Nuclear Single-Particle Wave Functions

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An approximate single-particle nuclear wave function is derived by solving an appropriate single-particle Schrödinger equation in a Hartree approximation with two-body central forces. The result of the first iteration is obtained in analytic form by means of the Thomas approximation in analogy with the Hulthén deuteron function. Because the wave function exhibits the correct asymptotic form for large particle separation, while maintaining the expected smooth behavior at small separations, it obtains significant improvement in the accuracy of the normalization over previously used functions for light nuclei. Comparison is made with the single-particle momentum distribution of He^4 obtained from recent (p,d) pickup experiments, and applications to the analysis of various inelastic scattering processes are discussed.

I. INTRODUCTION

THE theory of the "direct" interaction in inelastic scattering processes has contributed in large amount to the need for reliable single-particle nuclear wave functions. Usually in such a process, a particle initially bound in a target nucleus is "directly" scattered by the incident particle into an (nuclear) unbound state (or vice versa), and the angular distributions of the final-state particle(s) can only be understood if the initial single-particle state of the struck particle in the target nucleus is known. However, not all "direct" processes require that a single target nucleon be "picked out" by the incident projectile. In many scattering processes which employed the impulse approximation, it is often assumed, for example, that the incident particle interacts "directly" with *all* target nucleons *except* one "spectator" particle. In this case, the scattering results can be explained in terms of the experimentally known results of the simpler system (in which the spectator nucleon is absent) times a factor that gives the effect of the spectator on the original process. This factor, of course, requires the use of the spectator's single-particle behavior in the original nucleus. Of course, there are other fields of nuclear physics not directly related to scattering theory that also make use of single-particle wave functions. The nuclear shell model, for example, is based entirely on the assumption that such wave functions give a good description of many nuclear properties. And the direct-interaction stripping experiments of the nuclear spectroscopists have given the empirical means by which shell-model eigenfunctions can be determined.

The purpose of the present investigation, then, is to explore a method for deriving reasonable single-particle wave functions that will be useful in the analysis of the various inelastic scattering processes outlined above. Although the point of view is directed toward applications in high-energy scattering, the methods are general enough for further investigations with quite different applications. In order to understand the reason for choosing the particular methods outlined below, it is of interest to look at a typical process to which the wave function will be applied. Since the present investigation was motivated by a study of pion production in light nuclei[1] (H^3,He^3,He^4), a description of the theoretical treatment of the reaction, $p+d\rightarrow He^3+\pi^0$, will be given.

The usual analysis[2,3] of this process involves the use of the impulse approximation. This method obtains an expression involving an overlap integral of the deuteron wave function and the appropriate single-particle wave function for He^3. Now it is generally considered that the Hulthén function (with a hard core when applicable) represents an adequate description for the deuteron in such problems; however, this is unfortunately not true of the functions for the light nuclei. There are at present no adequate wave functions for $A>2$. In fact, the theoretical cross sections for the process mentioned above and for other inelastic scattering processes involving light nuclei show wide variations in both the angular distributions and the absolute magnitudes, when different forms of "usually acceptable" nuclear wave functions are used in the overlap integrals.

What are these "usually acceptable" wave functions, and why do they not give adequate single-particle wavefunctions? In the first place, most of them are not chosen to give an independent particle representation of the nucleus. Usually, they are functions which are symmetric in all interparticle distances, and which depend on a parameter that is fixed by minimizing the binding energy of the system. An example of such a form is that of Fröhlich *et al.*[4]: $\psi=\exp(-\frac{1}{2}\alpha\sum_{i>j}r_{ij})$. Other similar functions are given by Pease and Feshbach,[5] and Irving.[6] Although these functions may describe the many-body system adequately at small particle separations (since the function is made to satisfy conditions on the binding energy), they cannot represent the single-particle behavior at large sepa-

