Application of Dispersion Relations to the Photodisintegration of the Deuteron*

BUNJI SAKITA
Department of Physics, University of Rochester, Rochester, New York and Department of Physics, University of Wisconsin, Madison, Wisconsin

AND

CHARLES J. GOEBEL†‡
Department of Physics, University of Rochester, Rochester, New York

(Received April 4, 1961)

The calculation of the matrix element of the process $\gamma+d\rightarrow n+p$ by dispersion techniques is considered. There are twelve invariant amplitudes; the covariant form of the transition amplitude is related to the noncovariant (Pauli matrix) form, and we further relate this to partial wave amplitudes, keeping, however, only the dipole amplitudes. The Born terms of the dipole amplitudes are derived, and the dispersion relations for the dipole amplitudes are written down and solved in a low-energy approximation in which the n-p final-state rescattering is taken into account, but no other higher-order effects. In an Appendix these calculations are performed directly in the nonrelativistic limit to illustrate the essential simplicity of the technique. No light is shed on the well-known discrepancy between theory and experiment for the threshold $M1$ amplitude; the nearest (anomalous) singularities, at least, will have to be included in order for the dispersion calculation to be sufficiently accurate. But we remark that the form of the amplitude implies a correlation between the threshold value of the amplitude and its energy dependence, a correlation that would be interesting to check experimentally.

1. INTRODUCTION

MANY theoretical studies[1] of the photodisintegration of the deuteron have been made since Bethe and Peierls gave their quantum mechanical calculation of the electric dipole transition. These calculations, however, are based on nonrelativistic quantum mechanics, and an electromagnetic interaction which includes the phenomenological magnetic moment of the nucleon. We want to show here a different approach from the usual one, namely, through the application of relativistic dispersion relations to this process.

Chew *et al.*[2] applied relativistic dispersion relations at fixed momentum transfer to the photoproduction of a π meson from a nucleon. Our approach to our problem is similar, except that we use the dispersion relation in energy at a fixed *difference* of the momentum transfers (i.e., the difference between squares of the momentum transfers of the photon to the proton and to the neutron), in order to have all poles appear explicitly in the dispersion relation. The situation is that the momentum transfer between the photon and, say, the proton, is the momentum of the exchanged proton in the one-proton pole diagram [see Fig. 1(c)]; if the γ-p momentum transfer were held fixed, the one-proton pole would not appear explicitly in the dispersion relation. Our dispersion relation is of course equally as valid as the fixed momentum transfer dispersion relation if the amplitude is simultaneously analytic in both energy and momentum transfer, i.e., if the Mandelstam representation[3] for our process is valid.

Our treatment, as remarked above, is in the same spirit as Chew *et al.*[2] We limit ourselves to low energies

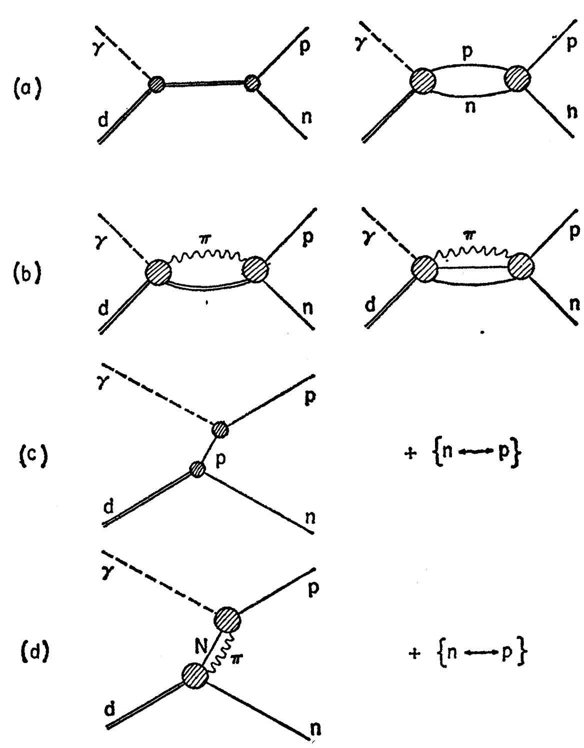

FIG. 1. Diagrams for the process $\gamma+d\rightarrow n+p$ which have the nearest singularities: (a) Deuteron pole and elastic cut; (b) inelastic cut; (c) one-nucleon cross poles; (d) $\pi-N$ crossed cuts.

* Supported in part by the U. S. Atomic Energy Commission.
† Alfred P. Sloan Research Fellow.
‡ Present address: University of Wisconsin, Madison, Wisconsin.
[1] For instance: H. A. Bethe and R. Peierls, Proc. Roy. Soc. (London) **A148**, 146 (1935); H. A. Bethe and C. Longmire, Phys. Rev. **77**, 647 (1950); N. Austern, *ibid.* **108**, 973 (1957); J. J. deSwart and R. E. Marshak, ibid. **110**, 272 (1958) and Physica **25**, 1001 (1959).
[2] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).
[3] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

and neglect all except the dipole amplitudes. Then, the use of the dispersion relation at just one momentum transfer suffices; the obvious choice is zero, i.e., photoproduction at 90°, for then, for example, there is no unphysical region for the final state.

In order to form and solve the dispersion relations, we use physical and invariant principles as much as possible. We give in Sec. 2 these kinematical considerations as a preliminary. The approximate expressions for magnetic dipole ($M1$) and electric dipole ($E1$) are given.

In Sec. 3 we give the dispersion relations which will be used in later sections. In this dispersion relation approach the Born terms can be calculated as the contributions of isolated poles. For these, we must know the various vertex parts, including the deuteron-neutron-proton vertex part; this latter is calculated in Appendix A from the deuteron pole of the n-p scattering.

The dispersion relations thus obtained contain dispersion integrals which as usual run over both positive and negative energies. The imaginary part of the amplitude for negative energies is related not only to the process of antiproton absorption on the deuteron but also to the structure of the deuteron, which enters through the anomalous singularities of the np-d vertex as a function of mass of one of the nucleons. However, we shall neglect these complications and remain only the isolated pole terms (Born terms). The imaginary amplitudes at positive energies larger than the threshold of the π-meson production are determined by inelastic processes as well as by elastic scattering in the final state. We shall neglect these inelastic processes in the final state since we apply our dispersion relations only to the low-energy photodisintegration of the deuteron.

The most important amplitudes of the photodisintegration at low energies are magnetic dipole and electric dipole. In Sec. 4 we derive the formulas for these amplitudes; these are functions of the n-p scattering phase shifts in the ${}^1S_0$ state for $M1$, and in the ${}^3P$ states for $E1$. We use the effective-range formula for the ${}^1S_0$ phase shift to estimate the value of the $M1$ amplitude.

In the final section we give a discussion and criticism of this calculation. The correction due to the effect of the singularities at negative energies is discussed.

2. KINEMATICAL CONSIDERATIONS

In this section we set forth some kinematical preliminaries which we need to write the transition amplitude for the photodisintegration of the deuteron in a form suitable for relativistic dispersion relations.

The transition matrix element, in general, is proportional to the polarization vectors of photon and deuteron, which are denoted by e_μ and U_μ, respectively. We treat the deuteron as a pseudovector particle in the framework of the quantum field theory, therefore, U_μ is a pseudovector while e_μ is a vector. Moreover, U_μ should satisfy the Lorentz condition

$$d\cdot U=0, \tag{2.1}$$

in order that the deuteron be in the triplet state in its rest system, where d is the four-vector momentum of deuteron. We also take e_μ to satisfy the usual Lorentz condition.

Let the four-vector momenta of the incident photon and deuteron be denoted by k and d, respectively. while those of the final nucleons are p and p'. Momentum-energy conservation,

$$k+d=p+p', \tag{2.2}$$

means that of these four momenta only three are independent. We choose to consider the combinations

$$q=\tfrac{1}{2}(p-p'),\quad Q=\tfrac{1}{2}(p+p'),\quad \text{and}\quad k \tag{2.3}$$

as the three independent four-vectors.

The mass shell restrictions, $p^2=p'^2=-m^2$, $k^2=0$, and $d^2=-M^2$, mean that only two independent scalars can be formed from other three independent vectors, where m and M are the mass of nucleon and deuteron, respectively. We choose

$$\nu=-Q\cdot k/m \tag{2.4}$$

and

$$\Delta=-q\cdot k/m. \tag{2.5}$$

This choice of the four-vector momenta q, Q and scalars ν, Δ is convenient because in the center-of-mass system,

$$q=(0,\mathbf{p}),\quad \nu=(E/m)\omega,$$
$$Q=(E,0),\quad \Delta=-\mathbf{p}\cdot\mathbf{k}/m,$$

where $\mathbf{p}$ and $\mathbf{k}$ are the momenta of one of the outgoing nucleons and of the incident photon, respectively, while E and ω are their energies.

