Here, $\mathfrak{F}_1(\eta,k,0)$ is the initial value of the singlet distribution (untransformed in time).

On substitution of (61) into (70), one obtains an equation for the perturbed singlet distribution, which in principle determines the transport properties of the system. An exact solution appears to be extremely difficult because of the formidable nature of the expression for the correlation function. However, it is hoped that (70) may prove useful in the approximate determination of high-frequency transport properties of a plasma, as well as the correlation effect on the damping of plasma oscillations. A more detailed discussion of these problems is reserved for future work.

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Elementary Excitations in Liquid Helium*

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An analysis of some recent experimental and theoretical investigations of the elementary excitation spectrum of liquid He II is carried out. The spectrum of density fluctuations, which is measured in an inelastic neutron scattering experiment, is shown to consist of two distinct parts: direct excitation of single quasi-particles from the condensed zero-momentum state; and excitation of more complex configurations arising from the interaction of two, three, or more quasi-particles. With the aid of general sum-rule arguments it is demonstrated that: (1) in the long-wavelength limit, a single quasi-particle excitation exhausts the *f*-sum rule; the resulting excitation spectrum is identical to the phonon spectrum proposed by Feynman and found experimentally by Henshaw and Woods; (2) this asymptotic behavior of the density fluctuation spectrum may be used to normalize the experimental results of Henshaw and Woods; one thereby obtains a somewhat altered liquid structure-factor curve, detailed information on the efficiency of quasi-particle excitation from the condensed state of an incident slow neutron, and an estimate of the depletion of the zero-momentum state as a consequence of particle interaction; (3) the backflow introduced by Feynman and Cohen corresponds to taking into account the coupling between a Feynman excitation and higher configurations involving several elementary excitations.

The physical picture of backflow is clarified by means of a study of impurity atom motion in an interacting-boson system. It is shown that in the Bogoliubov approximation the backflow around the impurity atom corresponds to a cloud of moving virtual-phonon excitations which act to increase the impurity effective mass as well as to conserve current in the system. The generalization of these results to higher-order approximations, and to the coupling between quasi-particles in liquid helium, is discussed.

The importance of accounting properly for depletion effects in a microscopic theory is emphasized; it is shown that such effects are neglected in the microscopic calculations of the interacting-boson excitation spectrum which have thus far been carried out, although they are of decisive importance in the determination of the density-fluctuation excitation spectrum.

I.

Considerable progress in our understanding of the elementary excitation spectrum of liquid He^4 has been made in recent years. Both neutrons[1] and charged particles[2] have been used as probes to provide a direct experimental measurement of the energy vs momentum curve for the elementary excitations. The spectrum so obtained is in good agreement with the theoretical spectrum calculated by Feynman[3] and Feynman and Cohen[4] using a trial wavefunction for the elementary excitations.

Progress has also been made in the development of a microscopic (as distinct from variational) theory of the elementary excitations in a system of Bose particles interacting via a repulsive potential. Bogoliubov[5] showed that a system of weakly interacting bosons will possess a phonon-like elementary excitation spectrum in the low-momentum region. Lee, Huang, and Yang[6] used the method of pseudopotentials to extend Bogoliubov's calculation to the case of a hard-sphere gas at low density.

* Work supported in part by the U. S. Army Research Office (Durham) and the Air Force Office of Scientific Research.

[1] H. Palevsky, K. Otnes, and K. E. Larsson, Phys. Rev. **112**, 11 (1959); J. L. Yarnell, G. P. Arnold, P. J. Bendt, and E. C. Kerr, *ibip.* **113**, 1379 (1959); D. G. Henshaw and A. D. B. Woods, *ibid.* **121**, 1266 (1961).

[2] G. Careri, *Progress in Low-Temperature Physics*, edited by J. C. Gorter (North-Holland Publishing Company, Amsterdam, 1961), Vol. 3, p. 58.

[3] R. P. Feynman, Phys. Rev. **94**, 267 (1954).

[4] R. P. Feynman and M. Cohen, Phys. Rev. **102**, 1189 (1956), frequently referred to as FC.

[5] N. N. Bogoliubov, J. Phys. (U.S.S.R.) **11**, 23 (1947).

[6] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

Beliaev[7] and Hugenholtz and Pines[8] have developed a field-theoretic formulation for the problem of bosons with an arbitrarily strong repulsive interaction, and have shown that the phonon-like character of the low-momentum excitation spectrum may be expected to be a general feature of such a system. Their method also permits one to take into account the depletion of the zero-momentum state as a consequence of the particle interaction.

The great triumph of the microscopic calculations is that they provide us with a well-defined model for a superfluid system: a low-density gas of bosons with repulsive interactions. Unfortunately, liquid helium is not a low-density gas, nor are the interactions between a pair of He atoms purely repulsive, so that one cannot expect to account for its properties on the basis of the aforementioned low-density calculations. It is therefore necessary to consider with some care the general features which a microscopic theory must possess in order that it might hope to account for the experimentally observed excitation spectrum of liquid He II. The present study began as an attempt to use the elegant physical arguments developed by Feynman and Cohen as a guide in this direction; that is, to understand what is the analog, in a microscopic theory, of the backflow of atoms about an excitation moving through the system. In the course of our investigations we were led to use certain general sum-rule arguments both to clarify the physical picture of backflow, and to make precise the information which is actually contained in the beautiful neutron scattering experiments of Henshaw and Woods. We were also led to recognize the importance, in all microscopic theories of intermediate direct boson systems, of taking into account the depletion of the zero-momentum state as a consequence of particle interaction. Such depletion effects make particularly difficult the calculation of the density-fluctuation excitation spectrum in these systems.

The principle results of the present paper are the following:

(1) It is shown that the zero-temperature liquid structure factor quite likely possesses a hitherto unsuspected hump for wavevectors in the vicinity of 0.6 Å^{-1}.

(2) There is derived from the experiments of Henshaw and Woods detailed information on the absolute efficiency of quasi-particle excitation from the condensed zero-momentum state; their experimental results for the efficiency are found to be in good agreement with the theory of Cohen and Feynman[9] for wavevectors up to about the roton minimum. There exists a rapid falloff in the efficiency for large momentum transfers; we are led to attribute this decrease to the depletion of the zero-momentum state arising from particle interaction, and to place a lower limit of some 92% on this depletion.

(3) Backflow is shown to arise from the interaction between different quasi-particle configurations; the backflow in the Feynman-Cohen theory corresponds to taking into account the interaction between the rotons and phonons obtained using the Feynman variational wavefunction.[10] Thus, as a Feynman roton moves through helium it is surrounded by a cloud of virtual phonons and rotons; the backflow represents a configuration space description of the cloud of virtual excitations.[11]

(4) Neither backflow nor depletion of the zero-momentum state are accounted for in the simple Bogoliubov-like approximations of most of the current microscopic theories; it is essential to take both effects properly into account in order to obtain a microscopic theory of the elementary excitation spectrum of He II.

In Sec. II, sum-rule arguments are developed and applied in a discussion of the theoretical and experimental investigations of the elementary excitation spectrum of He II. In Sec. III, the physical picture of backflow as arising from a coupling between different quasi-particle configurations is clarified with the aid of a microscopic study, based on the Bogoliubov theory, of the motion of an impurity atom in an interacting boson system. In Sec. IV, the generalization of these arguments to the motion and interaction of quasi-particles in liquid helium is carried out. In Sec. V, effects associated with the depletion of the zero-momentum state are considered, while in Sec. VI certain speculations are presented on the form that a successful field-theoretic treatment of both backflow and depletion might take. In the Appendix, a brief resumé of the Bogoliubov approximation is presented.

II.

We consider a system of N interacting bosons of mass M enclosed in a cubic box of volume Ω at $T=0$. We follow Hugenholtz and Pines[8] and take as our Hamiltonian $H-\mu N$, where H is the particle Hamiltonian and μ is the chemical potential. We write

$$\begin{aligned} H' &= H-\mu N = H_0+V, \\ H_0 &= \sum_{\mathbf{k}}(\epsilon_{\mathbf{k}}-\mu)a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}, \\ V &= \Omega^{-1}\sum_{\mathbf{k},1,1'}(V_{\mathbf{k}}/2)a_{1-\mathbf{k}}^{\dagger}a_{1'+\mathbf{k}}^{\dagger}a_{1'}a_{1}. \end{aligned} \tag{2.1}$$

In Eq. (2.1), $\epsilon_k=\hbar^2k^2/2M$ is the kinetic energy of particles of momentum $\hbar\mathbf{k}$. The operators $a_{\mathbf{k}}^{\dagger}$ and $a_{\mathbf{k}}$ are

[7] S. T. Beliaev, Soviet Phys.—JETP **7**, 289, 299 (1958).

[8] N. M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959).

[9] M. Cohen and R. P. Feynman, Phys. Rev. **107**, 13 (1957).

[10] A calculation of this interaction was carried out using Rayleigh-Schrödinger perturbation theory by C. G. Kuper, Proc. Roy. Soc. (London) **A233**, 223 (1955); a calculation based on Brillouin-Wigner perturbation theory has recently been carried out by H. W. Jackson and E. Feenberg (to be published).

[11] Such a suggestion was first put forth at the International Conference on Theoretical Physics at Seattle in 1956 by one of us (D.P.) on the basis of a treatment of the phonon modes by means of collective coordinates, as in the Bohm-Pines theory of electron interactions.

creation and annihilation operators for these particles and satisfy the commutation relations

$$[a_{1'},a_1]=0=[a_{1'}^\dagger,a_1^\dagger]; \quad [a_{1'},a_1^\dagger]=\delta_{1,1'}.$$

V_k is the Fourier transform of the central two-body interaction $V(x)$:

$$V_{\mathbf{k}}=\int d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}}V(\mathbf{x}).$$

As shown in reference 8, the introduction of the chemical potential μ permits one to consider non-particle-conserving processes, provided the number of particles is conserved on the average. Thus, one has

$$N_0+\langle 0|\sum_{\mathbf{k}\neq 0} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}|0\rangle=N, \tag{2.2}$$

where N_0 is the number of particles of zero momentum, and $|0\rangle$ is the ground-state wavefunction. The remaining equation which determines N_0 and μ is

$$\mu=\partial E_0/\partial N=\partial E_0'/\partial N_0, \tag{2.3}$$

where E_0 is the ground-state energy, while E_0' is the ground-state energy corresponding to the Hamiltonian H'.

