Kinetic Theory of Quantum Plasmas

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A unified theory of equilibrium and nonequilibrium quantum plasmas is presented, using an approach originally developed by Bogoliubov. The results, which are valid in both high temperature-low density and (for Fermi gases) low temperature-high density regions, include: (1) an expression for the equilibrium correlation function (and, hence, the thermodynamic functions) which agrees with results previously obtained by other methods, (2) the kinetic equation for homogeneous "steady state" systems recently derived by Balescu using a different approach, and (3) a new kinetic equation which is valid for inhomogeneous and/or rapidly varying systems.

I. INTRODUCTION

THE theory of quantum plasmas has been attacked by a number of different methods. In the equilibrium theory, the principal results have been obtained by summation over ring diagrams,[1–4] by the random phase approximation,[5,6] and by the self-consistent field approach.[7] For nonequilibrium, homogeneous, "steady state" systems, a kinetic equation has been derived by Balescu,[8] using a method developed by Prigogine and Balescu.[9]

In the present work, it is shown how all of these results may be obtained on the basis of some very simple physical considerations, using a method[9a] originally developed by Bogoliubov[10] which has proved successful in the classical theory.[11–16] Furthermore, a new kinetic equation is derived which is valid for inhomogeneous systems and arbitrary time scale, provided the system is not too far from equilibrium. The results are valid for both Bose and Fermi systems in the high temperature-low density limit, and also for Fermi gases in the low temperature-high density limit (the regions of validity will be discussed more precisely below).

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[9a] *Note added in proof*. The author's attention has been called to some recent work on this subject by V. P. Silin [Soviet Phys.—JETP **13**, 1244 (1961); Phys. Metals Metallog. (USSR) **11**, 805 (1961)]. An approach similar to that used in this paper has been employed to derive an expression for the uniform "steady" state correlation function and the corresponding kinetic equation. The results are in agreement with those of Sec. IV.
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II. THE GENERAL FORMALISM

We consider a gas of an arbitrary number of types of ions, with N_σ of type σ. Type σ will be specified by charge e_σ, mass m_σ, and statistics $\alpha(\sigma)$ ($=1$ for Bose, -1 for Fermi, 0 for Boltzmann statistics). The macroscopic properties of the gas may be described in terms of a density operator ρ_N, which has the properties

$$\mathrm{Tr}[\rho_N]=1 \tag{1}$$

and

$$\mathrm{Tr}[\rho_N D]=\langle D\rangle, \tag{2}$$

where D is any dynamical variable and $\langle D\rangle$ its expectation value. ρ_N satisfies the equation

$$\partial\rho_N/\partial t=[H_N,\rho_N]/i\hbar, \tag{3}$$

analogous to the classical Liouville equation, where H_N is the Hamiltonian and $[A,B]$ denotes the commutator as usual. We will consider a system of point charges interacting only through the Coulomb potential.

In order to develop an approximation method for determining the properties of the gas, it is convenient to define reduced density operators by

$$\rho_s=(V/N)^s\prod_{i=1}^{s}N_{\sigma_i}\cdot\mathrm{Tr}_{s+1}{}^N[\rho_N], \tag{4}$$

where the trace is taken over all but the first s particles (these may be of different types, with the ith particle of type σ_i). Taking the corresponding traces in (3), and letting

$$N=\textstyle\sum_\sigma N_\sigma\rightarrow\infty,\quad V\rightarrow\infty,\quad V/N\rightarrow v, \tag{5}$$

so that the ratio s/N may be neglected, one finds the quantum B-BG-K-Y hierarchy, which may be written in the position representation as

$$(\partial\rho_s/\partial t)(x_1\cdots x_s)=(-\mathcal{T}_s+\mathcal{U}_s)\rho_s(x_1\cdots x_s)+\frac{1}{v}\int dx_{s+1}\,\mathcal{V}_s(x_1\cdots x_s;x_{s+1})\rho_{s+1}(x_1\cdots x_{s+1}), \tag{6}$$

where we have used the shorthand

$$x_i=(\mathbf{q}_i,\mathbf{q}_i',\sigma_i), \tag{7}$$

$$\int dx_{s+1}=\textstyle\sum_{\sigma_{s+1}}\mathrm{Tr}_{(s+1)}, \tag{8}$$

and

$$\mathcal{T}_s(x_1\cdots x_s)=\frac{\hbar}{2i}\sum_{j=1}^{s}\frac{1}{m_\sigma}(\nabla_{\mathbf{q}_j}{}^2-\nabla_{\mathbf{q}_j'}{}^2), \tag{9}$$

$$\mathcal{U}_s(x_1\cdots x_s)=\frac{1}{i\hbar}\sum_{j<k\leq s}\sum e_{\sigma_j}e_{\sigma_k}\left(\frac{1}{|\mathbf{q}_j-\mathbf{q}_k|}-\frac{1}{|\mathbf{q}_j'-\mathbf{q}_k'|}\right), \tag{10}$$

$$\mathcal{V}_s(x_1\cdots x_s;x_{s+1})=\frac{e_{\sigma_{s+1}}}{i\hbar}\sum_{j=1}^{s}e_{\sigma_j}\left(\frac{1}{|\mathbf{q}_j-\mathbf{q}_{s+1}|}-\frac{1}{|\mathbf{q}_j'-\mathbf{q}_{s+1}'|}\right). \tag{11}$$