The transition matrix R is defined by

$$S=1+iR, \tag{2.6}$$

where S is the scattering matrix. The transition matrix element for this process can be written

$$\begin{aligned}\langle p,r;p',r'|R|k,d\rangle &= (2\pi)^4\delta^4(p+p'-k-d)[m^2/(2\pi)^6 2\omega 2WEE']^{\frac{1}{2}}\\ &\quad\times \bar{u}_\alpha{}'(\mathbf{p})M_{\alpha\beta}{}^{\mu\nu}(p,p';k,d)\mathcal{C}_{\beta\beta'}\bar{u}_{\beta'}{}'(p')\\ &\quad\times e_\mu(\mathbf{k})U_\nu(\mathbf{d}),\end{aligned} \tag{2.7}$$

where ω and W are the energies of the initial photon and deuteron, while E and E' are the energies of the final nucleons, and $u_\alpha{}^r(p)$ is the usual Dirac spinor which satisfies the Dirac equation and is normalized

$$\bar{u}^r(\mathbf{p})u^{r'}(\mathbf{p})=\delta_{rr'}. \tag{2.8}$$

The matrix $\mathcal{C}$ is defined by

$$\mathcal{C}=i\tau_2 C, \tag{2.9}$$

where C is a charge conjugation Dirac matrix.

The most general transition matrix element (2.7) must be a function of Lorentz invariants. Three substantial further restrictions on the form of the matrix element result if we consider, in addition, the requirements: (1) the generalized Pauli principle for the two final nucleons which demands that

$$[M(p,p';k,d)\mathcal{C}]_{\alpha\beta}=-[M(p',p;k,d)\mathcal{C}]_{\beta\alpha}, \quad (2.10)$$

(2) gauge invariance, which demands that

$$k_\mu M^{\mu\nu}=0, \quad (2.11)$$

(3) invariance under space inversion.

We construct the independent convariant forms of $M^{\mu\nu}$ from the four-vectors Q, q, k, and the γ matrix so as to satisfy the above conditions. The Lorentz conditions for e_μ and U_μ (2.1) and the Dirac equation for the Dirac spinors, in addition to the above conditions, limit the number of the independent covariant forms of $M^{\mu\nu}$ to be twelve, which are listed in Table I. If we denote them by $I_{\alpha\beta}{}^{\mu\nu}(Q,q,k,\gamma)$, where i runs from 1 to 12, we can expand $M^{\mu\nu}$ in terms of the $I^{\mu\nu}(i)$ as follows:

$$M_{\alpha\beta}{}^{\mu\nu}=\sum_{i=1}^{12} I_{\alpha\beta}{}^{\mu\nu}(i)H_i(\nu,\Delta), \quad (2.12)$$

where the H_i are scalar functions of the two scalar variables ν and Δ.

The number of invariants, 12 in this case, is the number of transitions possible, starting from a given orbital momentum; equivalently, it is the number of transitions in a given total angular momentum state. For instance, consider a photon and a deuteron in a orbital angular momentum state l; this orbital momentum can be compounded with the spin of the photon to form two multipoles: $\lambda=l$ or $l+1$; each of these two can be compounded with the spin of the deuteron to form three total angular momentum states; and finally, each of these six states can decay into two np states, i.e., into ${}^1J_J$ and ${}^3J_J$ if $\mathcal{P}=(-)^J$ or into ${}^3(J-1)_J$ and ${}^3(J+1)_J$ if $\mathcal{P}=(-)^{J+1}$. The same result follows if we start with an orbital state of the neutron-proton system. Also, since the number of orbital states per total angular-momentum state must be equal to the number of total angular momentum states per orbital state, we can start in the middle and calculate 12 as the number of $\gamma-d$ states per value of the total angular momentum with a given parity $(=2\times3)$ times the number of np states $(=2)$.

Since we take the electromagnetic interaction only to first order, the transition matrix element can be split into two parts, of which one is a contribution from the isoscalar part of the electromagnetic interaction, and the other is from the isovector part; these lead, respectively, to the charge singlet state and to the charge triplet state of the final nucleons. The sign in the second column of Table I denotes the change of each invariant under the interchange of p and p'.

TABLE I. Relativistic invariant forms.

i	I_i	Sign
1	$(1/2m^2)[(e\cdot U)(Q\cdot k)-(e\cdot Q)(U\cdot k)]$	$-$
2	$(1/2m^2)[(e\cdot U)(q\cdot k)-(e\cdot q)(U\cdot k)]$	$+$
3	$(1/m^4)[(q\cdot k)(e\cdot Q)-(e\cdot q)(Q\cdot k)](U\cdot q)$	$-$
4	$(1/m^3)[(q\cdot k)(e\cdot Q)-(e\cdot q)(Q\cdot k)]i\mathbf{U}$	$-$
5	$(1/2m)[(e\cdot U)i\mathbf{k}-(U\cdot k)i\mathbf{e}]$	$+$
6	$(1/4m^3)[(Q\cdot k)i\mathbf{e}-(e\cdot Q)i\mathbf{k}](U\cdot k)$	$+$
7	$(1/4m^3)[(q\cdot k)i\mathbf{e}-(e\cdot q)i\mathbf{k}](U\cdot k)$	$-$
8	$(1/2m^3)[(Q\cdot k)i\mathbf{e}-(e\cdot Q)i\mathbf{k}](U\cdot q)$	$-$
9	$(1/2m^3)[(q\cdot k)i\mathbf{e}-(e\cdot q)i\mathbf{k}](U\cdot q)$	$+$
10	$(1/2m^2)[\frac{1}{2}(\mathbf{eU}-\mathbf{Ue})(Q\cdot k)-\frac{1}{2}(\mathbf{kU}-\mathbf{Uk})(Q\cdot e)]$	$+$
11	$(1/2m^2)[\frac{1}{2}(\mathbf{eU}-\mathbf{Ue})(q\cdot k)-\frac{1}{2}(\mathbf{kU}-\mathbf{Uk})(q\cdot e)]$	$-$
12	$(1/2m)\epsilon_{\mu\nu\rho\sigma}k_\rho\gamma_\sigma\gamma_5 e_\mu U_\nu$	$-$

Since Δ changes to $-\Delta$ upon interchanging p and p', the generalized Pauli principle demands that if the sign listed is $+$, the isoscalar part of $H^{(i)}$ must be a symmetric function with respect to Δ whereas the isovector part must be antisymmetric, while the opposite holds if the sign listed is $-$.

The standard invariance requirements and symmetry considerations have now been exhausted but one general principle still remains unexploited, the unitarity of the S matrix. It is well known that for the photodisintegration of the deuteron, unitarity implies that the phase of the production amplitude in a single partial wave is the scattering phase shift of the two-nucleon final state (this is known as Watson's theorem in case of the photoproduction of π meson). The above decomposition of 24 parts of H (12 each for isoscalar and isovector), is not, however, an angular-momentum eigenstate expansion. In order to apply unitarity, it is necessary to find the relation between the amplitudes $H^{(i)}$ and eigenamplitudes (partial waves).

As the first step we write the amplitude in terms of Pauli, instead of Dirac, matrices. The photodisintegration amplitude $\mathfrak{F}$ is defined so that the differential cross section for disintegration in the center-of-mass system is

$$d\sigma/d\Omega=\bar{\sum}(p/\omega)|\langle\mathfrak{F}\rangle|^2, \quad (2.13)$$

where the matrix element indicated is taken between the two final Pauli spinors and $\bar{\sum}$ is the abbreviation for the sum of spin and isospin in the final state and the average of polarizations for the initial state. It is possible to expand $\mathfrak{F}$ as follows:

$$\mathfrak{F}=\sum_{i=1}^{12}\lambda^{(i)}i\sigma_2[i\tau_2 f_i{}^S+i\tau_2\tau_3 f_i{}^V], \quad (2.14)$$

where the $\lambda^{(i)}$ are 2×2 matrices which depend on the direction of polarization of photon and deuteron and on the direction of initial and final momentum. The $\lambda^{(i)}$, like the $I(i)$, comprise twelve independent forms, which are listed in Table II.