We first recall some general properties of the excitation spectrum as measured by, say, neutron scattering. As shown by Van Hove[12] and Cohen and Feynman,[9] what one can measure in an inelastic neutron scattering experiment is the probability per unit time that a slow neutron transfer energy ω and momentum $\mathbf{k}$ to the boson system in its ground state; this probability is given by[13]

$$W(\mathbf{k},\omega)=AS(\mathbf{k},\omega). \tag{2.4}$$

A is a constant which characterizes the neutron-boson interaction while $S(\mathbf{k},\omega)$ is the dynamic structure factor which describes the elementary excitation spectrum of the density fluctuations of the system, and is defined as

$$S(\mathbf{k},\omega)=\sum_n (\rho_{\mathbf{k}}^\dagger)_{n0}^2\delta(\omega-\omega_{n0}). \tag{2.5}$$

In Eq. (2.5), $\rho_{\mathbf{k}}^\dagger$ is the density fluctuation of momentum $\mathbf{k}$,

$$\rho_{\mathbf{k}}^\dagger=\int d^3x\,\rho(\mathbf{x})e^{+i\mathbf{k}\cdot\mathbf{x}}=\sum_i e^{+i\mathbf{k}\cdot\mathbf{x}_i}$$
$$=\sum_{\mathbf{q}} a_{\mathbf{q}+\mathbf{k}}^\dagger a_{\mathbf{q}},$$

while the $(\rho_{\mathbf{k}}^\dagger)_{n0}$ and ω_{n0} are the exact matrix elements and excitation frequencies corresponding to the state n.

[12] L. Van Hove, Phys. Rev. **95**, 249 (1954).
[13] We take $\hbar=1=\Omega$ throughout this paper.

Two moments of $S(\mathbf{k},\omega)$ are of interest. We have

$$\int_0^\infty d\omega\, S(\mathbf{k},\omega)=NS(\mathbf{k}), \tag{2.6}$$

$$\int_0^\infty d\omega\, \omega S(\mathbf{k},\omega)=N\hbar k^2/2M. \tag{2.7}$$

In Eq. (2.6), $S(\mathbf{k})$ is the liquid structure factor, and is the Fourier transform of the pair correlation function $p(\mathbf{r})$. Thus

$$S(\mathbf{k})=N^{-1}\langle 0|\rho_{\mathbf{k}}\rho_{\mathbf{k}}^\dagger|0\rangle=\int d^3r\, p(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{2.8}$$

$$p(\mathbf{r})=N^{-1}\langle 0|\rho^\dagger(0)\rho(\mathbf{r})|0\rangle. \tag{2.9}$$

Equation (2.7) is a statement of the longitudinal f-sum rule

$$\sum_n f_{0n}=(2M/k^2)\sum_n \omega_{n0}|(\rho_{\mathbf{k}}^\dagger)_{n0}|^2=N, \tag{2.10}$$

which applies if the potential V is velocity independent.[14]

Let us consider the dynamic structure factor $S(\mathbf{k}\omega)$ in more detail. The intermediate states $|\psi_n\rangle$ coupled to $|\psi_0\rangle$ by the operator $\rho_{\mathbf{k}}$ belong essentially to two distinct classes:

(i) $|\psi_n\rangle$ may correspond to a single elementary excitation, or quasi-particle, of momentum $\mathbf{k}$, excited directly from the condensed zero-momentum state. Let $\omega(\mathbf{k})$ be the energy of this quasi-particle mode. The single quasi-particle contribution to $S(\mathbf{k}\omega)$ then takes the form

$$NZ(\mathbf{k})\delta[\omega-\omega(\mathbf{k})],$$

where $Z(\mathbf{k})$ is a positive constant. We remark that as long as there is macroscopic occupation of the state of momentum zero, we may expect that the single quasi-particle contribution to $S(\mathbf{k}\omega)$ will be appreciable.

(ii) As a consequence of quasi-particle interaction, $|\psi_n\rangle$ may also correspond to some higher configuration, involving two, three, or more quasi-particles, with a total momentum $\mathbf{k}$. These configurations will have a finite excitation energy of the order of the roton energy. They will obviously form a continuum, as the momentum of any single-component excitation is allowed to vary.

Actually, the above discussion oversimplifies the problem somewhat. The discrete mode (i) will, in fact, be immersed in the continuum of possible excitation modes, and will thus have a finite lifetime. Consequently, the discrete line at $\omega=\omega(\mathbf{k})$ will be broadened; its width yields an estimate of the lifetime of the excitation. We know from experiment that this damping is

[14] The sum rule, Eq. (2.7), which was employed by Cohen and Feynman (see reference 9), is identical to that derived for the electron gas: P. Nosières and D. Pines, Phys. Rev. **109**, 741 (1958), ff., Eqs. (2.3) to (2.6). It can be obtained by computing the expectation value of the double commutator $[[H,\rho_{\mathbf{k}}],\rho_{-\mathbf{k}}]$ in the ground state. If the result obtained by using the explicit form of the Hamiltonian, Eq. (2.1), is compared to the result in terms of the energies ω_{n0}, then Eq. (2.7) is obtained.

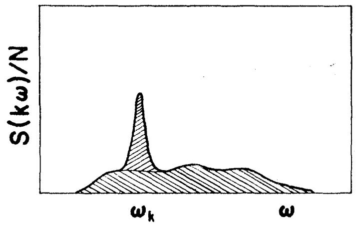

FIG. 1. Schematic plot of $S(\mathbf{k}\omega)$ vs ω.

small (and perhaps even negligible) in the low-**k** region, up to a wavevector of the order of 2.7 Å^{-1}, since over this region it is not possible for a single excitation to decay into a pair of lower energy excitations with overall conservation of momentum and energy.

Our discussion is summarized in Fig. 1, where $S(\mathbf{k}\omega)/N$ is shown to consist of a narrow peak centered at $\omega(k)$, enclosing an area $Z(k)$, superimposed on a continuum. (We remark that the continuum starts from a finite positive energy, of the order of ck.) We shall accordingly write

$$S(\mathbf{k}\omega)=NZ(\mathbf{k})\delta[\omega-\omega(\mathbf{k})]+S^{(1)}(\mathbf{k}\omega), \tag{2.11}$$

where $S^{(1)}(\mathbf{k}\omega)$ refers to the continuum contribution to $S(\mathbf{k}\omega)$. Equations (2.6) and (2.7) then take the form

$$\int_0^\infty d\omega\, S^{(1)}(\mathbf{k}\omega)=N[S(\mathbf{k})-Z(\mathbf{k})], \tag{2.12}$$

$$\int_0^\infty d\omega\, \omega S^{(1)}(\mathbf{k}\omega)=N[k^2/2M-Z(\mathbf{k})\omega(\mathbf{k})]. \tag{2.13}$$

We now consider the relative importance of the quasi-particle and continuum contributions in the limit of long wavelengths.

For low values of **k** we expect $S^{(1)}(\mathbf{k}\omega)$ to go to zero, as conjectured by Feynman and Cohen and found experimentally. To establish the limiting behavior of $S^{(1)}(\mathbf{k}\omega)$, we first remark that

$$\omega_{n0}(\rho_{\mathbf{k}}^\dagger)_{n0}=(\mathbf{k}\cdot\mathbf{j}_{\mathbf{k}}^\dagger)_{n0}, \tag{2.14}$$

where $\mathbf{j}_{\mathbf{k}}^\dagger$ is the kth Fourier component of the current density

$$\mathbf{j}_{\mathbf{k}}^\dagger=\frac{1}{2M}\sum_i\{\mathbf{p}_i e^{i\mathbf{k}\cdot\mathbf{x}_i}+e^{i\mathbf{k}\cdot\mathbf{x}_i}\mathbf{p}_i\}. \tag{2.15}$$

Equation (2.14) follows at once provided the particle interaction is velocity independent (and is indeed an intermediate step in the establishment of the sum rule). The sum rule (2.10) may thus be written

$$\sum_n(\mathbf{k}\cdot\mathbf{j}_{\mathbf{k}}^\dagger)_{n0}^2/\omega_{n0}=k^2N/2M. \tag{2.16}$$

Let us now consider (2.16) in the limit $k=0$. $j_{\mathbf{k}}$ then approaches the total current **J**. Again, we distinguish two kinds of intermediate states.

(i) The state $|n\rangle$ has a single elementary excitation with wavevector **k**: the excitation energy is of order k, the matrix element $(j_{\mathbf{k}})_{n0}$ depends on subtle correlation effects, and turns out to be of order $(k)^{1/2}$. The contribution to the sum rule is thus of order k^2 and therefore important.

(ii) The state $|n\rangle$ is a higher configuration. Because the state represents a superposition of quasi particles of net momentum **k**, it is quite likely that there is no special k dependence of ω_{n0}. Hence, ω_{n0} is finite in the limit of $k\rightarrow 0$. Furthermore, the matrix element $(j_{\mathbf{k}})_{n0}$ tends toward $\langle 0|\mathbf{J}|n\rangle$. As a consequence of translational invariance, **J** commutes with the Hamiltonian and is therefore a good quantum number: $\langle 0|\mathbf{J}|n\rangle$ is thus zero. We may expand $(j_{\mathbf{k}})_{n0}$ in a Taylor series; the first term is of order k. The corresponding contribution to the sum rule is thus of order k^4. [Again, it seems plausible that for these higher configurations correlation effects do not yield $(j_{\mathbf{k}})_{n0}$ proportional to some fractional power of k.]