It is convenient to introduce directly the symmetry requirements on the functions ρ_s by means of

$$\rho_s=\gamma_s F_s, \tag{12}$$

where γ_s is a symmetrization (antisymmetrization) operator defined by

$$\gamma_s=\prod_{j=2}{}^{s}(1+\sum_{k=1}{}^{j-1}\delta_{\sigma_j\sigma_k}\alpha(\sigma_j)P_{jk}), \tag{13}$$

where P_{jk} permutes the variables $\mathbf{q}_i$, $\mathbf{q}_j$. Since γ_s satisfies the relation

$$\gamma_{s+1}=\gamma_s(1+\sum_{j=1}{}^{s}\alpha(\sigma_j)\delta_{\sigma_j\sigma_{s+1}}P_{j,s+1}), \tag{14}$$

and commutes with the operators $\mathcal{T}_s$, $\mathcal{U}_s$, $\mathcal{V}_s$, one may substitute (12) into (6) and factor out γ_s to obtain the equation

$$\frac{\partial F_s}{\partial t}=-\mathcal{T}_sF_s+\mathcal{U}_sF_s+\frac{1}{v}\int dx_{s+1}\,\mathcal{V}_sF_{s+1}+\frac{1}{v}\int dx_{s+1}\,\mathcal{V}_s\sum_{j=1}^{s}\alpha(\sigma_j)\delta_{\sigma_j\sigma_{s+1}}P_{js+1}F_{s+1}. \tag{15}$$

The remainder of the report is devoted to the approximate solution of this hierarchy.

III. THE APPROXIMATION METHOD

Equation (15) very closely resembles the classical B-BG-K-Y hierarchy, to which it becomes equivalent as $\hbar\to 0$. Just as in the classical case, the second term on the right, representing the mutual interaction of the s-tuple under consideration, may be treated as small compared to the first, provided[17] the average Coulomb energy is much less than the average kinetic energy. In the nondegenerate case

$$\theta>E_F=(6\pi^2)^{\frac{2}{3}}\hbar^2/2mv^{\frac{2}{3}}, \tag{16}$$

where $\theta=kT$, this is equivalent to the classical condition

$$e^2/\theta\lambda_D\approx(e^3/\theta)(4\pi/v\theta)^{\frac{1}{2}}\ll 1. \tag{17}$$

For a degenerate Fermi gas the average kinetic energy is of the order of magnitude of E_F, and the average Coulomb energy may be approximated by

$$e^2/[(3/4\pi)v]^{\frac{1}{3}},$$

so we have the conditions[18]

$$\theta\ \ E_F,$$
$$e^2/[(3/4\pi)v]^{\frac{1}{3}}E_F=(2me^2v^{\frac{1}{3}}/3\pi\hbar^2)\ll 1. \tag{18}$$

The second of conditions (18) requires that the average interparticle spacing be much less than the Bohr radius, and the first that it be less than the de Broglie wavelength of the electrons (times a factor of order unity).

The third term on the right of (15) represents the interaction of the s-tuple with the rest of the gas. Even when the Coulomb potential is "weak" in the sense of (17) or (18), this term may not be considered small, due to the infinite range of the Coulomb field. It is this term which produces the effective "shielding" or "screening" of the potential.

Finally, the last term is an exchange type interaction; the integrand, unlike that of the third term, has a finite range.[19] For nondegenerate systems, F_s will vanish if any $|\mathbf{q}_i-\mathbf{q}_i'|$ is much greater than a de Broglie wavelength.[20] It follows that this term is of the order of the average Coulomb energy times a factor λ^3/v, where

$$\lambda=\hbar/(2m\theta)^{\frac{1}{2}} \tag{19}$$

is the electron de Broglie wavelength. But according to (16) this factor is $\lesssim 1$, so the fourth term will be of the same order of magnitude as or smaller than the second. The same holds for a degenerate Fermi gas, where the cutoff occurs approximately at

$$\hbar/P_F\sim v^{\frac{1}{3}}.$$

These considerations show that for the regions under discussion [defined by (16) and (17) or (18)], one may write (15) formally as

$$\left(\frac{\partial}{\partial t}+\mathcal{T}_s\right)F_s=\frac{1}{v}\int dx_{s+1}\,\mathcal{V}_sF_{s+1}+\epsilon\left[\mathcal{U}_sF_s+\frac{1}{v}\int dx_{s+1}\,\mathcal{V}_s\sum_{j=1}^{s}\alpha(\sigma_j)\delta_{\sigma_j\sigma_{s+1}}P_{js+1}F_{s+1}\right] \tag{20}$$

[17] An implicit assumption that the particles are effectively prevented from approaching too close to each other is involved. For a classical gas of charged hard spheres, the approximation will be a good one if the radius of the hard core is not too small with respect to e^2/θ. For a fuller discussion of this point, see reference 12, pp. 27–30.

[18] Requirements (18) may be thought of as defining what we mean by high density–low temperature, while (16) and (17) define the low density–high temperature region.

[19] The case of Bose gases below the condensation temperature is excluded.

[20] This condition and the one while follows for degenerate Fermi gases are a consequence of the vanishing of the Fourier transform of F_s (the Wigner distribution) for $|\mathbf{p}_i|\gtrsim[2m(\theta+E_F)]^{\frac{1}{2}}$.