It is possible to relate the $\lambda^{(i)}$ and $I^{\mu\nu}(i)$ by decompos-

ing the Dirac spinors to Pauli spinors. By a straightforward comparison, one then arrives at a set of linear equations relating the 12 amplitudes H_i to the 12 amplitudes f_i:

$$16\pi f_1=\frac{\omega p}{m^2}\left[-H_1+\frac{p}{E}zH_2+\frac{m}{E+m}\frac{p}{E}zH_5\right], \tag{2.15i}$$

$$16\pi f_2=-(\omega/m)H_5, \tag{2.15ii}$$

$$16\pi f_3=-\frac{\omega p}{m^2}\left[\frac{2E}{m}H_4-\frac{m}{E}H_{11}+\frac{m}{E}H_{12}\right], \tag{2.15iii}$$

$$16\pi f_4=\frac{p^3\omega}{m^4}\left[2H_3+\frac{2m}{E+m}H_4+\frac{m^2}{E(E+m)}H_{11}-\frac{m}{E+m}H_8\right], \tag{2.15iv}$$

$$16\pi f_5=(p^2\omega/m^3)H_9, \tag{2.15v}$$

$$16\pi f_6=-\frac{p\omega}{m^2}\frac{p}{E}\left[H_2+\frac{m}{E+m}H_5+\frac{E\omega}{2m(E+m)}H_6\right], \tag{2.15vi}$$

$$16\pi f_7=-(p\omega^2/2m^3)H_7, \tag{2.15vii}$$

$$16\pi f_8=\frac{\omega p}{m^2}\left[\frac{E}{m}\left(H_8-\frac{p}{E}zH_9\right)+\frac{m}{E}H_{12}\right], \tag{2.15viii}$$

$$16\pi f_9=\frac{\omega}{m}\left[H_5+\frac{E\omega}{2m^2}H_6-\frac{p\omega}{2m^2}zH_7\right], \tag{2.15ix}$$

$$16\pi f_{10}=-(p^2\omega/m^2E)H_{11}, \tag{2.15x}$$

$$16\pi f_{11}=-\frac{p\omega}{m^2}\left[H_{10}-\frac{p}{m}zH_{11}\right], \tag{2.15xi}$$

$$16\pi f_{12}=(\omega/E)H_{12}, \tag{2.15xii}$$

where $z=\mathbf{p}\cdot\mathbf{k}/|\mathbf{p}||\mathbf{k}|$.

We denote the electric and magnetic 2^λ-pole transition amplitudes by $E_\lambda({}^SL_j)$, $M_\lambda({}^SL_j)$, where j is total angular momentum and L and S are the orbital angular momentum and total spin of the final two nucleons. Keeping only the dipole transitions, we obtain the following formulas for the f_i:

$$f_1=-(3/\sqrt{2})zM_1({}^3D_1)+E_1({}^3P_0)-(1/\sqrt{3})E_1({}^3P_2)+(1/\sqrt{2})E_1({}^3F_2), \tag{2.16i}$$

$$f_2=M_1({}^3S_1)+(1/\sqrt{2})M_1({}^3D_1), \tag{2.16ii}$$

$$f_3=-(\sqrt{3}/2)E_1({}^3P_1)+(\sqrt{3}/2)E_1({}^3P_2)+(1/\sqrt{2})E_1({}^3F_2), \tag{2.16iii}$$

$$f_4=-(5/\sqrt{2})E_1({}^3F_2), \tag{2.16iv}$$

$$f_5=0, \tag{2.16v}$$

$$f_6=(3/\sqrt{2})M_1({}^3D_1), \tag{2.16vi}$$

$$f_7=0, \tag{2.16vii}$$

$$f_8=(\sqrt{3}/2)E_1({}^3P_1)+(\sqrt{3}/2)E_1({}^3P_2)+(1/\sqrt{2})E_1({}^3F_2), \tag{2.16viii}$$

$$f_9=-M_1({}^3P_0)-(1/\sqrt{2})M_1({}^3D_1), \tag{2.16ix}$$

$$f_{10}=3M_1({}^1D_2), \tag{2.16x}$$

$$f_{11}=E_1({}^1P_1)-3zM_1({}^1D_2), \tag{2.16xi}$$

$$f_{12}=2M_1({}^1D_2)+M_1({}^1S_0). \tag{2.16xii}$$

Continuing to neglect the higher multipole transitions, we finally obtain by the help of Eqs. (2.15) and (2.16) the following approximate expression for the magnetic and electric dipole amplitudes in terms of linear combination of functions H at $\Delta=0$:

$$M_1({}^1S_0)\simeq\frac{1}{16\pi}\frac{\omega}{E}\left[H_{12}+\frac{p^2}{m^2}H_{11}\right]_{\Delta=0}, \tag{2.17}$$

$$E_0\simeq\frac{1}{16\pi}\frac{\omega p}{m^2}\left[3H_1-2\frac{p^2}{m^2}H_3+2H_4-H_8-H_{11}\right]_{\Delta=0}, \tag{2.18a}$$

$$E_1\simeq\frac{1}{16\pi}\frac{\omega p}{m^2}\left[\frac{E}{m}(2H_4+H_8)+\frac{m}{E}(2H_{12}-H_{11})\right]_{\Delta=0}, \tag{2.18b}$$

$$E_2\simeq\frac{1}{16\pi}\frac{\omega p}{m^2}\left[-\frac{4}{5}\frac{p^2}{m^2}H_3+\frac{3E+2m}{5m}(2H_4-H_8)-\frac{2E+3m}{5E}H_{11}\right]_{\Delta=0}, \tag{2.18c}$$

$$E_f\simeq\frac{1}{16\pi}\frac{\omega p}{m^2}\left(-\frac{2}{5}\frac{p^2}{m^2}\right)\left[2H_3+\frac{2m}{E+m}H_4-\frac{m}{E+m}H_8+\frac{m^2}{E(E+m)}H_{11}\right]_{\Delta=0}, \tag{2.18d}$$

TABLE II. $\lambda^{(i)}$.

i	$\lambda^{(i)}$
1	$(\mathbf{e}\cdot\mathbf{U})(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})$
2	$(\mathbf{e}\cdot\mathbf{U})(\boldsymbol{\sigma}\cdot\hat{\mathbf{k}})$
3	$(\mathbf{e}\cdot\hat{\mathbf{p}})(\mathbf{U}\cdot\boldsymbol{\sigma})$
4	$(\mathbf{e}\cdot\hat{\mathbf{p}})(\mathbf{U}\cdot\hat{\mathbf{p}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})$
5	$(\mathbf{e}\cdot\hat{\mathbf{p}})(\mathbf{U}\cdot\hat{\mathbf{p}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{k}})$
6	$(\mathbf{e}\cdot\hat{\mathbf{p}})(\mathbf{U}\cdot\hat{\mathbf{p}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})$
7	$(\mathbf{e}\cdot\hat{\mathbf{p}})(\mathbf{U}\cdot\hat{\mathbf{k}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{k}})$
8	$(\mathbf{e}\cdot\boldsymbol{\sigma})(\mathbf{U}\cdot\hat{\mathbf{p}})$
9	$(\mathbf{e}\cdot\boldsymbol{\sigma})(\mathbf{U}\cdot\hat{\mathbf{k}})$
10	$i(\mathbf{e}\cdot\hat{\mathbf{p}})(\mathbf{U}\cdot\hat{\mathbf{p}}\times\hat{\mathbf{k}})$
11	$i(\mathbf{e}\cdot\mathbf{U}\times\hat{\mathbf{p}})$
12	$i(\mathbf{e}\cdot\mathbf{U}\times\hat{\mathbf{k}})$

where

$$E_0=-3E_1(^3P_0),\quad E_1=\sqrt{3}E_1(^3P_1),$$
$$E_2=-\sqrt{3}E_1(^2P_2),\quad\text{and}\quad E_f=\sqrt{2}E_1(^3F_2).\qquad(2.19)$$

Because of charge independence and parity conservation in nucleon-nucleon scattering there is no mixing between triplet and singlet states, but there is mixing between states of different orbital angular momentum, but the same total angular momentum and parity. We use the Blatt-Biedenharn[4] convention to specify the phase shift of each angular momentum state and the mixing. Then, by using unitarity, we arrive at

$$\mathrm{Im}\{E_1(^3P_2)\cos\epsilon+E_1(^3F_2)\sin\epsilon\}$$
$$=(\tan\delta_2)\mathrm{Re}\{E_1(^3P_2)\cos\epsilon+E_1(^3F_2)\sin\epsilon\},\qquad(2.20\text{a})$$

$$\mathrm{Im}\{-E_1(^3P_2)\sin\epsilon+E_1(^3F_2)\cos\epsilon\}$$
$$=(\tan\delta_f)\mathrm{Re}\{-E_1(^3P_2)\sin\epsilon+E_1(^3F_2)\cos\epsilon\},\qquad(2.20\text{b})$$

for $E_1(^3P_2)$ and $E_1(^3F_2)$, where δ_2 is a eigen phase shift relating to $^3P_2$ state, and δ_f to $^3F_2$, and ϵ is a mixing parameter.

3. DISPERSION RELATIONS

We shall assume that the scalar functions H_i with fixed Δ are analytic on the entire complex ν plane except for possible singularities on the real axis. We will look for these singularities by graphs.