We conclude that in the limit of $\mathbf{k}=0$, the single-excitation contribution exhausts the sum rule. We then have

$$Z(\mathbf{k})=S(\mathbf{k}) \quad\text{and}\quad \omega(\mathbf{k})=[k^2/2MS(\mathbf{k})], \quad \text{for}\quad k\rightarrow 0. \tag{2.17}$$

The continuum contribution, $S^{(1)}(\mathbf{k}\omega)$, first enters at a finite value of k, and most likely contributes a term of order k^4 to the sum rule.

The above discussion offers a new perspective on the classic work of Feynman,[3] and on its improvement by Feynman and Cohen.[4] Feynman gave very convincing physical arguments that the wavefunction in configuration space for low-lying excitations should be of the form

$$\psi_{\mathbf{k}}=\sum_i f_{\mathbf{k}}(\mathbf{x}_i)|0\rangle. \tag{2.18}$$

He determined $f_{\mathbf{k}}(\mathbf{x})$ by a variational calculation and found it to be $e^{+i\mathbf{k}\cdot\mathbf{x}}$. Thus the Feynman wavefunction is given by

$$\psi_{\mathbf{k}}=\rho_{\mathbf{k}}^\dagger|0\rangle. \tag{2.19}$$

The excitation spectrum appropriate to (2.11) was shown by Feynman to be

$$E_{\mathrm{F}}(\mathbf{k})=k^2/2mS(\mathbf{k}). \tag{2.20}$$

In the light of our previous discussion (2.20) becomes nearly obvious. Indeed, the choice of the wavefunction (2.19) implies that $\rho_{\mathbf{k}}$ couples the ground state to a unique excited state, namely that with a single quasi-particle of momentum **k**. Within this approximation, $S^{(1)}(\mathbf{k}\omega)=0$: (2.20) thus reduces to (2.17). One may view the Feynman excitation spectrum as the most general result one can obtain, consistent with Eqs. (2.6) and (2.7), with the assumption that the density fluctuations possess a unique excitation frequency. To put it another way, with the Feynman wavefunction, Eq. (2.19), a single excited state exhausts the f-sum rule.

The Feynman excitation spectrum, $E_{\mathrm{F}}(k)$, obtained from (2.20), yields an energy vs momentum curve which possesses the same qualitative shape as that de-

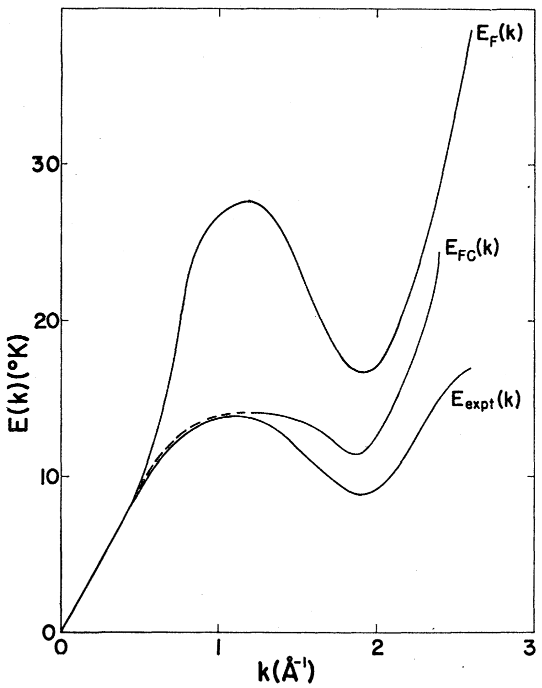

FIG. 2. Comparison of Feynman $[E_F(\mathbf{k})]$ and Feynman and Cohen $[E_{FC}(\mathbf{k})]$ predicted elementary excitation spectrum with that measured by Henshaw and Woods (see reference 1) $[E_{expt}(\mathbf{k})]$.

termined experimentally by neutron scattering,[1] as may be seen from Fig. 2. However, the agreement is not quantitative everywhere; thus near the roton minimum one finds $E(k)\cong 17°K$ instead of the observed value 8.6°K. In order to overcome this difficulty, Feynman and Cohen were led to an improved wavefunction for the elementary excitations on the basis of physical arguments to the effect that current must be conserved in the motion of an excitation through the liquid, and hence there must be a backflow of atoms (or elementary excitations) about an atom as it moves along. The resultant wavefunction (which we consider in some detail in Sec. IV) yields the excitation curve $E_{FC}(\mathbf{k})$ shown in Fig. 2, a curve which is in good quantitative agreement with experiment in the region up to and including the roton minimum.

The considerations of the present section lead to the same conclusion. According to (2.11), the disparity between $E_F(\mathbf{k})$ and experiment arises from the continuum contribution to the correlation function, $S^{(1)}(\mathbf{k}\omega)$. We have seen that $S^{(1)}(\mathbf{k}\omega)$ originates in the coupling of $\rho_{\mathbf{k}}^\dagger|\psi_0\rangle$ to higher configurations involving several elementary excitations. If we view $\rho_{\mathbf{k}}^\dagger|\psi_0\rangle$ as a "bare" quasi-particle, the coupling expresses the physical fact that the bare quasi-particle is surrounded by a cloud of virtual elementary excitations. As it moves, the quasi-particle drags this cloud along with it; the cloud is equivalent to the backflow proposed by Feynman and Cohen. In the limit of small k, $S^{(1)}\rightarrow 0$: the self-energy cloud gradually disappears, thus restoring the original result of Feynman. The variational approach of Feynman and Cohen has therefore a simple microscopic counterpart: the backflow around any given atom is nothing but the motion of the self-energy cloud arising from the interaction between the "bare" excitations (2.19).

Let us now turn to a more quantitative discussion. In principle, the three quantities $S(\mathbf{k})$, $\omega(\mathbf{k})$, $Z(\mathbf{k})$ are directly measurable. $S(\mathbf{k})$ refers to the total cross section for a scattering event with momentum transfer $\mathbf{k}$, irrespective of the energy transfer. It may be obtained in straightforward fashion from either x-ray scattering measurements or from slow neutron scattering experi-

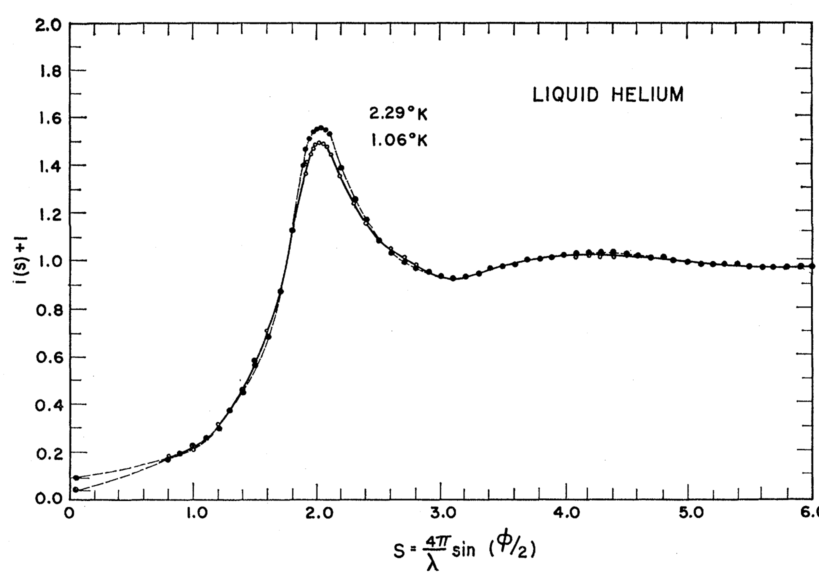

FIG. 3. Experimental results obtained by Henshaw (see reference 17) for $S(\mathbf{k})$.

ments,[15,16] provided the momentum of the incident x ray or neutron is sufficiently large that a measurement of the scattered intensity at a fixed angle is equivalent to a measurement of the probability per unit time of a given momentum transfer. In Fig. 3 we reproduce the most recent measurement of $S(\mathbf{k})$, obtained by Henshaw[17] using slow neutrons as the scattering probe.

In those experiments in which both the energy and the angle of the scattered neutrons are measured, analysis of the results shows a pronounced peak in the number of neutrons which have been scattered through a certain angle θ, and suffered an energy loss, ω; this peak is generally superimposed on a continuous background. The peak may be identified as representing those neutrons which have excited a single quasi particle from the condensed phase. The position of the peak yields $\omega(\mathbf{k})$, its strength (i.e., the area enclosed) gives $Z(\mathbf{k})$. $Z(\mathbf{k})$ is thus equal to the differential cross section for the production of a single quasi-particle; in Fig. 4 we reproduce the experimental results obtained by Henshaw and Woods[1] for $Z(\mathbf{k})$. One can thus compute the first two moments of $S^{(1)}(\mathbf{k}\omega)$ and estimate the importance of backflow. In fact, the available experimental data are incomplete and sometimes not very precise. We shall now try to analyze the data in the light of the preceding discussion.

We first consider $\omega(\mathbf{k})$. We expect the limiting value for small k to be equal to ck, where c is the sound velocity (237 m/sec). This limit is indeed observed experimentally. We remark that in Fig. 2 the linear portion of $\omega(\mathbf{k})$ extends up to a wave vector $k \cong 0.6$ Å^{-1}, which is

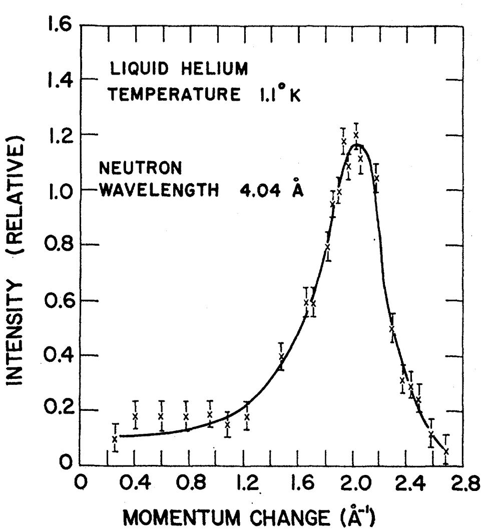

FIG. 4. Experimental results obtained by Henshaw and Woods (see reference 1) for $Z(\mathbf{k})$.