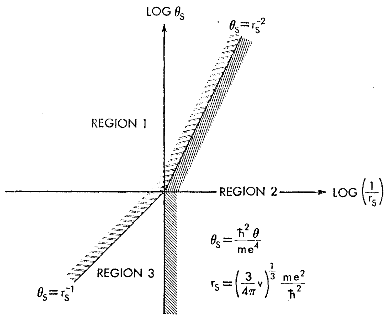

FIG. 1. Approximate regions of validity of results. The graph shows regions of validity of the method used in this paper: region 1 for both Fermi and Bose systems, region 2 for Fermi systems, and region 3 where the method breaks down.

and expand[21] $F_s (s \geq 2)$ in powers of ϵ. The approximate regions of validity of this scheme are illustrated in Fig. 1. Although we will not go beyond first order in ϵ, it should be emphasized that the method may be systematically generalized to arbitrary order.

It is readily verified that the entire hierarchy (20) is satisfied to zeroth order in ϵ by the uncorrelated distributions

$$F_s^{(0)}(x_1 \cdots x_s) = \prod_{j=1}^{s} F_1(x_j). \tag{21}$$

Accordingly, we put

$$F_s(x_1 \cdots x_s) = \prod_{j=1}^{s} F_1(x_j) + \epsilon g_s(x_1 \cdots x_s), \tag{22}$$

and find

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathcal{T}_s\right) g_s &= \mathcal{U}_s \prod_{j=1}^{s} F_1(x_i) \\ &+ \frac{1}{v} \int dx_{s+1}\, \mathcal{V}_s \sum_{j=1}^{s} \delta_{\sigma_j \sigma_{s+1}} \alpha(\sigma_j) P_{js+1} \prod_{k=1}^{s+1} F_1(x_k) \\ &- \sum_{j=1}^{s} \frac{1}{v} \int dx_{s+1}\, \mathcal{W}_{js+1} \alpha(\sigma_j) \delta_{\sigma_j \sigma_{s+1}} P_{js+1} \prod_{k=1}^{s+1} F_1(x_k) \\ &+ \frac{1}{v} \int dx_{s+1}\, \mathcal{V}_s g_{s+1} - \frac{1}{v} \sum_{j=1}^{s} \prod_{k \neq j}^{s} F_1(x_k) \mathcal{W}_{js+1} \\ &\times g_2(x_j, x_{s+1}) + O(\epsilon),\end{aligned} \tag{23}$$

where

$$\mathcal{W}_{jk} = \frac{1}{i\hbar} e_{\sigma_j} e_{\sigma_k} \left(\frac{1}{|\mathbf{q}_j - \mathbf{q}_k|} - \frac{1}{|\mathbf{q}_j' - \mathbf{q}_k'|} \right). \tag{24}$$

This system is satisfied by

$$g_s = \sum_{j<k\leq s} \sum \prod_{n \neq j,k} F_1(x_n) g_2(x_j, x_k) + O(\epsilon). \tag{25}$$

This fact, which may be checked directly by substituting (25) into (23), is a consequence of the similar property possessed by the inhomogeneous terms in (23), which is in turn due to the weakness of the interaction.

Because of the relation (25), we need only deal with the first two equations in the hierarchy, which may be written [using (25) for g_3]

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathcal{T}_2\right) g_2(x_1, x_2) &= \mathcal{W}_{12} F_1(x_1) F_1(x_2) + \frac{1}{v} \int dx_3 \{ [\mathcal{W}_{13} \delta_{\sigma_2 \sigma_3} \alpha(\sigma_2) P_{23} \\ &+ \mathcal{W}_{23} \delta_{\sigma_1 \sigma_3} \alpha(\sigma_1) P_{13}] F_1(x_1) F_1(x_2) F_1(x_3) \\ &+ \mathcal{W}_{13} g_2(x_2, x_3) F_1(x_1) + \mathcal{W}_{23} g_2(x_1, x_3) F_1(x_2) \\ &+ g_2(x_1, x_2)(\mathcal{W}_{13} + \mathcal{W}_{23}) F_1(x_3) \} + O(\epsilon),\end{aligned} \tag{26}$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathcal{T}_1\right) F_1 &+ \frac{1}{v} \int dx_2 \{ \mathcal{W}_{12} [1 + \epsilon \alpha(\sigma_1) \delta_{\sigma_1 \sigma_2} P_{12}] F_1(x_1) F_1(x_2) \\ &+ \epsilon \mathcal{W}_{12} g_2(x_1, x_2) \} + O(\epsilon^2).\end{aligned} \tag{27}$$

In the following sections it is shown how (26) may be solved in some important special cases. For the equilibrium state the determination of g_2 enables one to compute the thermodynamic properties, while for nonequilibrium systems one substitutes into (27) to obtain a kinetic equation from which one may eventually determine the transport properties of the gas, etc.

IV. HOMOGENEOUS SYSTEMS; THE STEADY STATE

A considerable simplification occurs in (26) for spatially homogeneous systems. In this case

$$F_1(x_i) = F_1(\mathbf{q}_i - \mathbf{q}_i', \sigma_i) \tag{28}$$

and

$$g_2(x_i, x_j) = g_2(\mathbf{q}_i - \mathbf{q}_i', \mathbf{q}_j - \mathbf{q}_j', \tfrac{1}{2}(\mathbf{q}_i + \mathbf{q}_i' - \mathbf{q}_j - \mathbf{q}_j'), \sigma_i, \sigma_j), \tag{29}$$

and it is convenient to introduce the Fourier-transformed Wigner distributions,

$$f(\eta) = \frac{1}{(2\pi)^3} \int d\boldsymbol{\tau} \exp(-i\mathbf{P} \cdot \boldsymbol{\tau}) F_1(\hbar \boldsymbol{\tau}, \sigma), \tag{30}$$

$$\begin{aligned}\mathcal{G}(\eta, \eta'; \mathbf{k}) &= \frac{1}{(2\pi)^6} \int\int\int d\mathbf{q} d\boldsymbol{\tau} d\mathbf{q}' \\ &\times \exp\{ i[\mathbf{k} \cdot \mathbf{q} - \mathbf{P} \cdot \boldsymbol{\tau} - \mathbf{P}' \cdot \tau'] \} \\ &\times g_2(\hbar \boldsymbol{\tau}, \hbar \boldsymbol{\tau}', \mathbf{q}, \sigma, \sigma'),\end{aligned} \tag{31}$$