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[1] K. R. Greider, Phys. Rev. **122**, 1919 (1961).
[2] M. A. Ruderman, Phys. Rev. **87**, 383 (1952).
[3] S. A. Bludman, Phys. Rev. **94**, 1722 (1954).
[4] H. Fröhlich, K. Huang, and I. N. Sneddon, Proc. Roy. Soc. (London) **A191**, 61 (1947).
[5] R. L. Pease and H. Feshbach, Phys. Rev. **88**, 945 (1952); **81**, 142 (1951).
[6] J. Irving, Phil. Mag. **42**, 338 (1951).

ration. For large separation, the single-particle function for A nucleons must behave asymptotically like $e^{-\beta r}/r$, where β is related to the separation energy E_s, by $\beta^2=2m[(A-1)/A]E_s$, and r is the distance from the particle to the center-of-mass of the $(A-1)$ system. None of the functions given in the above references exhibit such an asymptotic behavior. Even if one is interested only in the small-separation part of the single-particle function, the symmetric functions above are not completely adequate. They may produce reasonable angular distributions for these cases, but the absolute magnitudes will, in general, be quite bad. This is due to the fact that even if a particular analysis emphasizes the small separation part, the asymptotic tail of the wavefunction is still important to obtain the correct normalization.

On the other hand, if one is interested in only the large-separation part of the wave function, the asymptotic formula may be sufficient (except for the normalization problem), although its singular behavior as r goes to zero precludes its use at small distances. Certainly the particular problem under consideration may emphasize the importance of certain regions of space over others. For instance, in the analysis of low-energy deuteron pickup reactions ($\lesssim$20 MeV) only low-momentum transfers are involved, and the large separation part ($>10^{-13}$ cm) of the wave function is emphasized. Then the asymptotic formula is usually quite adequate to give angular distributions. Similarly, in high-energy pion-production processes, the small-separation behavior of the particle becomes more important. But unfortunately, none of these problems really "pick out" a selected portion of configuration space; all regions contribute to a greater or lesser extent in the integrals, and certainly all regions contribute in an important way in the normalization. Therefore, it is desirable to have a function that adequately represents the particle for all of configuration space.

Thus, the philosophy we use is determined by our desire to solve certain specific problems that involve products of our function with the Hulthén deuteron function. Then, it is reasonable to require only that our function be of the same order of accuracy as the Hulthén function. We wish then to present a rather simple procedure: to find and solve an appropriate single-particle wave equation to the *same order* of the same approximation as the Hulthén function. The first step of this program, the finding, is described partly in Sec. II, where the formalism is developed, and partly in Sec. III, where the Hartree averaging process is called on to find an effective single-particle potential for the wave equation. The second step, the solving of the wave equation, is accomplished by the Thomas approximation, which is the same method that obtains the Hulthén deuteron function. Since there are iteration procedures involved in both the Thomas and the Hartree approximations, we combine these methods together in Sec. III. Sec. IV will be concerned with various tests of reliability and validity of the resulting wavefunction.

II. TWO-BODY WAVE EQUATION

Let the Hamiltonian for a system of A nucleons be H_A, and the binding energy, E_A. Then the wave function $\Psi_A{}^0$ for the ground state is defined by the relation

$$H_A\Psi_A{}^0=E_A\Psi_A{}^0, \tag{1}$$

where the superscript "0" refers to the ground state of the system. The Hamiltonian H_A and the wavefunction $\Psi_A{}^0$ are expressed in terms of a suitable set of $A-1$ *relative* coordinate vectors $\boldsymbol{\varrho}_i$ and their conjugate relative momenta $\boldsymbol{\pi}_i$ $(i=1, 2, \cdots, A-1)$. The explicit nonrelativistic form for H_A with central two-body nucleon-nucleon forces is

$$H_A=\sum_{i=1}^{A-1}\frac{\boldsymbol{\pi}_i{}^2}{2\mu_i}+\sum_{i>j}V_{ij}(\boldsymbol{\varrho}_1,\cdots\boldsymbol{\varrho}_{A-1}). \tag{2}$$

The μ_i are the effective reduced masses corresponding to the chosen set of relative momenta, $\boldsymbol{\pi}_i$. The potential $V_{ij}(\boldsymbol{\varrho}_1, \cdots, \boldsymbol{\varrho}_{A-1})$ is merely the two-body central potential $V_{ij}(|\mathbf{r}_i-\mathbf{r}_j|)$ expressed in terms of the *relative* coordinates $\boldsymbol{\varrho}_i$ rather than the ordinary coordinate differences.