[15] C. F. A. Beaumont and J. Reekie, Proc. Roy. Soc. (London) **A228**, 363 (1955); A. G. Tweet, Phys. Rev. **93**, 15 (1954); W. L. Gordon, C. H. Shaw, and J. G. Daunt, J. Phys. Chem. Solids **5**, 117 (1958).

[16] D. G. Hurst and D. G. Henshaw, Phys. Rev. **100**, 994 (1955).

[17] D. G. Henshaw, Phys. Rev. **119**, 9 (1960).

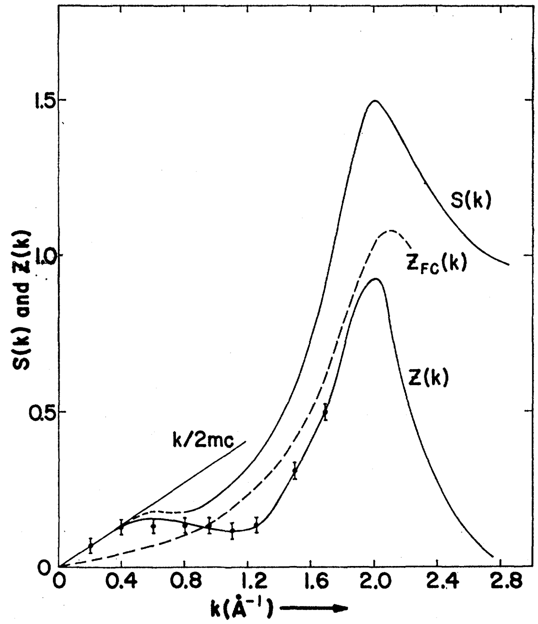

FIG. 5. Extrapolation of experimental results for $S(\mathbf{k})$ and $Z(\mathbf{k})$; comparison with the Cohen-Feynman calculation, $Z_{FC}(\mathbf{k})$.

rather surprising: one might expect a stronger dispersion of the phonon frequency. We also note that the slope $d\omega(k)/dk$ is always lower than the phonon velocity, which ensures the stability of the quasi-particle against the emission of real phonons.

We next consider the experimental results obtained by Henshaw[17] for $S(\mathbf{k})$ and by Henshaw and Woods[1] for $Z(\mathbf{k})$, which are shown in Figs. 3 and 4. We first remark that the data for $S(\mathbf{k})$ extend down only to 0.8 Å^{-1}. It is tempting to extrapolate the experimental curve linearly between the origin and $k=0.8$ Å^{-1}; the result so obtained, however, is incorrect. For by doing so, one finds that the slope at the origin dS/dk is 0.21 Å, a value which is in contradiction with the result 0.36 Å obtained from (2.17) and the experimental $\omega(\mathbf{k})$. As we know that (2.17) *must* be right in the limit $k\rightarrow 0$, we are led to draw the curve for $S(\mathbf{k})$ near the origin in such a way as to yield the correct slope at the origin. The next question is how far toward $k=0.8$ Å^{-1} does the straight line portion of $S(k)$ extend. This question can best be considered in the light of the behavior of $Z(\mathbf{k})$.

The measurements of $Z(\mathbf{k})$ shown in Fig. 4 extend to 0.2 Å^{-1}; however, they are relative, so that it is necessary to determine indirectly a suitable scale. We do this with the aid of our result, (2.17), which asserts that for small k, $Z(\mathbf{k})$ should be equal to $S(\mathbf{k})$ and approach the origin with the same slope as $S(\mathbf{k})$. We therefore extrapolate between 0.2 Å^{-1} and the origin by drawing a straight line (which also passes through the experimental value at 0.4 Å^{-1}); the scale is then fixed by requiring that the slope be 0.36 Å. The extrapolated value of $Z(\mathbf{k})$ so obtained is shown in Fig. 5, where it is compared with the theoretical value of Cohen and Feynman.[9]

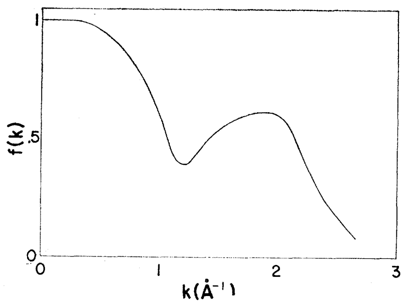

FIG. 6. $f(\mathbf{k})$ vs k.

We return to the extrapolation of the curve for $S(\mathbf{k})$ between the origin and 0.8 Å^{-1}. By comparison with the $Z(\mathbf{k})$ curve, it is clear that the straight line portion of $S(\mathbf{k})$ must extend at least to 0.4 Å^{-1}. Moreover, $S(\mathbf{k})$ must possess some sort of shoulder between 0.4 and 0.8 Å^{-1}. Whether that shoulder is marked [as it will be if one takes $S(\mathbf{k})$ linear to 0.6 Å^{-1}] or mild [assuming $S(\mathbf{k})=Z(\mathbf{k})$ up to 0.6 Å^{-1}], one cannot say; we have chosen an extrapolation of $S(\mathbf{k})$ in this region which lies between these extremes.

We estimate that the errors to be attached to our extrapolation of $S(\mathbf{k})$ and the scaling of $Z(\mathbf{k})$ do not exceed 20%. We further remark that the necessary existence of a shoulder in the $S(\mathbf{k})$ curve leads us to believe that the experimental points obtained by Henshaw and Woods, which show a similar shoulder in the $Z(\mathbf{k})$ curve, are reliable, and do not represent a systematic experimental error. We also note that given the form of our extrapolation for $S(\mathbf{k})$ in Fig. 5, the linearity in the phonon excitation curve observed by Henshaw and Woods between 0.4 and 0.6 Å^{-1} is to be attributed to a cancellation of two opposing effects: backflow, which acts to decrease $\omega(\mathbf{k})$ and the shoulder in $S(\mathbf{k})$, which would act to increase $\omega(\mathbf{k})$.

Another quantity of interest is the ratio $Z(\mathbf{k})/S(\mathbf{k})$ which we shall call $f(\mathbf{k})$:

$$f(\mathbf{k})=Z(\mathbf{k})/S(\mathbf{k}). \tag{2.21}$$

$f(\mathbf{k})$ furnishes a relative measure of the efficiency of quasi-particle excitation from the condensed state by an incident slow neutron. A plot of $f(\mathbf{k})$, based on the experimental measurements (and our extrapolations thereof in Fig. 5), is given in Fig. 6. We first remark that $f(\mathbf{k})=1$ for values of k extending from the origin to 0.4 Å^{-1}; that is scarcely surprising, since we have extrapolated $S(\mathbf{k})$ and $Z(\mathbf{k})$ in such a way that this would be true. We next remark that the behavior of $f(\mathbf{k})$ between 0.4 Å^{-1} and 2.7 Å^{-1} can be understood as a superposition of two effects:

(i) The depletion of the zero-momentum state as a consequence of particle interaction. We shall see in Sec. V that for large $\mathbf{k}$, one expects $f(\mathbf{k})\cong N_0/N$, the relative number of particles in the condensed state. This number may be quite small. For example, Onsager and Penrose[18] estimate $N_0/N\cong 0.08$. For small values of $\mathbf{k}$, coherence effects act to oppose such a substantial reduction; for intermediate values of $\mathbf{k}$ that opposition fades away, probably more or less uniformly, as backflow effects associated with the continuum part of $S(\mathbf{k}\omega)$ begin to play a role.

(ii) Coherence effects associated with the probability of exciting a quasi-particle with its associated "backflow" cloud of elementary excitations. This mechanism, suggested to us by Bardeen,[19] would tend to favor production of rotons, and so lead to a peaking of $f(\mathbf{k})$ in the region of the roton minimum. We note that the good agreement between the Cohen-Feynman result for $Z(\mathbf{k})$ and experiment indicates that their theory properly accounts for this phenomenon.

One way to investigate whether the proposed behavior of $f(\mathbf{k})$ is reasonable is to compute the quantity

$$\bar{\omega}(\mathbf{k})=\int_0^\infty d\omega\, \omega S^{(1)}(\mathbf{k}\omega)\Big/\int_0^\infty d\omega\, S^{(1)}(\mathbf{k}\omega)$$
$$=[(k^2/2m)-\omega(\mathbf{k})Z(\mathbf{k})]/[S(\mathbf{k})-Z(\mathbf{k})].$$

$\bar{\omega}(\mathbf{k})$ is the average excitation energy of the "higher configurations" entering in the spectral analysis of $S(\mathbf{k}\omega)$. We expect these configurations to contain typically two or three rotons. Using the preceding data, we find that the values of $\omega(\mathbf{k})$ range between 30 and 60°K, for $\mathbf{k}$ varying between 0.8 Å^{-1} and 2.8 Å^{-1}. This result is not unreasonable and supports our extrapolation procedure.

To summarize this discussion, the available experimental data yield a fair estimate of $Z(\mathbf{k})$ and $S(\mathbf{k})$. The general variation of their ratio, $f(\mathbf{k})$, with $\mathbf{k}$ is consistent with the observed excitation spectrum. The occurrence of the bump in $S(\mathbf{k})$, however, still requires a microscopic explanation.

III.

We have seen on the basis of general sum rule considerations that there is a close relationship between the backflow introduced by Feynman and Cohen and the coupling between $\rho_{\mathbf{k}}^\dagger|\Psi_0\rangle$ and higher configurations involving several elementary excitations. We have argued that this coupling acts to reduce the quasi-particle energy below the Feynman value and gives rise to a continuum contribution to $S(\mathbf{k}\omega)$. We now wish to develop a link between these considerations and the microscopic calculations of the excitation spectrum of an interacting boson system. Our aim is both to clarify the physical picture of backflow and to obtain useful clues as to the ingredients required for a successful microscopic theory of the elementary excitations in liquid helium.