[21] It should be remarked that we do not expand F_1 at this point, since it is desired to determine the higher distributions as functionals of F_1 for arbitrary F_1. On the other hand, $\partial F_1/\partial t$ will be expanded in powers of ϵ. The real expansion parameter is, of course, given by (17) or (18).

where

$$\eta=(\mathbf{P},\sigma),\quad \eta'=(\mathbf{P}',\sigma'). \tag{32}$$

Performing the indicated operations on (26), one finds

$$\left[\frac{\partial}{\partial t}-i\mathbf{k}\cdot\left(\frac{\mathbf{P}}{m_\sigma}-\frac{\mathbf{P}'}{m_{\sigma'}}\right)\right]\mathcal{G}(\eta,\eta';\mathbf{k})$$
$$=\frac{4\pi e_\sigma e_{\sigma'}}{i\hbar k^2}[f^+(\eta)f^-(\eta')-f^-(\eta)f^+(\eta')]$$
$$+\frac{4\pi}{i\hbar v k^2}\int d\eta'' e_{\sigma''}\{e_\sigma[f^+(\eta)-f^-(\eta)]$$
$$\times\mathcal{G}(\eta',\eta'';-\mathbf{k})-e_{\sigma'}[f^+(\eta')-f^-(\eta')]$$
$$\times\mathcal{G}(\eta,\eta'';\mathbf{k})\}+O(\epsilon), \tag{33}$$

where

$$f^\pm(\eta)=f(\mathbf{P}\pm\tfrac{1}{2}\hbar\mathbf{k},\sigma)\left[1+\frac{(2\pi\hbar)^3}{v}\alpha(\sigma)f(\mathbf{P}\mp\tfrac{1}{2}\hbar\mathbf{k},\sigma)\right] \tag{34}$$

and

$$\int d\eta=\int d\mathbf{P}\,\textstyle\sum_\sigma. \tag{35}$$

Similarly, from (27)

$$\frac{\partial f}{\partial t}(\eta)=\frac{\epsilon e_\sigma}{2i\pi^2\hbar v}\int d\eta'\, e_{\sigma'}\int\frac{d\mathbf{k}}{k^2}[\mathcal{G}(\mathbf{P}-\tfrac{1}{2}\hbar\mathbf{k},\sigma;\eta';\mathbf{k})$$
$$-\mathcal{G}(\mathbf{P}+\tfrac{1}{2}\hbar\mathbf{k},\sigma;\eta';\mathbf{k})]+O(\epsilon^2). \tag{36}$$

The solution of (33) will in general involve the initial value of $\mathcal{G}$. Furthermore (33) is nonlinear in the time dependent f's. However if one assumes, following Bogoliubov,[10] that sufficient time has elapsed so that $\mathcal{G}$ depends on time only through f, one has

$$\frac{\partial\mathcal{G}}{\partial t}(\eta,\eta';\mathbf{k};t)=\int d\eta''\frac{\delta\mathcal{G}(\eta,\eta';\mathbf{k};t)}{\delta f(\eta'',t)}\frac{\partial f}{\partial t}(\eta'',t)=O(\epsilon), \tag{37}$$

so that $\partial\mathcal{G}/\partial t$ may be ignored in (33). One also remarks that (33) determines $\mathcal{G}$ in terms of

$$G(\eta,\mathbf{k})=\int d\eta'\, e_{\sigma'}\mathcal{G}(\eta,\eta';\mathbf{k}), \tag{38}$$

which is also the quantity occurring in (36). Multiplying (33) by[22]

$$e_{\sigma'}\left[-i\mathbf{k}\cdot\left(\frac{\mathbf{P}}{m_\sigma}-\frac{\mathbf{P}'}{m_{\sigma'}}\right)\right]^{-1}\to 2\pi e_{\sigma'}\delta^+\left(\frac{\mathbf{k}\cdot\mathbf{P}}{m_\sigma}-\frac{\mathbf{k}\cdot\mathbf{P}'}{m_{\sigma'}}\right), \tag{39}$$

where

$$\delta^+(x)=\frac{1}{2\pi}\lim_{\gamma\to 0}\int_0^\infty dy\, e^{iy(x+i\gamma)}=\tfrac{1}{2}\delta(x)+\frac{i}{2\pi}P(1/x), \tag{40}$$

neglecting terms of order ϵ, and integrating over η', one finds for G

$$\Delta(\mathbf{k}\cdot\mathbf{P}/km_\sigma,k)G(\eta,\mathbf{k})=q(\eta,\mathbf{k})+\frac{2iD(\eta,\mathbf{k})}{k^2}$$
$$\times\int d\eta'\, e_{\sigma'}\delta^+\left(\frac{\mathbf{k}\cdot\mathbf{P}}{km_\sigma}-\frac{\mathbf{k}\cdot\mathbf{P}'}{km_{\sigma'}}\right)G(\eta',-\mathbf{k}), \tag{41}$$