First, the class of diagrams shown in Fig. 1(a) have singularities for positive ν. The first diagram, which has a deuteron in its intermediate state, has an isolated pole at $\nu=0$, while the others have continuous singularities (branch cut) for ν larger than the physical threshold of this reduction, namely $\nu=B$, the binding energy of the deuteron.

Second, the class of diagrams shown in Fig. 1(c) have singularities for negative $\nu-|\Delta|$. The first two diagrams, which have a one-nucleon intermediate state, give the isolated poles at $\nu=\pm\Delta$ plus continuous singularities which start from $\nu=\nu_0\pm\Delta$, where $\nu_0\approx-\mu(\mu+2\gamma)/m$. These last cuts result from the anomalous threshold of the np-d vertex, considered as a function of the square of the four-momentum of the intermediate nucleon. The second pair of diagrams of Fig. 1(d) have the same anomalous threshold.

In the present paper, we shall neglect all singularities except the poles and the np rescattering cut, starting at $\nu=B$, and will discuss the correction due to these anomalous singularities in the final section.

In determining the pole contributions, it is convenient to replace ν and Δ by α and β, defined by

$$\alpha=-(p\cdot k)/m,\quad \beta=-(p'\cdot k)/m;\qquad(3.1)$$

thus, they are related to ν and Δ by

$$\nu=(\alpha+\beta)/2,\quad \Delta=(\alpha-\beta)/2.\qquad(3.2)$$

The R matrix element which was given in the previous section Eq. (2.7) can be written as follows:

$$\mathcal{R}=e_\mu\left[\frac{m}{(2\pi)^6 2\omega E}\right]^{\frac{1}{2}}\bar{u}^r(\mathbf{p})i\int dx\langle p',r'|T(f(x)j_\mu(0))|d\rangle \times e^{-ip\cdot x}+P,\qquad(3.3)$$

where

$$\langle p,r;p',r'|R|k,d\rangle=(2\pi)^4\delta^4(p+p'-k-d)\mathcal{R}.$$

γ_μ is the electromagnetic current operator and $f(x)$ is the nucleon current defined by

$$(\gamma\partial+m)\psi=f.\qquad(3.4)$$

The second term of (3.3) is a contribution from a singularity of equal time in T product so that it is a function of $(p-k)^2$ and independent of β. The absorptive part of (3.3) is related to $H(\alpha+i\epsilon;\beta)-H(\alpha-i\epsilon;\beta)$ and can be calculated for its one-particle intermediate state as the absorptive part of the pole term. The singularity of H in the variable β can be obtained by the symmetric property of H by interchanging p and p'. Since the imaginary part of H on the real ν axis is obtained by

$$\mathrm{Im}H(\nu,\Delta)=(1/2i)[H(\alpha+i\epsilon;\beta+i\epsilon)-H(\alpha-i\epsilon;\beta+i\epsilon)+H(\alpha-i\epsilon;\beta+i\epsilon)-H(\alpha-i\epsilon;\beta-i\epsilon)],\qquad(3.5)$$

in the region of the isolated poles, we can determine the pole terms if we know the γ-p, γ-n, γ-d, and np-d vertex functions on the mass shell.

Isolating its pole terms, we write the function $H(\nu,\Delta)$ as

$$H(\nu,\Delta)=\frac{B_+(\Delta)}{\nu+\Delta}+\frac{B_-(\Delta)}{\nu-\Delta}+\frac{B_0(\Delta)}{\nu}+h(\nu,\Delta),\qquad(3.6)$$

where $h(\nu,\Delta)$ is analytic in the complex ν plane except for possible singularities on the real axis for $\nu>B$ and $\nu<\nu_0$. We obtain B_+, B_-, and B_0, which are listed in Table III, by the help of the following vertex functions

[4] J. Blatt and L. Biedenharn, Phys. Rev. 24, 258 (1952).

TABLE III. Born terms (charge singlet).[a]

1	$-\eta^+A+[\frac{1}{2}+(\Delta/2m)\mu^+]B$	$-(m\Delta Q/\sqrt{10})B$
2	$-\eta^+A+[\frac{1}{2}+(\Delta/2m)\mu^+]B$	$-\frac{1}{2}(M/m)\mu_D B$
3	$-(m/2\Delta)B$	$(m/\Delta)B$
4	$-(m/2\Delta)A-(\mu^+/2)[1+(\Delta/2m)]B$	$(m/\Delta)A$
5	$\mu^+[A+(\Delta/2m)[\gamma^2/m^2+\Delta/m)B]$	$-\frac{1}{2}(M/m)\mu_D A$
6	$\frac{1}{2}B$	$(2m^2Q/\sqrt{10})A$
7	$[\frac{1}{2}-(\Delta/2m)\mu^+]B$	
8	$-[\frac{1}{2}-(\Delta/2m)\mu^+]B$	
9	η^+B	
10	$-\eta^+A+(\Delta/2m)\mu^+B$	
11	$-(\eta^+A+\mu^+B)$	
12	$-\mu^+[A+(3\Delta/2m+\gamma^2/m^2)B]$	

[a] The $B_-^{(i)}$ can be obtained from $B_+^{(i)}$ by using the symmetry of Δ. The charge triplet terms can be obtained by replacing η^+ by η^- and putting all $B_0^{(i)}$ to zero.

which are derived in Appendix A.

(1) γ-d vertex

$$\langle d'|j_\mu|d\rangle=[(2\pi)^6 2W_d 2W_{d'}]^{-\frac{1}{2}}U_\rho^*(d')U_\nu(d)\times[(d+d')_\mu\delta_{\rho\nu}\alpha(k^2)+(k_\rho\delta_{\mu\nu}-k_\nu\delta_{\mu\rho})\beta(k^2)+(d+d')_\mu k_\rho k_\nu\gamma(k^2)] \quad (3.7)$$

and

$$\alpha(0)=e,\quad \beta(0)=e(M/m)\mu_D,\quad \gamma(0)=eQ/2(10)^{\frac{1}{2}}; \quad (3.8)$$

where μ_D is the deuteron magnetic moment in units of nucleon Bohr magnetons and Q is the deuteron quadrupole moment.

(2) γ-N vertex

$$\langle p'|\gamma_\mu|p\rangle=[m^2/(2\pi)^6EE']^{\frac{1}{2}}\bar{u}(\mathbf{p}')\times[i\gamma_\mu f(k^2)-i\sigma_{\mu\nu}k_\nu g(k^2)]u(\mathbf{p}) \quad (3.9)$$

and

$$f(0)=\tfrac{1}{2}(1+\tau_3)e,\quad g(0)=e(\eta^++\eta^-\tau_3)/2m,$$
$$\eta^+=\tfrac{1}{2}(\eta_p+\eta_n),\quad \eta^-=\tfrac{1}{2}(\eta_p-\eta_n), \quad (3.10)$$

where η_p and η_n are the proton and neutron anomalous magnetic moments in units of the nucleon Bohr magneton.

(3) np-d vertex

$$\left[\frac{m}{(2\pi)^3E}\right]^{\frac{1}{2}}\bar{u}(p)\langle p'|f|d\rangle =\left[\frac{m^2}{(2\pi)_9 2W_dEE'}\right]^{\frac{1}{2}}\bar{u}_\alpha(\mathbf{p})\left[\left\{-\gamma_\nu A((d-p')^2)+\frac{(p-p')_\nu}{2m}B((d-p')^2)\right\}\mathfrak{C}\right]_{\alpha\beta}\bar{u}_\beta(\mathbf{p}')U_\nu(\mathbf{d}), \quad (3.11)$$

where $\mathfrak{C}=i\tau_2C$ [see Eq. (2.9)] and

$$A(-m^2)\simeq-\Gamma(1+\alpha),\quad B(-m^2)\simeq3m^2\Gamma\alpha/\gamma^2;$$
$$\Gamma=\left[\frac{8\pi\gamma/m}{(1+2\alpha^2)(1-\rho\gamma)}\right]^{\frac{1}{2}},\quad \alpha=\left[\frac{1}{\sqrt{2}}\tan\epsilon\right]_{p^2\to-\gamma^2}. \quad (3.12)$$

The meaning of the notation in these expressions is explained in the Appendix.

Defining $\mathfrak{M}$ and $\mathcal{E}_i$ by

$$M_1(^1S_0)=(1/16\pi)(\omega/E)\mathfrak{M}, \quad (3.13)$$
$$E_i=(p/8\pi E)\mathcal{E}_i, \quad (3.14)$$

we can see from Eq. (2.17), (2.18), and (3.6) that $\mathfrak{M}$ and $\mathcal{E}_i$ are analytic functions with respect to ν, if we neglect the small ν dependence of E in Eq. (2.18). We use Cauchy's integral formula to construct the dispersion relations for these amplitudes and obtain a set of equations of the form

$$\mathfrak{M}(\nu)=\mathfrak{M}^{\mathrm{B}}+\frac{1}{\pi}\int_B^\infty\frac{\mathrm{Im}\mathfrak{M}(\nu')}{\nu'-\nu-i\epsilon}d\nu', \quad (3.15)$$

where we assumed that $\mathfrak{M}$ vanishes as ν goes to infinity in any direction in the complex ν plane and we have neglected the cut at negative energies.