We show in the Appendix that in the Bogoliubov theory[5] the density-fluctuation excitation spectrum is

[18] O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956).
[19] J. Bardeen (private communication).

identical to that of the quasi-particles, and that a single quasi-particle excitation exhausts the f-sum rule, (2.7). Hence no backflow effects are contained in this approximation; in order to take into account backflow, one must take into account the terms of H_2, Eq. (A3). Such terms give rise to a coupling between the Bogoliubov quasi-particles. The coupling is a microscopic analog of the coupling between the Feynman quasi-particles we have considered earlier; we would therefore anticipate that it gives rise to (a) backflow, (b) a continuum contribution to $S(\mathbf{k}\omega)$. The backflow, in the microscopic theory, would correspond to a self-energy cloud around each quasi-particle—a cloud of co-moving virtual excitations which would serve to reduce the quasi particle energy as well as to conserve current. A perturbation-theoretic treatment of H_2 is not so much difficult as it is complicated because of the appearance everywhere of the coherence factors, $u_\mathbf{k}$ and $v_\mathbf{k}$, of Eq. (A6). We can, moreover, obtain a good physical picture of the role played by the interaction between the excitations by considering a closely related problem, that of an impurity atom weakly coupled to a gas of weakly interacting bosons.[20]

We consider an impurity atom, of mass M_I, interacting with the boson system via the same law of interaction as the bosons interact with each other. It suffices then to add to the basic Hamiltonian, (2.1), the following terms:

$$P^2/2M_I+\sum_\mathbf{k} V_\mathbf{k}\rho_\mathbf{k}^\dagger e^{-i\mathbf{k}\cdot\mathbf{R}}, \tag{3.1}$$

where $\mathbf{P}$ is the impurity atom momentum, and $\mathbf{R}$ its position. We first obtain the wavefunction for the coupled impurity-boson system.

The unperturbed wavefunction of the impurity-boson system is

$$\psi(0)=e^{i\mathbf{P}\cdot\mathbf{R}}\phi_0, \tag{3.2}$$

where ϕ_0 is the ground-state wavefunction of the boson system. We write the perturbed wavefunction of the coupled system as

$$\psi=e^{iS}\psi(0), \tag{3.3}$$

where S may be regarded as generating a canonical transformation which serves to eliminate the impurity-boson interaction,

$$H_{\text{int}}=\sum_\mathbf{k} V_k\rho_\mathbf{k}^\dagger e^{-i\mathbf{k}\cdot\mathbf{R}}.$$

Since the interaction is assumed to be weak, it suffices to eliminate H_{int} to first order; S is then determined by the equation

$$i[(H_1+P^2/2M_I),S]=-H_{\text{int}}, \tag{3.4}$$

where H_1 is the boson Hamiltonian (A2). (It suffices to keep only H_1 since we wish to treat the bosons in the Bogoliubov approximation.)

One way to solve Eq. (3.4) is to carry out a Bogoliubov canonical transformation on H_1 and H_{int}; after a certain amount of algebra the correct answer emerges. However, we can avoid the algebra and the transformation in the following way. We first note that S may be written as

$$S=\sum A_\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{R}}, \tag{3.5}$$

where $A_\mathbf{k}$ is an operator which contains the dependence on the boson variables; on substitution of (3.5) into (3.4) we find an equation involving boson variables only:

$$i\{[H,A_\mathbf{k}]+(-\mathbf{k}\cdot\mathbf{P}/M_I+k^2/2M_I)A_\mathbf{k}\}=-V_k\rho_\mathbf{k}^\dagger. \tag{3.6}$$

If we now take the matrix elements of (3.6) between the ground state and some nth excited state coupled to it via $\rho_\mathbf{k}$, we find

$$(A_\mathbf{k})_{n0}=\frac{+iV_k(\rho_\mathbf{k}^\dagger)_{n0}}{\omega_{n0}-(\mathbf{k}\cdot\mathbf{P}/M_I)+(k^2/2M_I)}, \tag{3.7}$$

where ω_{n0} is the excitation frequency appropriate to that state. Since (as shown in the Appendix) $\rho_\mathbf{k}$ possesses a unique excitation frequency, Ω_k, then one finds at once that

$$S=i\sum_\mathbf{k} V_k\rho_\mathbf{k}^+e^{-i\mathbf{k}\cdot\mathbf{R}} \times[\Omega_k+k^2/2M_I-(\mathbf{k}\cdot\mathbf{P})/M_I]^{-1}. \tag{3.8}$$

The explicit form of the coupled system wavefunction is thus

$$\psi=\{\exp[i\sum_{j=1}^{N}g(\mathbf{r}_j-\mathbf{R})]\}\varphi_0e^{i\mathbf{P}\cdot\mathbf{R}}, \tag{3.9}$$

where

$$g(\mathbf{r})=i\sum_\mathbf{k} V_ke^{i\mathbf{k}\cdot\mathbf{r}}\{\Omega_k+(k^2/2M_I)-\mathbf{k}\cdot\mathbf{P}/M_I\}^{-1}. \tag{3.10}$$

Equation (3.9) agrees in form with the wavefunction proposed by FC to describe the backflow about an impurity atom as it moves through a boson gas. Moreover, for a slowly moving impurity atom, $g(\mathbf{r})$ reduces, at long distances, to the dipolar form which, as shown by FC, brings about current conservation. To show this, we expand the factor

$$\{\Omega_k+k^2/2M_I-\mathbf{k}\cdot\mathbf{P}/M_I\}^{-1}$$

as

$$\{\Omega_k+k^2/2M_I-\mathbf{k}\cdot\mathbf{P}/M_I\}^{-1} =(\Omega_k+k^2/2M_I)^{-1}+(\mathbf{k}\cdot\mathbf{P}/M_I)(\Omega_k+k^2/2M_I)^{-2}+\cdots.$$

The leading term is velocity-independent; the second, velocity-dependent, term at long wavelengths ($k^2/2M_I\ll\Omega_k$) may be written as

$$\begin{aligned} g(\mathbf{r})&=i\sum_\mathbf{k} V_k(\mathbf{k}\cdot\mathbf{P}/M_I)(1/\Omega_k^2)e^{i\mathbf{k}\cdot\mathbf{r}} \\ &\cong i\sum_\mathbf{k} V_k(\mathbf{k}\cdot\mathbf{P}/M_I)[1/(Nk^2V_k/M)]e^{i\mathbf{k}\cdot\mathbf{r}} \\ &\cong-(1/4\pi N)(M/M_I)(\mathbf{P}\cdot\mathbf{r}/r^3), \end{aligned} \tag{3.11}$$

provided one makes use of the long-wavelength version of the Bogoliubov dispersion relation, (A10),

$$\Omega_k^2\cong Nk^2V_k/M.$$

[20] Similar methods have been found useful in the electron gas problem; see P. Nozières and D. Pines, Phys. Rev. 109, 762 (1958).

The last equality in Eq. (3.11) displays the desired dipolar form. The current carried by the impurity atom is

$$\mathbf{j}_{\mathrm{imp}}(\mathbf{r})=(1/2M_I)\sum_{\mathbf{k}}\{\mathbf{P}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{R})}+e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{R})}\mathbf{P}\}. \quad (3.12)$$

The current carried by the bosons is

$$\begin{aligned} j_b(\mathbf{r})&=(N/M)\nabla_{\mathbf{r}}g(\mathbf{r}-\mathbf{R})\\ &\cong-(1/M_I)\sum_{\mathbf{k}}\mathbf{k}(\mathbf{k}\cdot\mathbf{P}/k^2)e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{R})}. \end{aligned} \quad (3.13)$$

The boson current represents therefore a backflow about the impurity atom motion, such that far away from the impurity $\mathrm{div}\mathbf{j}_b(\mathbf{r})\cong-\mathrm{div}\mathbf{j}_{\mathrm{imp}}(\mathbf{r})$. The net current, impurity plus boson backflow, associated with (3.3), is divergence free at distances far from the impurity atom.

In the language of quantum field theory, the boson backflow corresponds to a cloud of virtual phonons which surround the impurity atom. (As long as the velocity of the impurity atom is less than C, the boson sound velocity, no real transitions will occur.) The phonon cloud acts to increase the effective mass of the impurity atom; the change in energy of the clothed impurity atom is simply calculated using second-order perturbation theory:

$$\Delta E=-\sum_{n\mathbf{k}}\left[\frac{V_k^2(\rho_{\mathbf{k}}^+)_{n0}^2}{(\omega_{n0}-\mathbf{k}\cdot\mathbf{P}/M_I+k^2/2M_I)}\right]. \quad (3.14)$$

As shown in the Appendix, in the Bogoliubov approximation

$$\begin{aligned} \omega_{n0}&=\Omega(k)=[Nk^2V_k/M+k^4/4M^2]^{1/2}\\ (\rho_{\mathbf{k}})_{n0}^2&\cong Nk^2/2M\Omega(k), \end{aligned} \quad (3.15)$$

so that

$$\Delta E=-\sum_{\mathbf{k}}\left\{\frac{V_k^2Nk^2/2M\Omega(k)}{[\Omega(k)-(\mathbf{k}\cdot\mathbf{P}/M_I)+k^2/2M_I]}\right\}. \quad (3.16)$$

A result similar to (3.16) has been obtained by Girardeau,[21] with the aid of the Bogoliubov canonical transformation.

Equation (3.16) simplifies when the impurity velocity is small compared to the boson sound velocity; we may then expand the denominator in powers of $(\mathbf{k}\cdot\mathbf{P}/M_I)/(\Omega_k+k^2/2M_I)$. The leading term is independent of impurity momentum; the first momentum-dependent term may be written as

$$-\alpha\frac{P^2}{2M_I}=\frac{P^2}{2M_I}\left\{-\frac{M}{3NM_I}\sum_{\mathbf{k}}\left(\frac{Nk^2V_k}{M}\right)^2\times\frac{1}{\Omega_k(\Omega_k+k^2/2M_I)^3}\right\}. \quad (3.17)$$

Thus, the effective mass of the impurity atom is increased to

$$M^*=M_I/(1-\alpha) \quad (3.18)$$

as a result of the backflow described by (3.9) and (3.10).