To obtain a single-particle equation, we specify the particle under consideration with the subscript "k," and write

$$H_A=H_{A-1}+h_k, \tag{3}$$

where H_{A-1} is the complete Hamiltonian for the system of $(A-1)$ nucleons expressed in terms of a set of their relative coordinates, and h_k is the only part of H_A that involves the coordinate $\boldsymbol{\varrho}_k$ (and $\boldsymbol{\pi}_k$). It follows then that $\boldsymbol{\varrho}_k$ is the difference of the position vector of the kth particle from that of the center-of-mass of the other $(A-1)$ particles, i.e.,

$$\boldsymbol{\varrho}_k=\sum_{i\neq k}\frac{\mathbf{r}_i}{A-1}-\mathbf{r}_k, \tag{4}$$

and the conjugate relative momentum is

$$\boldsymbol{\pi}_k=\frac{1}{A}\sum_{i\neq k}\mathbf{p}_i-\frac{A-1}{A}\mathbf{p}_k. \tag{5}$$

In terms of these variables, h_k is given by

$$h_k=\frac{\boldsymbol{\pi}_k{}^2}{2\mu_k}+\sum_{i\neq k}V_{ik}(\boldsymbol{\varrho}_1,\cdots\boldsymbol{\varrho}_{A-1}), \tag{6}$$

where μ_k, the reduced mass for the two-body system, is related to the particle mass m by

$$\mu_k=[(A-1)/A]m. \tag{7}$$

Since we are singling out the kth particle from the remainder of the system, we adopt for this section a shorthand notation whereby brackets appearing around a variable signify all variables, but that one included in

brackets. Thus, $[k]$ signifies all particles but the kth, and $[\varrho_k]$ means all relative coordinates except ϱ_k. In this notation, Eq. (3) reads $H_A=H_{[k]}+h_k$.

We proceed with our derivation by writing the expectation value for H_A in the ground state

$$E_A{}^0=\langle\Psi_A{}^0|H_A|\Psi_A{}^0\rangle$$
$$=\langle\Psi_A{}^0|H_{[k]}|\Psi_A{}^0\rangle+\langle\Psi_A{}^0|h_k|\Psi_A{}^0\rangle. \quad (8)$$

The energy difference between the ground state of A nucleons and the ground state of $(A-1)$ or $[k]$ nucleons is just the separation energy $\mathcal{E}^0=E_A{}^0-E_{A-1}{}^0$, which is a definite experimentally measurable quantity,

$$\mathcal{E}^0=\langle\Psi_A{}^0|h_k|\Psi_A{}^0\rangle+\langle\Psi_A{}^0|H_{[k]}-E_{[k]}{}^0|\Psi_A{}^0\rangle. \quad (9)$$

Now Eq. (9) is in a form suitable for approximations. For a single-particle solution we require an equation for the kth particle only; therefore, one method of achieving this goal is to somehow eliminate the second term of Eq. (9), the rearrangement energy. We can do this by equating it to zero directly, canceling it with certain parts of the first term, or finding its numerical value to subtract from $\mathcal{E}^0$. The justification for such procedures will depend on the detailed structure of $\Psi_A{}^0$, particularly its expansion in eigenfunctions $\psi_{A-1}{}^n$ of the $(A-1)$ system,

$$\Psi_A{}^0=\sum_n u^n(\varrho_k)\psi^n[\varrho_k]. \quad (10)$$

The u^n are the expansion coefficients and the sum implies an integration over any continuum states. In fact, since the very light nuclei (H^3,He^3,He^4) apparently exhibit only one bound state, the ground state, Eq. (10), would take the form

$$\Psi_A{}^0=u^0(\varrho_k)\psi^0([\rho_k])+\int d\boldsymbol{\kappa}\, u(\varrho_k,\boldsymbol{\kappa})\psi([\varrho_k],\boldsymbol{\kappa}), \quad (11)$$

where $\psi([\varrho_k],\boldsymbol{\kappa})$ are the continuum states of momentum $\boldsymbol{\kappa}$ for the $(A-1)$ system.

