutual friction is caused by the formation of vortex line in the superfluid was suggested by Feynman[4] and developed by Vinen.[52] These vortices are possible solutions of the equations of motion, and nothing in the present hydrodynamic analysis appears to conflict with this point of view. In thermal counterflow Eq. (25a) of the present theory is not at variance with Eq. (27) suggested by Atkins on the basis of vortex creation. Other qualitative features can also be given plausible explanation in terms of vortex theory, but the exact relationship of the present macroscopic theory to the microscopic motion will not be considered here, since the exact nature of these microscopic motions is not settled.

The above analysis has been applied to channels with $d \geq 4.8 \times 10^{-4}$ cm. However, the characteristic temperature dependence of the critical velocities appears to persist into somewhat narrower channels. Whether the critical velocities in narrower channels are of a quite different origin has yet to be determined, but the separation between the two regions is apparently not completely sharp.

## V. CONCLUSIONS

Starting from hydrodynamic equations which contain a mutual friction force, we have proposed an instability condition which corresponds to the measured critical velocities with considerable accuracy. In fact, in certain ways, this instability condition seems more general than the equations on which it was based. This phenomenological theory puts the data into a consistent pattern and shows that the critical velocities in thermal counterflow, boundary motion, and isothermal flow have much the same character. Complicating features which need further investigation are clearly differentiated from the rest of the pattern. Quite aside from any theoretical basis, the rather simple correlation of many experimental results by the Gorter number $\mathcal{G}$ should be useful in the design of future experiments.

The principle of minimum entropy production was used here in a negative way to show under what circumstances we could not justify the stability of laminar flow of the normal component of helium II. Perhaps this principle could be applied more generally to turbulence, but in any case the macroscopic nature of the argument and its success indicate that the critical velocities in helium II can be considered from a continuum point of view. This suggests that dynamic stability calculations and statistical methods, which have led to the recent progress in understanding the turbulence of ordinary liquids, can be usefully applied to helium II.

## ACKNOWLEDGMENTS

I am grateful to Professor Lars Onsager, Professor C. T. Lane, Professor John Wilks, and Professor C. C. Lin for helpful discussion and criticism. I also wish to acknowledge useful suggestions by Dr. C. E. Chase and J. C. Fineman.

[52] W. F. Vinen, Proc. Roy. Soc. (London) **A242**, 493 (1957).