An interesting feature of the result, (3.17), is the following. Let us consider only the long-wavelength contribution to α, arising from values of $k\leq k_m$, such that $\Omega>k_m^2/2M_I$. We then find

$$\alpha\cong\sum_{k\leq k_m}\frac{M}{3NM_I}\left(\frac{Nk^2V_k}{M}\right)^2\frac{1}{\Omega_k^4}=\frac{N'}{3N}\frac{M}{M_I}, \quad (3.19)$$

where N' is the number of phonon modes which satisfy this criterion. The result (3.10) is independent of the specific parameters of the problem. We therefore speculate that it is a model-independent result; i.e., it is as valid for a He^3 atom moving in He^4 as it is for an impurity weakly coupled to a weakly interacting boson gas. For example, if we take $M_I=(3/4)M$, and $N'=N$ (corresponding to $k_m\cong1.1\times\text{Å}^{-1}$), we find $\alpha=4/9$, and $M^*=1.8M_I$, a result which agrees well with that obtained by Feynman and Cohen with the wavefunction (3.9), in which

$$g(r)=A\mathbf{P}\cdot\mathbf{r}/r^3\cong-(1/4\pi N)\mathbf{P}\cdot\mathbf{r}/r^3.$$

As another example, consider $N'=N$, and $M_I=M$; one finds then $M^*=(3/2)M$, a result which agrees with the classical hydrodynamic calculation of the effective mass of a sphere moving through a liquid.

We remark that where α becomes large, perturbation theory breaks down. The problem considered here is directly analogous to that of a slow electron moving in a polar crystal, the so-called polaron problem. For large α, an intermediate coupling calculation[22] of the impurity mass can be performed. The result is

$$M^*=M_I(1+\alpha).$$

IV.

Feynman and Cohen proceed from a study of impurity atom motion to a variational calculation of the elementary excitation spectrum of liquid helium. They consider as a trial wavefunction a symmetrized version of (3.9), viz.

$$\Psi=\{\sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i}\exp[i\sum_{j\neq i}g(\mathbf{r}_j-\mathbf{r}_i)]\}|0\rangle, \quad (4.1)$$

where $g(\mathbf{r})$ is assumed to take the dipolar form,

$$g(\mathbf{r})=A\mathbf{k}\cdot\mathbf{r}/r^3, \quad (4.2)$$

which one expects it to have at long distances. FC then simplify (4.1) by expanding the backflow term to obtain

$$\Psi_{\mathrm{FC}}\cong\sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i}\left\{1+iA\sum_{j\neq i}\frac{k\cdot(\mathbf{r}_j-\mathbf{r}_i)}{(\mathbf{r}_j-\mathbf{r}_i)^3}\right\}|0\rangle. \quad (4.3)$$

[21] M. Girardeau, Phys. Fluids 4, 279 (1960).

[22] T. D. Lee, F. Low, and D. Pines, Phys. Rev. 90, 297 (1953).

It is for this latter form that their variational calculation is carried out.

The FC wavefunction, (4.3), lends further support to our picture of backflow in liquid helium as arising from an interaction between the Feynman quasi-particles, $\rho_{\mathbf{k}}^{\dagger}|0\rangle$. To see this, we Fourier analyze the backflow term, (4.3), and so obtain

$$\Psi_{\mathrm{FC}}=\{\rho_{\mathbf{k}}^{\dagger}+\sum_{q}\sum_{i,j\neq i}4\pi A[(\mathbf{k}\cdot\mathbf{q})/q^2] \times e^{i\mathbf{q}\cdot\mathbf{r}_j}e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}_i}\}|0\rangle, \quad (4.4)$$

which may also be written as

$$\Psi_{\mathrm{FC}}\cong\left\{\rho_{\mathbf{k}}^{\dagger}+\sum_{q\neq k}4\pi A\frac{\mathbf{k}\cdot\mathbf{q}}{q^2}\rho_{\mathbf{q}}^{\dagger}\rho_{\mathbf{k}-\mathbf{q}}^{\dagger}\right\}|0\rangle. \quad (4.5)$$

Equation (4.5) shows that the backflow term approximately corresponds to including a term which allows for a superposition of single quasi-particle and double quasi-particle excitations in the system wavefunction. In other words, it corresponds to an approximate treatment of the interaction between Feynman quasi-particles. [The term with $\mathbf{q}=\mathbf{k}$ is omitted in the above sum because otherwise the double excitation function would have a substantial single excitation component (of order $4\pi AN$) mixed in.]

In fact, independent of the work of FC, Kuper[10] carried out a perturbation-theoretic treatment of the interaction between the virtual Feynman quasi-particle excitations. We now derive the wavefunction used by Kuper, and compare it to (4.4). We begin by defining a Hamiltonian, H^1, which directly yields the Feynman excitation spectrum, when acting on the ground state:

$$H^1\rho_{\mathbf{k}}^{\dagger}|0\rangle=(E_{\mathrm{F}}(\mathbf{k})+E_0)\rho_{\mathbf{k}}^{\dagger}|0\rangle, \quad (4.6)$$

where E_0 is the ground-state energy, and $E_{\mathrm{F}}(k)$ is the Feynman quasi-particle excitation energy. Then, in Rayleigh-Schrödinger perturbation theory, the modified quasi-particle excitation takes the form[23]

$$(1/NS_{\mathbf{k}})\rho_{\mathbf{k}}^{\dagger}\Psi_0+\tfrac{1}{2}\sum_{\mathbf{q}\neq 0,\mathbf{k}}A_{\mathbf{kq}}\rho_{\mathbf{k}-\mathbf{q}}^{\dagger}\rho_{\mathbf{q}}^{\dagger}\Psi_0/N(S_{\mathbf{k}}S_{\mathbf{q}})^{1/2}, \quad (4.7)$$

where

$$A_{\mathbf{kq}}=\frac{\langle\Psi_0|\rho_{\mathbf{k}-\mathbf{q}}\rho_{\mathbf{q}}(H-H^1)\rho_{\mathbf{k}}^{\dagger}|\Psi_0\rangle}{E_{\mathrm{F}}(\mathbf{k})-E_{\mathrm{F}}(\mathbf{k}-\mathbf{q})-E_{\mathrm{F}}(\mathbf{q})}. \quad (4.8)$$

The numerator of A may be written as follows:

$$\langle\Psi_0|\rho_{\mathbf{k}-\mathbf{q}}\rho_{\mathbf{q}}\{[H,\rho_{\mathbf{k}}^{\dagger}]-E_{\mathrm{F}}(\mathbf{k})\rho_{\mathbf{k}}^{\dagger}\}|\Psi_0\rangle =\langle\Psi_0|\rho_{\mathbf{k}-\mathbf{q}}\rho_{\mathbf{q}}\{\mathbf{k}\cdot\mathbf{j}_{\mathbf{k}}-E_{\mathrm{F}}(\mathbf{k})\rho_{\mathbf{k}}^{\dagger}\}|\Psi_0\rangle. \quad (4.9)$$

We remark that in the limit of $k\rightarrow 0$ this numerator vanishes, since, as we have seen in the preceding section, the single Feynman excitation exhausts the sum rule. This is as it should be; i.e., the perturbation, $H-H^1$, should vanish in the limit $k\rightarrow 0$, since in this limit the excitation is properly described by $\rho_{\mathbf{k}}^{\dagger}\Psi_0$.[24]

Another way of saying this is that the strength of the perturbation $H-H^1$ is directly determined by the amount of backflow required in order to bring about longitudinal current conservation. Thus, when $\langle n|\mathbf{k}\cdot\mathbf{j}_k^{\dagger}|0\rangle=\omega_{n0}(\rho_{\mathbf{k}}^{\dagger})_{n0}\neq E_{\mathrm{F}}(\mathbf{k})(\rho_{\mathbf{k}}^{\dagger})_{n0}$, we get a contribution to the double excitation mode. We expect that $A_{\mathbf{kq}}$ is $O(k^2)$ in the limit of small k although we have not yet constructed a proof of this conjecture.

The wavefunction (4.9) is that considered by Kuper. Using it, he performed an approximate calculation for k in the roton region, and found $E_r\cong 11.5°\mathrm{K}$, a value in quite good agreement with that obtained by Feynman and Cohen. Quite recently, Jackson and Feenberg[25] have carried out calculations analogous to Kuper's, with the only difference residing in their use of Brillouin-Wigner perturbation theory, in which the energy denominators in (4.7) are to be replaced by the observed excitation energies. They obtain results which compare favorably with those of FC and with experiment. Such agreement is perhaps not surprising, in view of the close relationship between (4.7) and (4.5). Indeed, since backflow corrections are small for low-k excitation, it should be possible to show that (4.7) reduces to (4.5) in this limit, for in these circumstances the amount of the double excitation wavefunction required is small and should be accurately treated by means of perturbation theory. We have attempted to establish such an identity, but have not yet been able to do so; the principal stumbling block would appear to reside in approximations required to estimate the contribution to $A_{\mathbf{kq}}$ from the three-particle distribution function.

From a microscopic point of view, we conclude that the success of the FC theory is a consequence of perhaps two different factors:

(i) Allowing the virtual excitation of pairs of Feynman quasi-particles goes a long way toward including the many effects of the interaction between different quasi-particle configurations.

(ii) The Feynman quasi-particles for values of k lying between ~ 1 Å^{-1} and 2 Å^{-1} behave rather like slow moving impurity atoms in their coupling to the long wavelength excitations. Thus, the dipolar form of the backflow will be justified in treating, for example, the roton-phonon interaction (the coupling between a Feynman roton and Feynman phonons, which gives rise to the lowered roton energy).

We can also obtain at least a qualitative understanding of two aspects of the FC calculation. First, FC

[23] We are neglecting the overlap integrals, $(\Psi_0,\rho_{\mathbf{k}}\rho_{\mathbf{k}-\mathbf{q}}^{\dagger}\rho_{\mathbf{q}}^{\dagger}\Psi_0)$, which if included, would alter (4.7) by a small normalization constant.

[24] We remark that the apparent vanishing of the denominator in (4.8), which is nearly possible if $\mathbf{k}$ and $\mathbf{q}$ are both phonon wavevectors, therefore, leads to no difficulties; the numerator vanishes sufficiently rapidly for small k that $A_{\mathbf{kq}}\rightarrow 0$ in this limit.