Equation (3.15) is the basic equation which will be solved in the next section.

4. SOLUTION OF THE DISPERSION RELATIONS AT LOW ENERGY

In this section we consider the solution of the dispersion relations for the photodisintegration of the deuteron at low energy. Our criterion of "low energy" is that the momentum of the incident photon is much larger than the momentum of the outgoing nucleons; we consider only $M1$ and $E1$ transitions, leading to s- or p-wave n-p states.

$M1$ Transition

From Table IV, we can see that there are five magnetic dipole transitions; i.e., $M_1(^1S_0)$, $M_1(^3S_1)$, $M_1(^3D_2)$, $M_1(^3D_1)$, and $M_1(^1D_2)$. However, we do not consider the last three, D-wave amplitudes; they are proportional to p^2 so that we may neglect them near threshold. Moreover, the Born term of the amplitude $M_1(^3S_1)$ is proportional to

$$[\mu_p+\mu_n-(M/2m)\mu_D]\approx0.0233;$$

this is smaller by a factor of 1/500 than $\mu_p-\mu_n$, the corresponding factor for $M_1(^1S_0)$.[5] Therefore, we neglect $^3M_1(^3S_1)$ too, and will consider only the amplitude $M_1(^1S_0)$.

The approximate dispersion equation for this amplitude was given above by Eq. (3.15). The Born term $\mathfrak{M}^{\mathrm{B}}$ can be calculated from B_+ and B_- listed in Table III,

$$\mathfrak{M}^{\mathrm{B}}=e(\mu_p-\mu_n)\Gamma. \quad (4.1)$$

By the unitarity of the S matrix the phase of $M_1(^1S_0)$ for $\nu>B$ is the phase shift of n-p scattering in the $^1S_0$

[5] It might be noted that the amplitude $M_1(^3S_1)$ would vanish completely in the absence of the n-p tensor interaction; in the usual calculation the reason would be the orthogonality of the initial and final ^3S wavefunctions, and in the present calculation the equality of the magnetic moments of the initial and the final states.

TABLE IV. Dipole transition amplitudes.

Dipole amplitude	Final np state
$M_1(^1S_0)$	$^1S_0$
$E_1(^1P_0)$	$^1P_1$
$M_1(^1D_2)$	$^1D_2$
$M_1(^3S_1)$	$^3S_1$
$E_1(^3P_0)$	$^3P_0$
$E_1(^3P_1)$	$^3P_1$
$E_1(^3P_2)$	$^3P_2$
$M_1(^3D_1)$	$^3D_1$
$M_1(^3D_2)$	$^3D_2$
$E_1(^3P_2)$	$^3F_2$

state. This statement is exact only for $\nu \lesssim 140$ MeV; above this energy other channels, πd or $\pi 2N$, are then open so that this simple rule does not hold anymore. But as far as low energy is concerned, the contribution to the dispersion integral from the high-energy region is negligible, so we take the phase of $\mathfrak{M}$ to be everywhere δ, the $^1S_0$ n-p phase shift. Therefore, our dispersion equation becomes

$$\mathfrak{M}(\nu)=\frac{e(\mu_p-\mu_n)\Gamma}{\nu}+\frac{1}{\pi}\int_B^\infty d\nu'\frac{\tan\delta(\nu')\,\mathrm{Re}\mathfrak{M}(\nu')}{\nu'-\nu-i\epsilon}. \quad (4.2)$$

In order to have a unique solution of the singular integral equation (4.2) we impose the condition that the amplitude $\mathfrak{M}(\nu)$ is zero at infinite ν. The method of solving the integral equation is given by Omnes.[6] The solution is

$$\mathfrak{M}(p)=e^{i\delta(p)}e(\mu_p-\mu_n)\Gamma m\left[\frac{\cos\delta(p)}{p^2+\gamma^2}+e^{\tau(p)}\,\mathrm{P}\int_0^\infty dk^2\frac{\sin\delta(k)\,e^{-\tau(k)}}{(k^2+\gamma^2)(k^2-p^2)}\right], \quad (4.3)$$

$$\tau(p)=\frac{p^2+\gamma^2}{\pi}\mathrm{P}\int_0^\infty dk^2\frac{\delta(k)}{(k^2+\gamma^2)(k^2-p^2)}, \quad (4.4)$$

where p, the magnitude of outgoing nucleon momentum in the center-of-mass system, is related to ν by

$$m\nu=p^2+\gamma^2. \quad (4.5)$$

In order to evaluate the integrals appearing in (4.3) and (4.4), we use the effective range formula for the $^1S_0$ n-p scattering phase shift $\delta(p)$[7]:

$$p\cot\delta(p)=-a^{-1}+\tfrac{1}{2}rp^2. \quad (4.6)$$

Setting

$$\omega\mathfrak{M}(p)/mE=e(\mu_p-\mu_n)\Gamma e^{i\delta(p)}F(p), \quad (4.7)$$

we find $F(p)$ to be

$$F(p)=-\frac{\sin\delta(p)}{ap}\left[1-\gamma a-\tfrac{1}{2}p^2ra-\frac{(p^2+\gamma^2)\gamma^2ar}{\gamma^2[1+(1-2r/a)^{\frac{1}{2}}+\gamma r]}\right]. \quad (4.8)$$

In the zero-range approximation ($r=0$), $F(p)$ becomes simply

$$F(p)|_{r=0}=(1-\gamma a)[1+p^2a^2]^{-\frac{1}{2}}. \quad (4.9)$$

In terms of F, we have

$$\mathfrak{F}_{M1}=i(\mathbf{e}\cdot\mathbf{U}\times\mathbf{k})\sigma_{2}\tau_2\tau_3[e(\mu_p-\mu_n)/16\pi m]\times\Gamma F(p)e^{i\delta(p)} \quad (4.10)$$

for the magnetic dipole transition amplitude. Using this expression, we write the total cross section of the photodisintegration of the deuteron near threshold with the help of Eq. (2.13):

$$\sigma_{M1}=\left(\frac{2\pi}{3m^2}\right)\left(\frac{e^2}{4\pi}\right)(\mu_p-\mu_n)^2\frac{pa}{(1-\rho\gamma)(p^2+\gamma^2)}F^2(p). \quad (4.11)$$

In the zero-range approximation ($r=0$), this becomes, using (4.9),

$$\sigma_{M1}|_{r=0}=\left(\frac{2\pi}{3m^2}\right)\left(\frac{e^2}{4\pi}\right)(\mu_p-\mu_n)^2\frac{p\gamma(1-\gamma a)^2}{(p^2+\gamma^2)(1+p^2\gamma^2)}, \quad (4.12)$$

agreeing with the standard result, quoted in (zero-range approximation) Blatt and Weisskopf.[7]

Austern and Rost[7] have made a careful discussion of the numerical evaluation of the ordinary (i.e., nonrelativistic and nonmesonic) expression for the threshold M1 matrix element $\mathfrak{M}_{\text{nonmesonic}}$ where $\mathfrak{M}$ is a "reduced" matrix element some known constants having been factored out. According to them, the nonmesonic matrix element is given by the Bethe-Longmire (B-L) approximation plus three corrections which are each of the order of 1%, and almost cancel one another. Deciding on best values of the singlet scattering length a and the effective r (singlet) and ρ (triplet), they find

$$\mathfrak{M}_{\text{nonmesonic}}\approx\mathfrak{M}_{\text{B-L}}=(1-\gamma a)/\gamma^2a-(r+\rho)/4 \approx 5.10-1.10=4.00$$

in units of 10^{-13} cm. With this is to be compared our result in the elastic rescattering approximation [see Eq. (4.8) above, and Eq. (B22) below]

$$\mathfrak{M}^{\text{rs}}=\frac{1-\gamma a}{\gamma^2 a}-\frac{r}{1+(1-2r/a)^{\frac{1}{2}}+\gamma r}=4.12.$$

The experimental result is

$$\mathfrak{M}^{\text{exp.}}=4.18.$$

[6] R. Omnes, Nuovo cimento, **8**, 316 (1958).
[7] J. M. Blatt and V. F. Weisskopf *Theoretical Nuclear Physics* (J. Wiley & Sons, Inc., New York, 1952).
[8] N. Austern and E. Rost, Phys. Rev. **117**, 1506 (1960).