For many nuclei, we know from deuteron pickup or stripping reactions that the first term of Eq. (11) is much more important than the second. In such cases, one could neglect the contribution of this term compared with $\mathcal{E}^0$. Certainly this is not the only justification one should make for eliminating the second term of Eq. (9). A more detailed analysis would consider the structure of the continuum states $\psi([\varrho_k],\boldsymbol{\kappa})$ in terms of nucleon-nucleon scattering data, etc. However, for our elementary study of the single-particle wave functions, we neglect the second term of Eq. (9), and assume,

$$\mathcal{E}^0=\langle\Psi_A{}^0|h_k|\Psi_A{}^0\rangle. \quad (12)$$

In order to obtain a two-body wave equation in ϱ_k from Eq. (12), it is now necessary to perform the integration over all coordinates other than ϱ_k, i.e., over all the $[\varrho_k]$ coordinates. There are several methods that yield such a result. One is to construct $\Psi_A{}^0$ from simpler systems, assuming ground-state contributions only:

$$\Psi_A{}^0(\varrho_1,\cdots,\varrho_{A-1})=u^0(\varrho_k)\psi_{A-1}{}^0([\varrho_k]),$$
$$\psi_{A-1}{}^0(\varrho_1,\cdots,\varrho_{A-2})=v^0(\varrho_j)\psi_{A-2}{}^0([\varrho_j]),$$
$$\cdots,$$

until one reaches $\psi_2{}^0$ for the deuteron, whose approximate solution is known. But the building up of such a product of ground-state functions, although feasible, does not appear to provide a good solution, since the symmetric behavior of all A particles (for $A\leqslant 4$) is lost. Therefore, we choose to use a method which, though it has other drawbacks, is symmetric between the A particles.

III. HARTREE-THOMAS APPROXIMATION

Normally, we would use an approximation such as the Hartree-Fock method to find the effective single-particle potential, $\mathcal{U}(\varrho_k)$, by averaging the actual two-body potentials over the wave functions of the $(A-1)$ or $[k]$ particles. This problem implies a self-consistency procedure, since the wave function used in the averaging to obtain $\mathcal{U}(\varrho_k)$ must be the same as the solution of the Schrödinger equation with that $\mathcal{U}(\varrho_k)$. In addition to this iterative procedure, we wish to solve the two-body Schrödinger equation by the Thomas approximation, the same method that obtains the deuteron Hulthén function. This approximation also entails an iteration process; thus, our method will be to combine these two iteration processes into one.

The Hartree approximation assumes that $\Psi_A{}^0$ can be written as a product of A single-particle wave functions. We note that we must now necessarily use the usual coordinate system, $\mathbf{r}_i$, with the conjugate momenta $\mathbf{p}_i$ instead of the relative coordinates defined in Sec. II. Difficulties will now be encountered due to the lack of specification of the center of mass, but we shall ignore these for the present. We also will neglect the particle correlations due to interactions in the $A-1$ system, the spin and isotopic spin dependence of the nucleon-nucleon forces, Coulomb effects, pairing forces, hard-core potentials, and the antisymmetrization of the wave function. Although these latter effects can be taken into account without too much difficulty, we will keep the following work simple and will not consider them here.