where

$$D(\eta,\mathbf{k})=-\frac{4\pi^2}{\hbar v k}e_\sigma[f^+(\eta)-f^-(\eta)], \tag{42}$$

$$\Delta(u,k)=1+\frac{2i}{k^2}\int d\eta\, e_\sigma\delta^+\left(u-\frac{\mathbf{k}\cdot\mathbf{P}}{km_\sigma}\right)D(\eta,\mathbf{k}), \tag{43}$$

$$q(\eta,\mathbf{k})=-\frac{2\pi i(4\pi e_\sigma)}{\hbar k^2}\int d\eta'\, e_{\sigma'}^2\delta^+\left(\frac{\mathbf{k}\cdot\mathbf{P}}{m_\sigma}-\frac{\mathbf{k}\cdot\mathbf{P}'}{m_{\sigma'}}\right)$$
$$\times[f^+(\eta)f^-(\eta')-f^-(\eta)f^+(\eta')]. \tag{44}$$

Equation (41) is a standard type of singular integral equation; in fact, it is a special case of (21) in I. (Cf. also work cited in footnotes 12 and 13.) Since the only properties used in obtaining the solution were the relation (43) between D and Δ and the symmetry relation

$$D(-\mathbf{P},\sigma;\mathbf{k})=-D(\mathbf{P},\sigma;\mathbf{k}), \tag{45}$$

one may immediately write down the solution. Using also the symmetry of

$$\bar{q}(u,\mathbf{k})=\int d\eta e_\sigma\delta\left(u-\frac{\mathbf{k}\cdot\mathbf{P}}{km_\sigma}\right)q(\eta,\mathbf{k})=\bar{q}(-u,-\mathbf{k}) \tag{46}$$

one finds[23]

$$G(\eta,\mathbf{k})=\frac{q(\eta,\mathbf{k})}{\Delta(\mathbf{k}\cdot\mathbf{P}/km_\sigma,k)}+\frac{iD(\eta,\mathbf{k})\bar{q}(\mathbf{k}\cdot\mathbf{P}/km_\sigma,\mathbf{k})}{k^2|\Delta(\mathbf{k}\cdot\mathbf{P}/km_\sigma,k)|^2}$$
$$+\frac{D(\eta,\mathbf{k})}{\pi k^2}P\int_{-\infty}^{\infty}\frac{du}{(u-\mathbf{k}\cdot\mathbf{P}/km_\sigma)}\frac{\bar{q}(u,\mathbf{k})}{|\Delta(u,k)|^2}. \tag{47}$$

Substituting (47) into (36), ignoring terms of order ϵ^2, and setting the formal parameter ϵ equal to unity, one

[22] The interpretation of the singularity in (38) is equivalent to Bogoliubov's condition of "weakening of correlation." (See work cited in reference 10.)

[23] As pointed out by Balescu, it is not necessary to obtain the explicit form of the correlation function in order to determine the contribution to (36); however, we will also be interested in the calculation of the thermodynamic functions where the complete correlation function is needed.

finds after some manipulation

$$\frac{\partial f(\eta)}{\partial t}=\frac{2e_\sigma^2}{\hbar^2 v}\int\frac{d\mathbf{k}}{k^4}(T^+-T^-)\int d\eta' e_{\sigma'}^2\delta\left(\frac{\mathbf{k}\cdot\mathbf{P}}{m_\sigma}-\frac{\mathbf{k}\cdot\mathbf{P}'}{m_{\sigma'}}\right)\times\frac{f^+(\eta)f^-(\eta')-f^-(\eta)f^+(\eta')}{|\Delta(\mathbf{k}\cdot\mathbf{P}/km_\sigma,k)|^2}. \quad (48)$$

where the T's are operators defined for any function $\psi(P)$ by

$$T^{\pm}(\mathbf{P},\mathbf{k})\psi(\mathbf{P})=\psi(\mathbf{P}\pm\tfrac{1}{2}\hbar\mathbf{k}). \quad (49)$$

Equation (48) is a generalization of the equation of Balescu[8] to a multicomponent plasma. We remark in passing that the H theorem is satisfied for systems obeying (48), and it is easily shown that the unique equilibrium distribution has the expected form

$$f_0(\eta)=[v/(2\pi\hbar)^3]/\{A(\sigma)\exp[(\mathbf{P}-m_\sigma\mathbf{V})^2/2m_\sigma\theta]-\alpha(\sigma)\}. \quad (50)$$

V. THE AVERAGE COULOMB ENERGY; THE EQUILIBRIUM STATE

In order to compare with previous treatments for the equilibrium state, which have proceeded from the partition function rather than the correlation function, it is convenient to calculate the average interaction energy of the system. This is given by

$$U_c=\mathrm{Tr}[\rho_N H_c]=\frac{2\pi N}{v}\int\frac{d\mathbf{k}}{k^2}\int d\eta\, e_\sigma\left[e_\sigma\alpha(\sigma)f(\eta)\hbar^3\times f(\mathbf{P}+\hbar\mathbf{k},\sigma)+\frac{1}{(2\pi)^3}G(\eta,\mathbf{k})\right], \quad (51)$$

where H_c is the total Coulomb energy of the system and we have used (4), (12), (22), (30), (31), and (38). Using the explicit form (47) for G, one may write

$$U_c=\frac{2\pi N}{v}\int\frac{d\mathbf{k}}{k^2}\left[\int d\eta\, e_\sigma^2 f(\eta)f(\mathbf{P}+\hbar\mathbf{k},\sigma)\alpha(\sigma)(\hbar)^3+\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}du\frac{\bar{q}(u,\mathbf{k})}{|\Delta(u,k)|^2}\right]. \quad (52)$$