$E1$ Transitions

From Table IV, we can see there are five electric dipole transition amplitudes: $E_1(^1P_1)$, $E_1(^3P_0)$, $E_1(^3P_1)$, $E_1(^3P_2)$, and $E_1(^3F_2)$. However, we see that the Born term of $E_1(^1P_1)$ (the spin-flip transition) is smaller by a factor (ω/m) than the others, and, therefore, we shall neglect it.

Defining E_0, E_1, E_2, and E_f by (2.19), the electric dipole transition amplitude becomes

$$\mathfrak{F}_{E1}=\sigma_2\tau_2\tau_3[\mathbf{e}\cdot\mathbf{U}\boldsymbol{\sigma}\cdot\hat{p}f_1+\mathbf{e}\cdot\hat{p}\boldsymbol{\sigma}\cdot\mathbf{U}f_3+\cdot\mathbf{e}\hat{p}\mathbf{U}\cdot\hat{p}\boldsymbol{\sigma}\cdot\hat{p}f_4+\mathbf{e}\cdot\boldsymbol{\sigma}\mathbf{U}\cdot\hat{p}f_8], \quad (4.14)$$

where f_1, f_3, f_4, and f_8 are given by

$$\begin{aligned} f_1&=-\tfrac{1}{3}(E_0-E_2)+\tfrac{1}{2}E_f,\\ f_3&=-\tfrac{1}{2}(E_1+E_2-E_f),\\ f_4&=-\tfrac{5}{2}E_f,\\ f_8&=\tfrac{1}{2}(E_1-E_2+E_f). \end{aligned} \quad (4.15)$$

Using the differential cross-section formula, (2.14), we obtain for an unpolarized photon beam,

$$(d\sigma/d\Omega)_{E1}=(p/\omega)(a+b\sin^2\theta), \quad (4.16)$$

where a and b are given by

$$\begin{aligned} a&=(4/3)[|f_1|^2+|f_8|^2],\\ b&=\tfrac{2}{3}[3|f_3|^2+|f_4|^2+2\,\mathrm{Re}\{f_1f_3^*+f_1f_4^*+f_1f_8^*+f_3f_4^*+f_3f_8^*+f_4f_8^*\}]. \end{aligned} \quad (4.17)$$

Using (4.15), we obtain[9]

$$\begin{aligned} a&=(1/27)\{4|E_2-E_0+\tfrac{3}{2}E_f|^2+9|E_1-E_2+E_f|^2\},\\ b&=(2/9)\{|E_0|^2+3|E_1|^2+5|E_2|^2+(12/5)|E_f|^2\}-\tfrac{3}{2}a. \end{aligned} \quad (4.18)$$

A set of dispersion equations for $\mathcal{E}_i$ ($i=0$, 1, 2, f) which are defined in (3.14) are given by an equation similar to (3.15), that is,

$$\mathcal{E}_i=\mathcal{E}_i^{\mathrm{B}}+\frac{1}{\pi}\int_B^\infty d\nu'\frac{\mathrm{Im}\mathcal{E}_i(\nu')}{\nu'-\nu-i\epsilon}. \quad (4.19)$$

The Born terms $\mathcal{E}_i^{\mathrm{B}}$ can be calculated from Table III and we find

$$\begin{aligned} &\mathcal{E}_0^{\mathrm{B}}=(1-2\alpha)/m\nu,\quad \mathcal{E}_1^{\mathrm{B}}=(1+\alpha)/m\nu,\\ &\mathcal{E}_2^{\mathrm{B}}=(1-\alpha/5)/m\nu,\quad \mathcal{E}_f^{\mathrm{B}}=-6\alpha/5m\nu. \end{aligned} \quad (4.20)$$

As a first approximation we neglect the integral in (4.19) (Born approximation). Since the imaginary part of $\mathcal{E}$ is proportional to the corresponding final n-p scattering phase shift, this approximation is a neglect of the final state interaction. Using (4.18), we obtain

$$a=0,\quad b=2p^2(1+2\alpha^2)[e\Gamma/8\pi(\gamma^2+p^2)]^2. \quad (4.21)$$

If we neglect α in the above expression, in other words if we neglect the D-state probability of the deuteron, and use (4.16), we obtain

$$\sigma_{E1}|_{\mathrm{Born}}=\frac{8\pi}{3}\left(\frac{e^2}{4\pi}\right)\left(\frac{1}{\gamma^2}\right)\left(\frac{p\gamma}{p^2+\gamma^2}\right)^3\left(\frac{1}{1-\rho\gamma}\right), \quad (4.22)$$

agreeing with the standard result, quoted in Blatt and Weisskopf.[7]

Proceeding now to the calculation of the integral in (4.19), we observe that the unitarity of the S matrix implies

$$\begin{aligned} \mathrm{Im}\mathcal{E}_0&=\tan\delta_0\,\mathrm{Re}\mathcal{E}_0,\\ \mathrm{Im}\mathcal{E}_1&=\tan\delta_1\,\mathrm{Re}\mathcal{E}_1,\\ \mathrm{Im}\mathcal{E}_2&=\tan\delta_2\,\mathrm{Re}(\mathcal{E}_2-\sqrt{3}\sin\epsilon\,\mathcal{E}_f),\\ \mathrm{Im}\mathcal{E}_f&=-(1/\sqrt{3})\sin\epsilon\tan\delta_2\,\mathrm{Re}\mathcal{E}_2, \end{aligned} \quad (4.23)$$

where δ_0, δ_1, and δ_2 are the phase shifts of np scattering in the $^3P_0$, $^3P_1$, and $^3P_2$ states, respectively, and ϵ is the mixing parameter of $^3P_2$ and $^3F_2$. In the above expressions we have omitted terms containing the $^3F_2$ phase shift because it is very small compared with δ_2. Putting (4.23) into (4.19) we obtain uncoupled singular integral equations for $\mathcal{E}_0$ and $\mathcal{E}_1$ and the following coupled integral equations for $\mathcal{E}_2$ and $\mathcal{E}_f$:

$$\begin{aligned} \mathcal{E}_2(p)&=\mathcal{E}_2^{\mathrm{B}}+\frac{1}{\pi}\int_0^\infty dk^2\frac{\tan\delta_2\,\mathrm{Re}(\mathcal{E}_2-\sqrt{3}\sin\epsilon\,\mathcal{E}_f)}{k^2-p^2-i\epsilon},\\ \mathcal{E}_f(p)&=\mathcal{E}_f^{\mathrm{B}}-\frac{p^2}{\sqrt{3}\pi}\int_0^\infty\frac{dk^2}{k^2}\frac{\tan\delta_2\sin\epsilon\,\mathrm{Re}\mathcal{E}_2}{k^2(k^2-p^2-i\epsilon)}. \end{aligned} \quad (4.24)$$

Here we have made a subtraction at $p^2=0$ for $\mathcal{E}_f$, because $\mathcal{E}_p$ should be zero at $p^2=0$. Since the coupled term is proportional to $\sin\epsilon$, an iteration procedure is not bad if $\sin\epsilon$ is small in the entire energy region. We make the first iteration and obtain

$$\mathcal{E}_2(p)\simeq\mathcal{E}_2^{\mathrm{B}}+\frac{\sqrt{3}}{\pi}\int_0^\infty dk^2\frac{k^2\tan\delta_2\,\mathcal{E}_f^{\mathrm{B}}\sin\epsilon}{\gamma^2(k^2+\gamma^2)(k^2-p^2-i\epsilon)}+\frac{1}{\pi}\int_0^\infty dk^2\frac{\tan\delta_2\,\mathrm{Re}\mathcal{E}_2}{k^2-p^2-i\epsilon}, \quad (4.25)$$

$$\mathcal{E}_f(p)=\mathcal{E}_f^{\mathrm{B}}-\frac{p^2}{\sqrt{3}\pi}\int_0^\infty dk^2\frac{\tan\delta_2\sin\epsilon\,\mathcal{E}_2^{\mathrm{B}}.}{k^2(k^2-p^2-i\epsilon)}. \quad (4.25)$$

These integral equations can be solved in the standard way. The solution is given by

$$\mathcal{E}_i=\left(\frac{\zeta_i}{p^2+\gamma^2}+K_i\right)\exp[\tau_i(p)+i\bar{\delta}_i(p)],\quad i=0,1,2; \quad (4.26)$$

$$\mathcal{E}_f=-\frac{p^2}{\gamma^2}\left(\frac{\zeta_f}{p^2+\gamma^2}+K_f\right)\exp i\bar{\delta}_f(p)_d$$

[9] This agrees with the expression obtained by J. deSwart. J. J. deSwart, thesis, University of Rochester, 1959 (unpublished) and Physica **25**, 233 (1959).