Using the Hartree approximation,

$$\Psi_A{}^0=\prod_{i=1}^{A} u_i(r_i), \quad (13)$$

and Eq. (5), we have for Eq. (12),

$$\mathcal{E}=\Big\langle \prod_{i=1}^{A} u_i(\mathbf{r}_i)\Big|\frac{A}{2(A-1)m}\Big[\frac{A-1}{A}\mathbf{p}_k-\frac{1}{A}\sum_{i\neq k}\mathbf{p}_i\Big]^2$$
$$+\sum_{i\neq k} V_{ik}(|\mathbf{r}_i-\mathbf{r}_k|)\Big|\prod_{i=1}^{A} u_i(\mathbf{r}_i)\Big\rangle.$$

[7] L. H. Thomas, Phys. Rev. 51, 202 (1937).

Next, we vary the u_i's to minimize the expectation value, $\mathcal{E}$, of the Hamiltonian above. All products $\mathbf{p}_i\cdot\mathbf{p}_j$ give zero when integrated over the single-particle wave functions. The variation of $u(\mathbf{r}_k)$ is straightforward and yields the expression $\langle u(r_k)|[(A-1)/A](p_k^2/2m)+\mathcal{U}(r_k)|u(r_k)\rangle$, which is a minimum only if $u(r_k)$ is the eigenfunction of $[(A-1)/A](p_k^2/2m)+\mathcal{U}(r_k)$ corresponding to the lowest eigenvalue of the separation energy. This lowest eigenvalue is just $\mathcal{E}^0$, the same separation energy that occurs in the relative-coordinate problem. Thus, we obtain the equation in $\mathbf{r}_k$,

$$\left(\frac{A-1}{A}\frac{p_k^2}{2m}+\mathcal{U}(\mathbf{r}_k)-\mathcal{E}^0\right)u(r_k)=0, \tag{14}$$

where

$$\mathcal{U}(\mathbf{r}_k)=\sum_{i\neq k}\int d\mathbf{r}_i\, u^*(\mathbf{r}_i)V_{ik}(|\mathbf{r}_i-\mathbf{r}_k|)u(\mathbf{r}_i). \tag{15}$$

Now, we set each potential, $V_{ik}(|\mathbf{r}_i-\mathbf{r}_k|)$, equal to a single expression,

$$V_{ik}(|\mathbf{r}_i-\mathbf{r}_k|)=V_0 f(|\mathbf{r}_i-\mathbf{r}_k|),$$

where V_0 is the strength of the potential and f gives its dependence on the variables. We define g as the average of this potential over $u_i(\mathbf{r}_i)$,

$$g(\mathbf{r}_k,u_k(\mathbf{r}_k))=\langle u_i(r_i)|f(|\mathbf{r}_i-\mathbf{r}_k|)|u_i(r_i)\rangle.$$

The content of the self-consistency in the Hartree approximation appears explicitly, because due to the averaging process, g is also a function of $u_k(\mathbf{r}_k)$. If we let $u_k(\mathbf{r}_k)=v_k(\mathbf{r}_k)/r_k$, the radial part of the wave equation is

$$\left(\frac{d^2}{dr^2}+\frac{A}{A-1}\frac{2m}{\hbar^2}\mathcal{E}_k^0\right)v_k(r_k)=\eta g(r_k,u_k(r_k))v_k(r_k) \tag{16}$$

with

$$\eta=\frac{A}{A-1}\frac{2m}{\hbar^2}V_0. \tag{17}$$

We are now ready to utilize the Thomas approximation to obtain a solution to Eq. (15).

As was stated in the introduction, our purpose is to carry out only one iteration to obtain an expression of the same order of approximation as the Hulthén deuteron function. We do this by choosing the same trial function as is used in the deuteron problem[8]:

$$u^{(0)}(r_k)=e^{-\beta r_k},$$

where β is our variational parameter. Using a Yukawa two-body potential, the effective potential $g^{(0)}$ for the zeroth iteration is

$$g^{(0)}(r_k)=\int d\mathbf{r}_i\, e^{-2\beta r_i}\frac{\beta^3}{\pi}\frac{e^{-|\mathbf{r}_k-\mathbf{r}_i|/r_0}}{|\mathbf{r}_k-\mathbf{r}_i|/r_0}.$$