The expression (52) is valid for all uniform states for which the "asymptotic" time restriction is valid, and the f's may, in general, be time dependent. Considerable simplification occurs for the equilibrium state, when the f's take on the equilibrium values given by (50). In this case one readily shows that

$$\bar{q}_0(u,\mathbf{k})=-\frac{v\Delta_2(u,k)}{\pi\sinh(\hbar ku/2\theta)}\times\int_{-\infty}^{\infty}du'\frac{\sinh[\hbar k(u'-u)/2\theta]}{u'-u}M(u',k), \quad (53)$$

where

$$M(u,\mathbf{k})=\int d\eta\, e_\sigma^2\delta\left(u-\frac{\mathbf{k}\cdot\mathbf{P}}{km_\sigma}\right)f_0^+(\eta)\exp\frac{\hbar\mathbf{k}\cdot(\mathbf{P}-m\mathbf{V})}{2m_\sigma\theta}. \quad (54)$$

It follows that we may write

$$\int_{-\infty}^{\infty}\frac{du\,\bar{q}_0(u)}{|\Delta_0(u)|^2}=\frac{v}{\pi}I_m P\int_{-\infty}^{\infty}du\frac{(1/\Delta_0(u)-1)}{\sinh(\hbar ku/2\theta)}\times\int_{-\infty}^{\infty}du'\frac{\sinh[\hbar k(u'-u)/2\theta]}{(u'-u)}M(u',k). \quad (55)$$

The analyticity properties of the integrand are such that the path may be closed in the upper half plane, and residues taken at the zeros of $\sinh(\hbar ku/2\theta)$. One then finds

$$U_c=V\int d\mathbf{k}\left\{\frac{2\pi}{k^2}\hbar^3\int d\eta\, e_\sigma^2\alpha(\sigma)f_0(\eta)f_0(\mathbf{P}+\hbar\mathbf{k},\sigma)-\frac{\theta}{2(2\pi)^3}\sum_{n=-\infty}^{\infty}\frac{[1-\Delta_0(2n\pi i\theta/\hbar k)]^2}{\Delta_0(2\pi in\theta/\hbar k)}\right\}, \quad (56)$$

which is a generalization of the result of Englert and Brout[6] to include Bose statistics and multicomponent systems. The function Δ_0 is readily identified as the dielectric constant.

VI. INHOMOGENEOUS AND RAPIDLY VARYING SYSTEMS

The transport properties of a plasma may be determined to a good degree of approximation from (48) provided (a) the scale of any spatial inhomogeneities is much greater than a Debye length, and (b) sufficient time has elapsed[24] for the higher distributions to depend on time only through F_1. Since these conditions are not satisfied for many problems of interest, it is desirable to derive a kinetic equation which is not subject to such restrictions. It was shown in I how this could be done for a classical plasma not too far from equilibrium. The present treatment is completely analogous.

In order to solve (26) approximately for inhomogeneous and/or rapidly varying systems, it is convenient to perform the following linearization:

$$F_1=F^{(0)}+F^{(1)}, \quad (57)$$

$$g_2=g^{(0)}+g^{(1)}, \quad (58)$$

where $F^{(0)}$ and $g^{(0)}$ are the equilibrium one-particle distribution and correlation function respectively and neglect second-order terms in $F^{(1)}$, $g^{(1)}$. As before, we introduce the Fourier-transformed Wigner distributions and integrate over one momentum argument. In addition, we take a one-sided Fourier transform in time.

[24] A condition on the frequencies of the processes involved is also implied; these should be much less than the electron plasma frequency.

Defining

$$f_0(\eta,\omega)=\frac{1}{(2\pi)^3}\int_0^\infty dt\int d\boldsymbol{\tau}\,\exp[i(\omega t-\boldsymbol{\tau}\cdot\mathbf{P})]F^{(0)}(\hbar\boldsymbol{\tau},\sigma,t),\tag{59}$$

$$\mathcal{G}_0(\eta,\eta',\mathbf{k},\omega)=\frac{1}{(2\pi)^6}\int_0^\infty dt\int d\boldsymbol{\tau}\int d\boldsymbol{\tau}'\int d\mathbf{q}\,\exp[i(\omega t+\mathbf{k}\cdot\mathbf{q}-\boldsymbol{\tau}\cdot\mathbf{P}-\boldsymbol{\tau}'\cdot\mathbf{P}')]g^{(0)}(\hbar\boldsymbol{\tau},\hbar\boldsymbol{\tau}',\mathbf{q},\sigma,\sigma',t),\tag{60}$$

$$f_1(\eta,\mathbf{k},\omega)=\frac{1}{(2\pi)^3}\int_0^\infty dt\int d\boldsymbol{\tau}\int d\mathbf{q}\,\exp[i(\omega t+\mathbf{k}\cdot\mathbf{q}-\boldsymbol{\tau}\cdot\mathbf{P})]F^{(1)}(\mathbf{q}+\tfrac{1}{2}\hbar\boldsymbol{\tau},\,\mathbf{q}-\tfrac{1}{2}\hbar\boldsymbol{\tau},\,\sigma,\,t),\tag{61}$$