[25] H. W. Jackson and E. Feenberg (to be published). We should like to thank Professor Feenberg for sending us a copy of their paper in advance of publication.

find that $E_{\text{FC}}(\mathbf{k}) \cong \frac{2}{3} E_{\text{F}}(\mathbf{k})$ in the vicinity of the roton minimum. That the result should be of this order of magnitude follows from our estimates at the close of the preceding section. Where coherence effects are not important, we could initially regard each He atom as surrounded by a phonon cloud, which acts to increase its mass to roughly $M^* \cong \frac{3}{2}M$; if we now superimpose the dressed-particle states, we would find

$$E(\mathbf{k}) \cong \frac{k^2}{2M^*S(\mathbf{k})} \cong \frac{k^2}{3MS(\mathbf{k})}, \tag{4.10}$$

a result which agrees rather well with FC for the region between 1 Å^{-1} and 2 Å^{-1}. A similar argument could *not* be used for the long-wavelength excitations, because here coherence effects dominate in determining the effective interaction between quasi-particles.

Second, we remark that it is easy to understand why FC, and all other theories, fail for the excitation spectrum between $k \cong 2$ Å^{-1} and 2.8 Å^{-1}. Here the long-wavelength approximation, which is at the heart of Eqs. (4.1) to (4.3), has begun to break down. The notion of a dipolar backflow is not particularly appropriate when one is considering distances of the order of an interparticle spacing.

V

Most of the microscopic calculations which have thus far been carried out on the excitation spectrum of an interacting boson system are equivalent to using a modified Bogoliubov theory in which an effective potential $\tilde{V}_k$ is introduced into the Bogoliubov excitation spectrum (3.15). The excitation spectrum in the low-density limit is obtained if one replaces V_k and V_0 by an effective potential,

$$V_k^{(1)} = V_0^{(1)} = \hbar^2 f_0/M, \tag{5.1}$$

where f_0 is the low-energy pair scattering amplitude. This form of the excitation spectrum is valid when the scattering length is small compared to the average particle spacing, that is, when

$$N f_0^3 \ll 1. \tag{5.2}$$

For a system of hard spheres for which Eq. (5.2) is *not* satisfied, Brueckner and Sawada[26] have proposed taking into account a selected further class of interactions beyond the multiple-scattering terms summed by means of the pseudopotential f_0. Their approximate summation is equivalent to replacing V_k by the effective potential

$$V_k^{(2)} = \lambda \sin(ka)/ka, \tag{5.3}$$

where a is the hard-sphere radius and λ is determined in self-consistent fashion.

In all of the microscopic calculations, Bogoliubov, low-density, Brueckner and Sawada, the effect of the depletion of the zero-momentum state as a consequence of particle interaction is neglected. Such neglect is justified for the weak-interaction and low-density limits; it is not justified in the intermediate-density calculation of Brueckner and Sawada. Recently, Parry and ter Haar[27] have shown that if one takes the Brueckner-Sawada form for the excitation spectrum, the depletion of the zero-momentum state amounts to some 270%. This is, of course, a clear indication of the need for carrying out a consistent calculation in which the effects of depletion are taken into account. When this is done, the depletion is reduced to some 50% (i.e., $N_0 \cong \frac{1}{2}N$); unfortunately the good qualitative agreement between the Brueckner-Sawada excitation spectrum and the experimentally measured one also disappears.

An estimate of the depletion of the zero-momentum state in liquid He has been made by Onsager and Penrose.[18] They find a depletion of some 92%. The likely quite considerable depletion of the zero-momentum state poses a particular problem in the calculation of $S(k,\omega)$. In the framework of Bogoliubov approximation, one has

$$\rho_{\mathbf{k}}^\dagger = a_0^\dagger a_{-\mathbf{k}} + a_{\mathbf{k}}^\dagger a_0. \tag{5.4}$$

Consequently, $\rho_{\mathbf{k}}$ couples the ground state to a single excited state, namely, that with one quasi-particle of momentum $\mathbf{k}$ and energy $\omega_{\mathbf{k}}$. The corresponding matrix element is

$$(\rho_{\mathbf{k}}^\dagger)_{0n} = (N_0 k^2/2M\Omega_k)^{1/2}.$$

The factor $(N_0)^{1/2}$ originates in the operators $a_0^\dagger$ and a_0. Hence

$$S(\mathbf{k}\omega) = (N_0 k^2/2M\Omega_k)\delta(\omega - \Omega_k). \tag{5.5}$$

Whatever the choice of Ω_k, this will yield

$$\int_0^\infty S(\mathbf{k}\omega)\omega d\omega = N_0 \frac{k^2}{2M}. \tag{5.6}$$

We thus see that the Bogoliubov approximation fails to satisfy (2.7), the sum rule, by an amount of order $(N-N_0)/N$, because in this approximation, the strength of the quasi-particle contribution to $S(\mathbf{k},\omega)$ contains a factor N_0, instead of the expected N. This is a basic shortcoming of the Bogoliubov method, independent of the neglect of backflow effects. In order to see what are the terms which are neglected in the Bogoliubov approximation, it is convenient to turn to the field-theoretic formulation of the interacting boson problem.[7,8] $S(\mathbf{k}\omega)$ is most readily obtained by calculating $F(\mathbf{k},\omega)$, the Fourier transform of the density-fluctuation propagator, $F(\mathbf{k},\tau)$. The latter is defined by

$$F(\mathbf{k},\tau) = -i\langle 0 | T\{\rho_{\mathbf{k}}(\tau)\rho_{\mathbf{k}}^\dagger(0)\} | 0 \rangle, \tag{5.7}$$

where $|0\rangle$ is the exact ground-state wavefunction, T is the Dyson chronological operator, which orders earlier times to the right, and the operators $\rho_{\mathbf{k}}(\tau)$ are given in

[26] K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1117 (1957).

[27] W. A. Parry and D. ter Haar (to be published).

the Heisenberg representation. One has

$$F(\mathbf{k},\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, F(\mathbf{k}\omega)e^{-i\omega\tau}; \tag{5.8}$$

it is straightforward to show from the definition of $F(\mathbf{k},\tau)$, (5.7), and of $S(\mathbf{k}\omega)$, (2.5), that

$$S(\mathbf{k}\omega)=-(1/\pi)\,\mathrm{Im}F(\mathbf{k}\omega). \tag{5.9}$$

As shown by Hugenholtz and Pines,[8] when one replaces the condensed state operators, a_0 and $a_0^\dagger$, by $\sqrt{N_0}$, one finds three distinct contributions to $F(\mathbf{k},\tau)$ corresponding to diagrams with two, three, and four external lines. Thus

$$F(\mathbf{k},\tau)=F^a(\mathbf{k},\tau)+F^b(\mathbf{k},\tau)+F^c(\mathbf{k},\tau), \tag{5.10}$$

where

$$F^a(\mathbf{k},\tau)=-iN_0\langle 0|T\{[a_{-\mathbf{k}}^\dagger(\tau)+a_{\mathbf{k}}(\tau)]\times[a_{\mathbf{k}}^\dagger(0)+a_{-\mathbf{k}}(0)]\}|0\rangle, \tag{5.11a}$$

$$F^b(\mathbf{k},\tau)=-iN_0^{1/2}\langle 0|T\{[a_{-\mathbf{k}}^\dagger(\tau)+a_{\mathbf{k}}(\tau)]\times[\textstyle\sum_{\mathbf{q}}'\, a_{\mathbf{q}+\mathbf{k}}^\dagger(0)a_{\mathbf{q}}(0)]\}|0\rangle, \tag{5.11b}$$

$$F^c(\mathbf{k},\tau)=-i\langle 0|T\{[\textstyle\sum_{\mathbf{q}}'\, a_{\mathbf{q}-\mathbf{k}}^\dagger(\tau)a_{\mathbf{q}}(\tau)]\times[\textstyle\sum_{\mathbf{q}'}'\, a_{\mathbf{q}'+\mathbf{k}}^\dagger(0)a_{\mathbf{q}}(0)]\}|0\rangle, \tag{5.11c}$$

and the primes on the summations denote the fact that creation and annihilation operators for zero momentum bosons no longer appear.

The first class of contributions, $F^a(\mathbf{k},\tau)$, are the only terms one would keep in a modified Bogoliubov calculation. These represent the direct quasi-particle contribution to $F(\mathbf{k},\tau)$, since their calculation requires only a knowledge of the single-particle Green's function,

$$G(\mathbf{p},\tau)=-i\langle 0|T\{a_{\mathbf{p}}(\tau)a_{\mathbf{p}}^\dagger(0)\}|0\rangle, \tag{5.12}$$

and the closely related quantity,

$$\tilde{G}(\mathbf{p},\tau)=-i\langle 0|T\{a_{\mathbf{p}}(\tau)a_{-\mathbf{p}}(0)\}|0\rangle. \tag{5.13}$$

In the modified Bogoliubov approximation we have been considering, one has[8]

$$G(\mathbf{k},\omega)=u_k^2/(\omega-\Omega_{\mathbf{k}}+i\delta)-v_k^2/(\omega+\Omega_{\mathbf{k}}-i\delta), \tag{5.14}$$

$$\tilde{G}(\mathbf{p},\epsilon)=-u_{\mathbf{k}}v_{\mathbf{k}}\{1/(\omega-\Omega_{\mathbf{k}}+i\delta)-1/(\omega+\Omega_{\mathbf{k}}-i\delta)\}, \tag{5.15}$$

where $G(\mathbf{k}\omega)$ and $\tilde{G}(\mathbf{k}\omega)$ are the Fourier transforms of $G(\mathbf{k},\tau)$ and $\tilde{G}(\mathbf{k},\tau)$. Hence,

$$F(k,\omega)=\frac{N_0k^2/M}{(\omega-\Omega_{\mathbf{k}}+i\delta)(\omega+\Omega_{\mathbf{k}}-i\delta)}, \tag{5.16}$$

which leads at once to (5.5). In order to reach the conclusion (5.16), we had first to neglect $F^{(b)}$ and $F^{(c)}$, then to use the approximate results (5.14) and (5.15).