The integration is straightforward, and yields

$$\begin{aligned}g^{(0)}(r_k)&=-2\beta^3\int_0^\infty dr_i\, r_i^2e^{-2\beta r_i}\int_1^{-1}\frac{e^{-|\mathbf{r}_k-\mathbf{r}_i|/r_0}}{|\mathbf{r}_k-\mathbf{r}_i|/r_0}d(\cos\theta)\\&=-\frac{2\beta^3}{r_k}\int_0^\infty dr_i\, r_ie^{-2\beta r_i}[e^{-|\mathbf{r}_k-\mathbf{r}_i|/r_0}-e^{-(r_k-r_i)/r_0}]\\&=\frac{e^{-r_k/r_0}-e^{-2\beta r_k/r_0}}{r_k/r_0}\frac{1}{[1-(1/2\beta)^2]^2}-\frac{\beta e^{-2\beta r_k/r_0}}{1-(1/2\beta)^2}.\end{aligned} \tag{18}$$

The value of β in Eq. (18) will be determined uniquely later by minimizing the potential strength η with respect to β. It is evident that the effective single-particle potential above exhibits a much smoother behavior for small separation than the Yukawa form. This is physically quite reasonable and is due to the averaging effect of the Hartree method. Yet for large separation, since $2\beta>1$, the effective potential falls off asymptotically like the Yukawa potential (excluding center-of-mass difficulties). Therefore, the form of Eq. (18) gives a reasonable behavior for $g(r_k)$.

With this potential function, the first iteration in the Thomas method yields in a straightforward manner

$$u^{(1)}=\frac{1}{[1-(1/2\beta)^2]^2}\frac{1}{[(1+\beta)^2-\alpha^2]}\times\left[\frac{e^{-\alpha x}-e^{-(\beta+1)x}}{x}-A\frac{e^{-\alpha x}-e^{-3\beta x}}{x}+Be^{-3\beta x}\right] \tag{19}$$

with $x=r/r_0$,

$$\alpha^2=[2mA/\hbar^2(A-1)]r_0^2\mathcal{E}^0, \tag{20}$$

and

$$\begin{aligned}A&=[15\beta^2-\alpha^2-3/2][(1+\beta)^2-\alpha^2]/[9\beta^2-\alpha^2]^2,\\B&=\beta[1-(1/2\beta)^2][(1+\beta)^2-\alpha^2]/9\beta^2-\alpha^2.\end{aligned} \tag{21}$$

A second iteration cannot be done analytically, since integrals over incomplete gamma functions are involved. However, a second iteraction and higher iterations could easily be carried out numerically. For our purposes, the desired analogy with the Hulthén deuteron function is complete with the first iteration, Eq. (19).

We then minimize the strength η with respect to β for this iteration. This has been carried out numerically for various values of α (corresponding to the separation energies in He^3 and He^4) using the method given elsewhere.[8] It is found that for each α, there is a minimum in η for only one value of β, and therefore the wave function is uniquely determined by the Thomas method.

IV. He^3, He^4 WAVE FUNCTIONS

Because our variation methods give a unique dependence of β on α, the wave function of Eqs. (19) and (21)

[8] For a clear presentation of the Thomas approximation as applied to the deuteron problem, see R. G. Sachs, *Nuclear Theory* (Addison Wesley Publishing Company, Cambridge, Massachusetts, 1953), p. 35.