$$G_1(\eta,\mathbf{k},\mathbf{k}',\omega)=\frac{1}{(2\pi)^6}\int d\eta'\,e_{\sigma'}\int_0^\infty dt\int d\mathbf{q}\int d\mathbf{q}'\int d\boldsymbol{\tau}\int d\boldsymbol{\tau}'\,\exp[i(\omega t+\mathbf{k}\cdot\mathbf{q}+\mathbf{k}'\cdot\mathbf{q}'-\boldsymbol{\tau}\cdot\mathbf{P}-\boldsymbol{\tau}'\cdot\mathbf{P}')]$$
$$\times g^{(1)}(\mathbf{q}+\tfrac{1}{2}\hbar\boldsymbol{\tau},\,\mathbf{q}-\tfrac{1}{2}\hbar\boldsymbol{\tau},\,\sigma;\,\mathbf{q}'+\tfrac{1}{2}\hbar\boldsymbol{\tau}',\,\mathbf{q}'-\tfrac{1}{2}\hbar\boldsymbol{\tau}',\,\sigma';\,t).\tag{62}$$

One finds for G an equation of the form

$$\Delta_0\left(\frac{\omega+\mathbf{k}\cdot\mathbf{v}}{k'},\,k'\right)G_1(\eta,\mathbf{k},\mathbf{k}',\omega)=q(\eta,\mathbf{k},\mathbf{k}',\omega)+\frac{2iD_0(\eta,\mathbf{k})}{kk'}\int d\eta'\,e_{\sigma'}\delta^+\left(\frac{\omega+\mathbf{k}\cdot\mathbf{v}+\mathbf{k}'\cdot\mathbf{v}'}{k'}\right)G_1(\eta',\mathbf{k}',\mathbf{k},\omega),\tag{63}$$

where D_0 and Δ_0 are defined by (42), (43), (34) with f replaced by f_0,

$$\mathbf{v}=\mathbf{P}/m_\sigma,\quad \mathbf{v}'=\mathbf{P}'/m_\sigma',\tag{64}$$

and

$$q(\eta,\mathbf{k},\mathbf{k}',\omega)=-\frac{8\pi^2 i}{\hbar}\int d\eta'\,e_{\sigma'}\delta^+(\omega+\mathbf{k}\cdot\mathbf{v}+\mathbf{k}'\cdot\mathbf{v}')\biggl\{\frac{e_\sigma e_{\sigma'}}{k'^2}[f_1^+(\eta,\mathbf{k},\mathbf{k}',\omega)f_0^+(\eta',\mathbf{k}')-f_1^-(\eta,\mathbf{k},\mathbf{k}',\omega)f_0^-(\eta',\mathbf{k}')]$$
$$+\frac{e_\sigma}{vk'^2}G_0(\eta',\mathbf{k}')[T^-(\mathbf{P},\mathbf{k}')-T^+(\mathbf{P},\mathbf{k}')]f_1(\eta,\,\mathbf{k}+\mathbf{k}',\,\omega)+\frac{e_\sigma}{(\mathbf{k}+\mathbf{k}')^2 v}\int d\eta''e_{\sigma''}f_1(\eta'',\,\mathbf{k}+\mathbf{k}',\,\omega)$$
$$\times[T^+(\mathbf{P},\,\mathbf{k}+\mathbf{k}')-T^-(\mathbf{P},\,\mathbf{k}+\mathbf{k}')]\mathcal{G}_0(\eta,\,\eta',\,-\mathbf{k}')+\frac{\hbar}{4\pi}\mathcal{G}_1(\eta,\eta',\mathbf{k},\mathbf{k}',0)\biggr\}.\tag{65}$$

Here, $f_0^\pm$ is defined by (34) with f replaced by f_0, $T^\pm$ by (49), and

$$f_1^\pm(\eta,\mathbf{k},\mathbf{k}',\omega)=f_1(\mathbf{P}\mp\tfrac{1}{2}\hbar\mathbf{k}',\,\sigma,\,\mathbf{k}+\mathbf{k}',\,\omega)\left[1+\frac{(2\pi\hbar)^3}{v}\alpha(\sigma)f_0(\mathbf{P}\mp\tfrac{1}{2}\hbar\mathbf{k},\,\sigma)\right]$$
$$+\frac{(2\pi\hbar)^3}{v}\alpha(\sigma)f_0(\mathbf{P}\pm\tfrac{1}{2}\hbar\mathbf{k},\,\sigma)f_1(\mathbf{P}\pm\tfrac{1}{2}\hbar\mathbf{k}',\,\sigma,\,\mathbf{k}+\mathbf{k}',\,\omega).\tag{66}$$

Once again the form of the equation is precisely the same as in the classical case [Eq. (21) of I] and we may use the results of I to write the solution as [cf. I, Eq. (61)]

$$G_1(\eta,\mathbf{k},\mathbf{k}')=\frac{q(\eta,\mathbf{k},\mathbf{k}')}{\Delta_0[(\omega+\mathbf{k}\cdot\mathbf{v})/k',\,k']}+\frac{D_0(\eta,\mathbf{k})}{\pi k^2}\int_C\frac{du[\Phi_2^+(u)-\Phi_1^-(u)]}{(\omega-\mathbf{k}\cdot\mathbf{v}/k)\Delta_0(\omega+ku/k',\,k')\Delta_0(u,k)},\tag{67}$$

where C is a path just below the real axis, and

$$\Phi_1^\pm(u)=\frac{1}{2\pi i}\lim_{\epsilon\to0+}\int d\eta'\,e_{\sigma'}\frac{q(\eta',\mathbf{k},\mathbf{k}')}{(\mathbf{k}\cdot\mathbf{v}'/k-u\mp i\epsilon)},\tag{68}$$

$$\Phi_2^\pm(u)=-\frac{1}{2\pi i}\lim_{\epsilon\to0+}\int d\eta'\frac{e_{\sigma'}q(\eta',\mathbf{k}',\mathbf{k})}{(\mathbf{k}'\cdot\mathbf{v}'+\omega)/k+u\pm i\epsilon}.\tag{69}$$