To summarize, the modified Bogoliubov theory is a one-parameter theory. If one choses that parameter (Ω_k, say) so as to yield agreement with experiment on the quasi-particle excitation spectrum, then one finds that the quasi-particle contribution to $F(\mathbf{k}\omega)$ and $S(\mathbf{k}\omega)$ is far too weak to exhaust the sum rule, a difficulty which may be directly attributed to the depletion of the zero-momentum state as a consequence of particle interaction.

We know from the arguments of Sec. II that higher order terms, either from $F^{(b)}$ and $F^{(c)}$, or from an improved calculation of $G(\mathbf{p}\epsilon)$, will act to increase the quasi-particle contribution to the sum rule for low momentum transfers in such a way that it exhausts the sum rule. We can think of such terms as arising from various kinds of coherence effects; a simple model for the role played by such effects has been discussed elsewhere by one of us.[28] Here we wish to point out that as one goes to large momentum transfers, such coherence effects will fade away—essentially because for sufficiently large momentum transfers, the interactions are unimportant, and the behavior of the system will resemble that of a gas of noninteracting particles. In other words, the quasi-particle contribution to $S(\mathbf{k}\omega)$ becomes

$$N_0\delta(\omega-k^2/2m);\quad k\gg 1\text{ Å}^{-1} \tag{5.17}$$

so that in this limit

$$f_k\rightarrow N_0/N,\quad (k\gg 1\text{ Å}^{-1}). \tag{5.18}$$

The experimental results of Henshaw and Woods, which we have plotted in Fig. 6, tend to offer support to this argument. We remark that already at $k=2.7$ Å^{-1}, f_k has reduced to 0.08. Indeed, it would seem likely that f_k is somewhat smaller than this. We further remark that if one accepts (5.18), then the experimental results for f_k are in good agreement with the value of 0.08 proposed by Penrose and Onsager.[18]

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APPENDIX

We present here a brief resumé of the Bogoliubov approximation.[5] To begin, we replace the operators a_0 and $a_0^\dagger$ in (2.1) by the "c"-number, $(N_0)^{1/2}$, where N_0 is the average number of particles in the zero momentum state. For weak interactions, or with a sufficiently dilute system, N_0 may be taken equal to N; more generally, it should be regarded as a parameter, along with μ, to be determined by Eqs. (2.2) and (2.3). The resulting Hamiltonian may be written as

$$H'=H_1+H_2, \tag{A1}$$

with

$$H_1=\sum_{\mathbf{k}}[\tilde{\epsilon}_{\mathbf{k}}a_{\mathbf{k}}^\dagger a_{\mathbf{k}}+(N_0V_k/2)(a_{\mathbf{k}}^\dagger a_{-k}^\dagger+a_{\mathbf{k}}a_{-\mathbf{k}})]. \tag{A2}$$

$$H_2=N_0^{1/2}\sum_{\mathbf{k},\mathbf{l}}{}'\, V_k(a_{\mathbf{l}-\mathbf{k}}^\dagger a_{-\mathbf{k}}a_{\mathbf{l}}+a_{\mathbf{l}-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger a_{\mathbf{l}})+\sum_{\mathbf{k},\mathbf{l},\mathbf{l}'}{}'\,\tfrac{1}{2}V_k a_{\mathbf{l}-\mathbf{k}}^\dagger a_{\mathbf{l}'+\mathbf{k}}^\dagger a_{\mathbf{l}'}a_{\mathbf{l}}. \tag{A3}$$

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[28] D. Pines, Proceedings of the Summer School on Liquid Helium, Varenna, 1961 (to be published).

The primes denote the fact that creation and annihilation operators for zero-momentum bosons no longer appear; we have omitted constant terms, and

$$\tilde{\epsilon}_{\mathbf{k}}=\epsilon_{\mathbf{k}}+N_0[V_k+V_0]-\mu. \quad \text{(A4)}$$

The Bogoliubov approximation consists in neglecting the terms H_2, (A3). The Hamiltonian, Eq. (A2), may then be diagonalized by the transformation

$$a_{\mathbf{k}}=u_{\mathbf{k}}\alpha_{\mathbf{k}}-v_{\mathbf{k}}\alpha_{-\mathbf{k}}^{\dagger}, \quad \text{(A5)}$$

where

$$u_{\mathbf{k}}^2=\tfrac{1}{2}[1+\tilde{\epsilon}_{\mathbf{k}}/\hbar\Omega_k], \qquad v_{\mathbf{k}}^2=\tfrac{1}{2}[-1+\tilde{\epsilon}_{\mathbf{k}}/\hbar\Omega_k]. \quad \text{(A6)}$$

The transformed Hamiltonian is (aside from a constant)

$$H_1'=\sum_{\mathbf{k}}{}' \hbar\Omega_k\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}. \quad \text{(A7)}$$

The energy of the elementary excitations is

$$\Omega_k=(\tilde{\epsilon}_{\mathbf{k}}^2-N_0^2V_k^2)^{1/2}. \quad \text{(A8)}$$

The chemical potential μ is given by[8]

$$\mu=N_0V_0, \quad \text{(A9)}$$

so that one finds the well-known excitation spectrum,

$$\Omega_k=[N_0k^2V_k/M+k^4/4M]^{1/2}, \quad \text{(A10)}$$

which displays phonon-like behavior for small k.

The relation between N_0 and N is given by

$$N-N_0=\langle 0|\sum_{\mathbf{k}\neq 0} a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}|0\rangle=\sum_{\mathbf{k}\neq 0} v_{\mathbf{k}}^2 = \sum_{k\neq 0}\left[-\frac{\epsilon_{\mathbf{k}}+N_0V_k}{2\Omega_k}-\tfrac{1}{2}\right]\cong O(V_k). \quad \text{(A11)}$$

The Bogoliubov approximation is a weak-coupling theory, which is valid only to lowest order in V_k; therefore, according to (A11), $N_0\cong N$; the depletion of the zero-momentum state arising from particle interaction is to be neglected.

The calculation of $S(\mathbf{k}\omega)$ is straightforward. We have

$$\rho_{\mathbf{k}}^{\dagger}=\sum_{\mathbf{q}} a_{\mathbf{q}+\mathbf{k}}^{\dagger}a_{\mathbf{q}}=N_0^{1/2}(a_{\mathbf{k}}^{\dagger}+a_{-\mathbf{k}})+\sum_{\mathbf{q}}{}' a_{\mathbf{q}+\mathbf{k}}^{\dagger}a_{\mathbf{q}}, \quad \text{(A12)}$$

where the prime denotes the fact that terms with $\mathbf{q}=-\mathbf{k}$ or $\mathbf{q}=0$ are to be omitted. In the Bogoliubov approximation, only the first term on the right-hand side of (A12), which is of order $N_0^{1/2}$, is kept. On applying the transformation (A5), we find

$$\rho_{\mathbf{k}}^{\dagger}=N_0^{1/2}(u_{\mathbf{k}}-v_{\mathbf{k}})(\alpha_{\mathbf{k}}^{\dagger}+\alpha_{-\mathbf{k}}) = (N_0k^2/2M\Omega_k)^{1/2}(\alpha_{\mathbf{k}}^{\dagger}+\alpha_{-\mathbf{k}}). \quad \text{(A13)}$$

The density-fluctuation excitation spectrum is identical to the quasi-particle spectrum; the excited state n coupled by $\rho_{\mathbf{k}}$ to the ground state $|0\rangle$ possesses a unique excitation frequency, Ω_k, and the matrix element,

$$(\rho_{\mathbf{k}})_{n0}=(N_0k^2/2M\Omega_k)^{1/2}\cong(Nk^2/2M\Omega_k)^{1/2}. \quad \text{(A14)}$$

Hence

$$S_B(\mathbf{k}\omega)=(N_0k^2/2M\Omega_k)\delta(\omega-\Omega_k) \cong(Nk^2/2M\Omega_k)\delta(\omega-\Omega_k), \quad \text{(A15)}$$

$$S_B(\mathbf{k})=(N_0/N)(k^2/2M\Omega_k)\cong(k^2/2M\Omega_k). \quad \text{(A16)}$$

Moreover, the sum rule (2.7) reads

$$\int_0^{\infty} d\omega\, \omega S_B(\mathbf{k}\omega)=N_0k^2/2M\cong Nk^2/2M. \quad \text{(A17)}$$

From (A16) and (A17) it follows directly that the Bogoliubov approximation is a weak-coupling version of the Feynman assumption that a single quasi-particle excitation exhausts the f-sum rule, (2.7).

Finally, we will verify (within the Bogoliubov theory) the statement made in Sec. II that the matrix elements of $(\mathbf{j}_k^{\dagger})_{n0}$ vary as $k^{1/2}$ for small k when the state $|n\rangle$ represents a single elementary excitation with wave-vector k. To do this, we write the current operator $\mathbf{j}_k^{\dagger}$ given by Eq. (2.15) for configuration space in second quantized form:

$$\mathbf{j}_k^{\dagger}=\frac{1}{2M}\sum_{\mathbf{q}}(2\mathbf{q}-\mathbf{k})a_{\mathbf{q}}^{\dagger}a_{\mathbf{q}-\mathbf{k}}. \quad \text{(A18)}$$

Using the Bogoliubov approximation, the terms with $\mathbf{q}=0$ and $\mathbf{q}=\mathbf{k}$ are the most important so that

$$(\mathbf{j}_{\mathbf{k}})_{n0}=(N_0^{1/2}k/2M)\langle n|a_{\mathbf{k}}^{\dagger}-a_{-\mathbf{k}}|0\rangle = (N_0^{1/2}k/2M)(u_{\mathbf{k}}+v_{\mathbf{k}}). \quad \text{(A19)}$$

Since $u_{\mathbf{k}}+v_{\mathbf{k}}\sim k^{-1/2}$ for small k, we see that $(\mathbf{j}_{\mathbf{k}})_{n0}\sim k^{1/2}$.