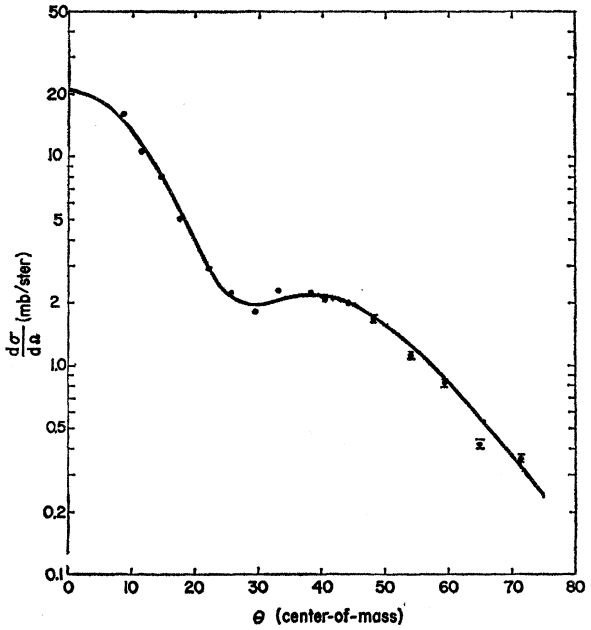

FIG. 1. Optical-model analysis of Selove and Teem's 95-MeV data for the reaction $p+He^4 \rightarrow d+He^3$. The analysis of reference 12 was used with the following parameters: $R_0=2.80$ F, $v_{Cp}=-(25+15i)$ MeV, $v_{Cd}=-(20+8i)$ MeV.

depends solely on α, or as seen from Eq. (20), on the separation energy $\mathcal{E}^0$ and the range of the potential, r_0.

There are several ways to check the validity of the foregoing procedures. One is to compare the value of $\eta_{\min}$ obtained from the variational principle with the phenomenological values of the potential strength required to fit nucleon-nucleon scattering data. We have chosen two such potentials for comparison: the singlet and triplet even Gammel-Thaler potential,[9] and the spin-averaged Pease-Feshbach potential.[5] The results are shown in Table I with the potential strength from the first iteration for three nuclei. It is apparent that the variation procedure requires a potential strength much smaller than that given from the scattering data. There is a simple reason for this. As was pointed out above, the Hartree method is an independent average over the wave function of each nucleon, and therefore it does not restrict the averaging process to account for a fixed center-of-mass of the A nucleons. The correct treatment of the center-of-mass problem does not appear simple; however, we have considered a classical treatment of it which obtains physically reasonable results. This method essentially places a delta function $\delta(\sum_i \mathbf{r}_i)$ in the Hartree average, and thereby a fixed position for the center-of-mass is insured. Such a procedure yields the same wave equation for the deuteron that is obtained by the usual method of separating the coordinates into relative and center-of-mass coordinates. The effect of the center-of-mass correction is to increase the required potential strength to a value closer to the phenomenological potentials. In addition, from the analogy with the deuteron problem, further iterations beyond the first one gives here will also give better agreement for the potential strength.

TABLE I. Comparison of phenomenological Gammel-Thaler and Pease-Feshbach potentials with results from Hartree-Thomas first iteration.

		Hartree-Thomas $r_0=0.813$ F		
	Gammel-Thaler $r_0=0.813$ F	He^4	He^3	H^2
V_0 (MeV)	100.7	32	42	57

		Hartree-Thomas $r_0=1.18$ F		
	Pease-Feshbach $r_0=1.18$ F	He^4	He^3	H^2
V_0 (MeV)	46	22	28	36

[9] J. C. Gammel and R. M. Thaler, Phys. Rev. **107**, 1337 (1957).

A second method of checking our results is to compare the Fourier transform of our configuration-space wavefunction (the "form factor") with experimental results on single-particle momentum distributions. The main source of such data is the pickup (or stripping) reaction. Depending on the approximations used in the analysis of such experiments, one may obtain either the momentum probability distribution of the picked up nucleon directly, or the momentum distribution, times a factor that represents the overlap of the initial-state and final-state nuclear wave functions. Unfortunately,