To obtain a kinetic equation, one substitutes into (27), whose linearized transformed version reads

$$(\omega+\mathbf{k}\cdot\mathbf{v})f_1(\eta,\mathbf{k},\omega)=\frac{4\pi e_\sigma}{\hbar v}\biggl\{\int d\eta'\,e_{\sigma'}\frac{f_1(\eta',\mathbf{k},\omega)}{k^2}[T^+(\mathbf{P},\mathbf{k})-T^-(\mathbf{P},\mathbf{k})]f_0(\eta,\mathbf{k})+\hbar^3 e_\sigma\alpha(\sigma)\int\frac{d\mathbf{k}'}{k'^2}\{f_1(\mathbf{P}+\hbar\mathbf{k}',\,\sigma,\,\mathbf{k},\,\omega)$$
$$\times[T^+(\mathbf{P},\mathbf{k})-T^-(\mathbf{P},\mathbf{k})]f_0(\eta,\mathbf{k})+f_1(\eta,\mathbf{k},\omega)[T^+(\mathbf{P},\,\mathbf{k}+2\mathbf{k}')-T^-(\mathbf{P},\,\mathbf{k}+2\mathbf{k}')]f_0(\eta,\mathbf{k})\}$$
$$-\frac{1}{(2\pi)^3}\int\frac{d\mathbf{k}'}{k'^2}[T^+(\mathbf{P},\mathbf{k}')-T^-(\mathbf{P},\mathbf{k}')]G_1(\eta,\,\mathbf{k}+\mathbf{k}',\,-\mathbf{k}',\,\omega)\biggr\}+\mathfrak{F}_1(\eta,\mathbf{k},0).\tag{70}$$

Here, $\mathfrak{F}_1(\eta,k,0)$ is the initial value of the singlet distribution (untransformed in time).

On substitution of (61) into (70), one obtains an equation for the perturbed singlet distribution, which in principle determines the transport properties of the system. An exact solution appears to be extremely difficult because of the formidable nature of the expression for the correlation function. However, it is hoped that (70) may prove useful in the approximate determination of high-frequency transport properties of a plasma, as well as the correlation effect on the damping of plasma oscillations. A more detailed discussion of these problems is reserved for future work.

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Elementary Excitations in Liquid Helium*

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An analysis of some recent experimental and theoretical investigations of the elementary excitation spectrum of liquid He II is carried out. The spectrum of density fluctuations, which is measured in an inelastic neutron scattering experiment, is shown to consist of two distinct parts: direct excitation of single quasi-particles from the condensed zero-momentum state; and excitation of more complex configurations arising from the interaction of two, three, or more quasi-particles. With the aid of general sum-rule arguments it is demonstrated that: (1) in the long-wavelength limit, a single quasi-particle excitation exhausts the f-sum rule; the resulting excitation spectrum is identical to the phonon spectrum proposed by Feynman and found experimentally by Henshaw and Woods; (2) this asymptotic behavior of the density fluctuation spectrum may be used to normalize the experimental results of Henshaw and Woods; one thereby obtains a somewhat altered liquid structure-factor curve, detailed information on the efficiency of quasi-particle excitation from the condensed state of an incident slow neutron, and an estimate of the depletion of the zero-momentum state as a consequence of particle interaction; (3) the backflow introduced by Feynman and Cohen corresponds to taking into account the coupling between a Feynman excitation and higher configurations involving several elementary excitations.

The physical picture of backflow is clarified by means of a study of impurity atom motion in an interacting-boson system. It is shown that in the Bogoliubov approximation the backflow around the impurity atom corresponds to a cloud of moving virtual-phonon excitations which act to increase the impurity effective mass as well as to conserve current in the system. The generalization of these results to higher-order approximations, and to the coupling between quasi-particles in liquid helium, is discussed.

The importance of accounting properly for depletion effects in a microscopic theory is emphasized; it is shown that such effects are neglected in the microscopic calculations of the interacting-boson excitation spectrum which have thus far been carried out, although they are of decisive importance in the determination of the density-fluctuation excitation spectrum.

I.

CONSIDERABLE progress in our understanding of the elementary excitation spectrum of liquid He^4 has been made in recent years. Both neutrons[1] and charged particles[2] have been used as probes to provide a direct experimental measurement of the energy vs momentum curve for the elementary excitations. The spectrum so obtained is in good agreement with the theoretical spectrum calculated by Feynman[3] and Feynman and Cohen[4] using a trial wavefunction for the elementary excitations.

Progress has also been made in the development of a microscopic (as distinct from variational) theory of the elementary excitations in a system of Bose particles interacting via a repulsive potential. Bogoliubov[5] showed that a system of weakly interacting bosons will possess a phonon-like elementary excitation spectrum in the low-momentum region. Lee, Huang, and Yang[6] used the method of pseudopotentials to extend Bogoliubov's calculation to the case of a hard-sphere gas at low density.

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