where

$$\zeta_0 = 1-2\alpha$$

$$\zeta_1 = 1-\alpha$$

$$\zeta_2 = 1-(1/5)\alpha$$

$$\zeta_f = -(6/5)\alpha$$

$$\tau_i(p) = \frac{p^2+\gamma^2}{\pi} \mathrm{P}\int_0^\infty \frac{dk^2\delta_i(k)}{(k^2+\gamma^2)(k^2-p^2)},$$

$$K_0 = K_1 = 0,$$

$$K_2 = \frac{\sqrt{3}}{\pi} \mathrm{P}\int_0^\infty dk^2 \frac{k^2 \sin\delta_2\ \mathcal{E}_f{}^{\mathrm{B}} \sin\epsilon\ e^{-\tau_2(k)}}{\gamma^2(k^2-p^2)},$$

$$K_f = \frac{1}{\sqrt{3}\pi} \mathrm{P}\int_0^\infty dk^2 \frac{\gamma^2 \tan\delta_2 \sin\epsilon\ \mathcal{E}_2{}^{\mathrm{B}}}{k^2(k^2-p^2)},$$

$$\tilde{\delta}_0 = \delta_0, \quad \tilde{\delta}_1 = \delta_1,$$

$$\tilde{\delta}_2 = \delta_2 + \sqrt{3}[k^2\mathcal{E}_f{}^{\mathrm{B}} \sin\epsilon \sin\delta_2\ e^{-\tau_2(p)}]/\gamma^2|\mathcal{E}_2|,$$

$$\tilde{\delta}_f = -[p^2\mathcal{E}_2{}^{\mathrm{B}} \tan\delta_2 \sin\epsilon]/\sqrt{3}\gamma^2|\mathcal{E}_f|.$$

5. DISCUSSION

We have shown in detail for the process $\gamma+d\rightarrow n+p$ that the deuteron can be treated as an ordinary particle.[10] But, in fact, the deuteron exhibits itself as a not quite "ordinary particle" by the existence of anomalous singularities, corresponding to an inner structure which arises from the long range nature of the n-p potential. Although it is clear in principle how these singularities can be included,[11] in the present work we have neglected them, keeping only the elastic rescattering cut. Thus, our result for the $M1$ matrix element Eq. (4.8) depends on the ^1S scattering parameters, but not on the ^3S parameters except through the binding energy and the npd-coupling constant.[8]

It is due to the crudeness of our calculation in neglecting the anomalous cuts that we cannot be said to have shed any light on the experimental-theoretical discrepancy in the threshold value of the $M1$ amplitude, even though our result agrees with experiment better than does the ordinary calculation. But we can conclude that a dispersion calculation which does take the nearest anomalous cut into account will agree better with experiment than does the ordinary calculation only if the jump on the nearest anomalous cut is exceptionally small, that is, smaller than expected from

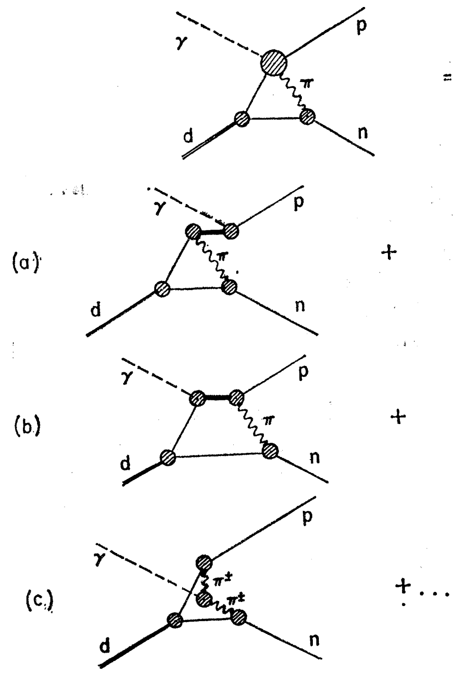

FIG. 2. A part of diagram (1d) which has the nearest anomalous singularity. (a,b,c): The Born contributions to this diagram, which may be described as representing the effects of (a) deuteron structure due to the long-range part of the n-p potential, (b) structure of the n-p final state due to the long-range part of the n-p potential, and (c) the long-range part of the meson current. The heavier lines are off the mass shell.

the ordinary calculation. This can result if, for instance, there is cancellation between the contributions of Figs. 2(a), (b), and 2(c). An experimental check on our results is in principle possible through the observation of the energy dependence of the $M1$ matrix element. The argument goes as follows.

For our process, $\gamma+d\rightarrow n+p$, we have, as usual, "threshold theorems," or better termed, "zero-energy theorems": At zero photon energy, the matrix elements are given exactly by Born approximation, hence in terms of charges and magnetic moments of the particles involved.[12] But to what extent is the photodisintegration cross section above threshold related to the zero energy, below threshold value? In the $E1$ case, where the final n-p state is P wave, the zero-energy pole term is dominant, and the matrix element extrapolates smoothly across threshold; but in the $M1$ case, where the final n-p state is S wave, we have a cusp as is usual with S waves, i.e., the slope of the matrix element is discontinuous at threshold. Thus, though zero energy is only 2.2 MeV below threshold, simple extrapolation of the $M1$ matrix element is nonsense. But the cusp depends only on the final-state (^1S) scattering, and, in fact, we have the following result: The matrix element can be written in the form

$$M = M^{\mathrm{rs}}[1+C_1k+C_2k^2+\cdots], \tag{5.1}$$

[10] In this connection, compare the field theory of composite particles developed by K. Nishijima, Phys. Rev. **111**, 995 (1958); R. Haag, *ibid.* **118**, 669 (1958); W. Zimmermann, Nuovo cimento **10**, 597 (1958).

[11] The inclusion of the anomalous singularities in an effective range potential approximation does not improve the agreement with experiment of the threshold $M1$ amplitude; nor, by the way, does it add to M^{rs} any terms of first order in the effective range and so does not lead when expanded to first order in the effective range to the Bethe-Longmire formula.

[12] B. Sakita, following paper [Phys. Rev. **127**, 1800 (1962)].

where M^{rs} is the solution of the dispersion relations taking into account only the n-p elastic rescattering, [i.e., Eqs. (4.7) and (4.8) or Eq. (B22)] and the series $1+C_1k+\cdots$ represents a function without the elastic cut, and so converges until the next branch point is reached, at $|k|=(\mu^2+2\mu\gamma)/m\approx 35$ MeV. It might be remarked again that the factor $1+C_1k+\cdots$ contains effects both contained in the ordinary static calculation [wavefunction structure, e.g., Figs. 2(a) and 2(b)], and not contained in it [mesonic "transition moments," e.g., Fig. 2(c)].

The thermal neutron capture experiment yields the following result:

$$[1+C_1k+C_2k^2+\cdots]_{k=2\cdot 2\text{ MeV}}\approx 1.015. \qquad (5.2)$$

which is not far from unity; that is, the "distant" effects on the matrix element are small. Now the ratio of the third to the second term of the series in Eq. (51) is k/E, where E is roughly the energy from which the "distant" contributions to the matrix element come, and is at least of the order of 40 MeV. Thus, Eq. (52) gives us an estimation of the energy dependence of the $M1$ matrix element,

$$M\approx(1+0.015\ k/B)M^{rs} \quad \text{for} \quad k\ll 40 \text{ MeV},$$

where B is the binding energy of the deuteron.

ACKNOWLEDGMENT

We wish to acknowledge the helpfulness of discussions with Professor R. E. Marshak, Dr. S. Okubo, and Dr. J. de Swart.

APPENDIX A

In this appendix we derive the $\gamma-d$ and np-d vertex functions.

γ-d Vertex

Lorentz convariance and gauge invariance determine the following unique form for the matrix element of the current operator between one deuteron states:

$$\langle d'|j_\mu|d\rangle=\frac{U_\rho{}^*(d')U_\nu(d)}{[2W_d2W_{d'}(2\pi)^6]^{\frac{1}{2}}}[(d+d')_\mu\delta_{\rho\nu}\alpha(k^2)$$
$$+(k_\rho\delta_{\mu\nu}-k_\nu\delta_{\mu\rho})\beta(k^2)$$
$$+(d+d')_\mu k_\rho k_\nu\gamma(k^2)], \qquad (A1)$$

where $k=d'-d$.