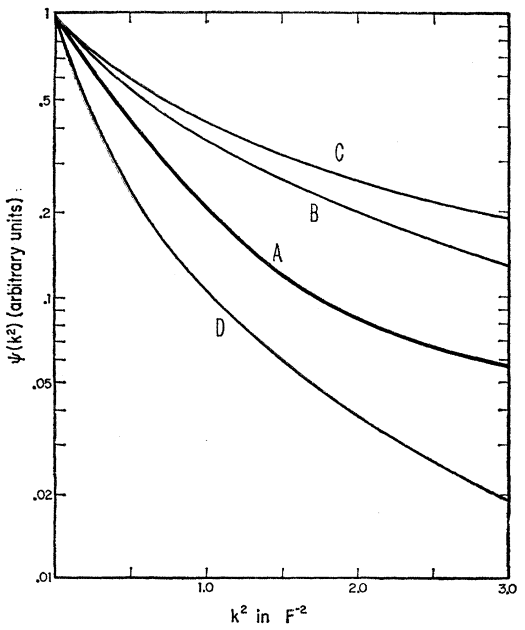

FIG. 2. Comparison of He^4 single-particle wave functions. Curve A is taken from the optical-model pickup analysis of the data in reference 11. Curve B gives the result of the Fourier transform of Eq. (19). Curve C is the asymptotic formula, and Curve D is that from reference 4.

the available data for He^3 are only at quite low energies[10] and are therefore rather inadequate for our purposes; however, our function reproduces these data as well as any of the "usual" wave functions. For He^4, the experimental situation is considerably better. Selove and Teem have measured the cross section for elastic $p-He^4$ scattering, and also the cross section for the (p,d) pickup reaction on He^4 at a laboratory energy of 95 MeV.[11] We have analyzed their data in a high-energy optical-model approximation,[12] and show our results in Fig. 1. The parameters used to describe the optical-model are inferred from a fit to the elastic data of protons on He^4. In Fig. 2, the momentum distribution we obtain from Selove's data is compared with the Fourier transform of Eq. (19) for the appropriate He^4 separation energy, the asymptotic function, and the Fröhlich wave function. The results show a reasonable agreement with experiment.

A third and rather weak test of the reliability of our wave function results from its use in the impulse approximation description of the reaction $p+d \rightarrow He^3+\pi^0$, already treated by Ruderman[2] and Bludman.[3] This will not produce a very definitive test of our wave functions and should not be considered as such too seriously, since the impulse approximation itself contains many uncertainties. However, the use of Eq. (19) instead of the usual symmetric He^3 function obtains a fit to the experimental results by requiring a deuteron hard-core radius of less than 0.4×10^{-13} cm rather than 0.7×10^{-13} cm as required by Bludman. Since the smaller radius better describes the known features of the hard core, the success of this fit can be used as a plausibility argument for the validity of the single-particle wave functions.

[10] A. I. Hamburger, Phys. Rev. **118**, 1271 (1960).
[11] W. Selove and J. M. Teem, Phys. Rev. **112**, 1658 (1958).
[12] The analysis is very similar to that by K. R. Greider, Phys. Rev. **114**, 786 (1959).

V. DISCUSSION

For simplicity, we have eliminated from our derivation all questions of spin dependence, nuclear two-body correlations, etc., in order to bring out the features of the Hartree-Thomas iteration procedure. Also we have not considered completely the question of the convergence of this iteration approximation. For the deuteron case, we know that the first iteration gives remarkably accurate results, and from preliminary numerical investigation of the higher iterations for our three- and four-body problems, we conclude that the convergence is also quite rapid. It should not be too difficult to test the convergence more thoroughly by numerical computer analysis. The one difficulty whose effect is not yet clearly understood is the same that plagues shell-model theory—the center-of-mass problem.[13,14] Unless harmonic oscillator wave functions are used, this will apparently remain a difficulty in any single-particle or independent-particle model.

Within these limitations, we feel that the simple Hartree-Thomas approach gives a rather good first approximation to the desired single-particle function. Most important, it allows at least a reasonable representation of the wave function in both the asymptotic and the small-separation regions simultaneously and, therefore, yields a more accurate normalization then heretofore possible. Furthermore, it appears to be a good framework into which the many refinements, neglected here, can be easily and naturally incorporated.

ACKNOWLEDGMENTS

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[13] S. Gartenhaus and C. Schwartz, Phys. Rev. **108**, 482 (1957).
[14] H. Lipkin, Phys. Rev. **110**, 1395 (1958).