To determine the mass-shell values of the parameters α, β, γ, let us consider the time component of

$$\int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}\langle d'|j_\mu(\mathbf{x})|d\rangle=(2\pi)^3\delta(\mathbf{d}'-\mathbf{k}-\mathbf{d})\langle d'|j_\mu|d\rangle \quad (A2)$$

in the limit of $\mathbf{k}\to 0$. Then the left-hand side is just the expectation value of total charge operator which should be diagonal; therefore, we can immediately show by comparison with (A1) that $\alpha(0)$ is equal to the total charge of the deuteron:

$$\alpha(0)=e. \qquad (A3)$$

Now let us operate with $-i\nabla_k\times$ on the space component of Eq. (A2) and take the limit of $\mathbf{k}\to 0$, then we obtain

$$\left\langle d'\left|\int d^3x\ \tfrac{1}{2}\mathbf{x}\times\mathbf{j}\right|d\right\rangle.$$

The operator inside of the above matrix element is just the magnetic moment operator so that this can be written

$$m_D\langle d'|\mathbf{S}|d\rangle=-im_D\mathbf{u}'\times\mathbf{u},$$

in the rest system of deuteron, where $\mathbf{u}$ and $\mathbf{u}'$ are the polarization vectors of the deuteron, $\mathbf{S}$ is its spin operator, and m_D is the magnetic moment of deuteron. Thus, by comparison with (A1) we find

$$\beta(0)=2Mm_D=e(M/m)\mu_D, \qquad (A4)$$

where μ_D is the deuteron magnetic moment in units of nuclear Bohr magnetons. Let us now operate $\nabla_k{}^2-3\nabla_{k_3}{}^2$ to the time component of Eq. (A2) and take the limit of $\mathbf{k}\to 0$, then we obtain

$$\left\langle d'\left|\int d^3x\ (3x_3{}^2-\mathbf{x}^2)\rho(x)\right|d\right\rangle. \qquad (A5)$$

The operator inside is just the quadrupole moment operator so that the above quantity is equal to

$$-(2/5)^{\frac{1}{2}}eQ, \qquad (A6)$$

when initial and final deuteron polarization are in z direction in its rest system. In above expression Q denote the quadrupole moment of the deuteron. Apply the same operation to (A1), and we find

$$\gamma(0)=eQ/2(10)^{\frac{1}{2}}, \qquad (A7)$$

by comparing with (A6).

np-d Vertex

Lorentz invariance and the Lorentz condition for the deuteron polarization vector u_μ (2.1) means that

$$\left[\frac{m}{E(2\pi)^3}\right]^{\frac{1}{2}}\bar{u}(\mathbf{p})\langle p'|f|d\rangle$$
$$=\left[\frac{m^2}{2WEE'(2\pi)^9}\right]^{\frac{1}{2}}\bar{u}_\alpha(\mathbf{p})\bar{u}_\beta(\mathbf{p}')U_\nu(\mathbf{d}')$$
$$\times\left[i\gamma_\nu A((d-p')^2)+\frac{(p-p')_\nu}{m}B((d-p')^2)\right]\mathcal{C}, \qquad (A8)$$

where $\mathcal{C}=i\tau_2C$ and A and B are real constants on the mass shell.

In order to determine the mass shell values of A and B, we shall compute the pole terms of np scattering by using (A8) in perturbation theory and compare this with the extrapolation to the pole of the actual scattering. The R matrix element of the nucleon-nucleon scattering, can be written as follows:

$$\langle qq'|R|pp'\rangle=(2\pi)^4\delta^4(p+p'-q-q')\times[m^4/(2\pi)^{12}p_0p_0'q_0q_0']^{\frac{1}{2}}F(q,q';p,p'), \quad (A9)$$

where F is given by

$$F(q,q';p,p')=\left[\frac{p_0p_0'(2\pi)^6}{m^2}\right]^{\frac{1}{2}}i\int d^4x e^{-i(q+q')x/2}\bar{u}_\alpha(q')\times\left\langle q\left|T\left[f_\alpha\left(\frac{x}{2}\right),f_\beta\left(-\frac{x}{2}\right)\right]-i\delta(x_0)\left\{f_\alpha\left(\frac{x}{2}\right),\psi_\beta^*\left(-\frac{x}{2}\right)\right\}\right|p\right\rangle u_\beta(p). \quad (A10)$$

The second term contains an equal-time anticommutator, which gives a 3-dimensional δ function. Therefore, we can integrate over x immediately and obtain a quantity depending only on $(p-q)^2$. In the center-of-mass system this term depends only on momentum transfer and is independent of the total energy. In the center-of-mass system the first term of (A10) can be written

$$F(E,\mathbf{p},\mathbf{q})=\left[\frac{(2\pi)^6E^2}{m^2}\right]^{\frac{1}{2}}\bar{u}_\alpha(\mathbf{q})u_\beta(\mathbf{p})\left[\sum_n(2\pi)^3\delta(\mathbf{n}-\mathbf{0})\times\frac{\langle-\mathbf{q}|f_\alpha|\mathbf{n}\rangle\langle\mathbf{n}|\bar{f}_\beta|-\mathbf{p}\rangle}{n_0-2E-i\epsilon}\sum_n(2\pi)^3\delta(\mathbf{n}-\mathbf{p}+\mathbf{q})\times\frac{\langle-\mathbf{q}|\bar{f}_\beta|\mathbf{n}\rangle\langle\mathbf{n}|f_\alpha|-\mathbf{p}\rangle}{n_0}\right]. \quad (A11)$$

The first term of (A11) contains the deuteron as an intermediate state so that the function F has a pole at $2E=M$ (M is the deuteron mass). The other term does not contain a pole at $2E=M$, and, of course, the equal-time anticommutator term also does not. Therefore, we obtain

$$\lim_{E\to M/2}(M-2E)F(E)=\left[\frac{(2\pi)^6E^2}{m^2}\right]^{\frac{1}{2}}\bar{u}_\alpha(\mathbf{q})\times\sum_{\mathrm{pol\,d}}\langle-\mathbf{q}|f_\alpha|\mathbf{d}\rangle\langle\mathbf{d}|\bar{f}_\beta|-\mathbf{p}\rangle\delta(\mathbf{d}-\mathbf{0}).$$

In a general coordinate system this can be written

$$\lim_{(p+p')^2\to-M^2}[(p+p')+M^2]F(q,q';p,p')=\left[\frac{p_0p_0'(2\pi)^6}{m^2}\right]^{\frac{1}{2}}\bar{u}_\alpha(\mathbf{q}')\sum\langle\mathbf{q}|f_\alpha|\mathbf{d}\rangle\langle\mathbf{d}|\bar{f}_\beta|\mathbf{p}\rangle(2\pi)^3\times\delta(\mathbf{p}-\mathbf{p}'-\mathbf{d})u_\beta(\mathbf{p}). \quad (A12)$$

Using (A8), we obtain

$$\lim_{(p+p')^2\to-M^2}[(p+p')^2+M^2]\langle q,q'|R|p,p'\rangle=(2\pi)^4\delta(q+q'-p-p')\bar{u}_{\alpha'}(\mathbf{q})\bar{u}_{\beta'}(\mathbf{q}')u_\alpha(\mathbf{p})u_\beta(\mathbf{p}')\times\left[\frac{m^4}{E_qE_{q'}E_pE_{p'}(2\pi)^{12}}\right]^{\frac{1}{2}}\left[\left(i\gamma_\nu A+\frac{(q-q')_\nu}{2m}B\right)\mathcal{C}\right]_{\alpha'\beta'}\times\left[\mathcal{C}^{-1}\left(i\gamma_\nu A-\frac{(p-p')_\nu}{2m}B\right)\right]_{\alpha\beta}. \quad (A13)$$

The R matrix element of n-p scattering is related to the scattering amplitude T in the center-of-mass system by the following equation:

$$\langle qq'|R|pp'\rangle=(2\pi)^4\delta^4(p+p'-q-q')\times\left[\frac{m^4}{(2\pi)^{12}E_qE_{q'}E_pE_{p'}}\right]^{\frac{1}{2}}\frac{4\pi E}{m^2}T, \quad (A14)$$

where T is defined such that

$$d\sigma/d\Omega=\tfrac{1}{4}\,\mathrm{Tr}\{TT^*\}.$$

In order to compare the constants A and B with the n-p scattering amplitude by Eq. (A9), we decompose the Dirac spinors on the right-hand side of this equation into Pauli spinors. Straightforward calculation gives

$$\bar{u}(\mathbf{p})[i\gamma A+(\mathbf{q}/m)B]\mathcal{C}\bar{u}(-\mathbf{p}')=\{\boldsymbol{\sigma}a+[3\hat{q}(\boldsymbol{\sigma}\cdot\hat{q})-\boldsymbol{\sigma}]b/\sqrt{2}\}\sigma_2\tau_2/2, \quad (A15)$$

where

$$a\approx-2A+\tfrac{2}{3}(p/m^2)B,\qquad b\approx(\sqrt{2}/3)(p/m)^2(A+2B); \quad (A16)$$

here $\hat{p}=\mathbf{p}/|\mathbf{p}|$ and $\hat{q}=\mathbf{q}/|\mathbf{q}|$. If we use the expression (A15) in the right-hand side of Eq. (A13), each σ connects the two final spinors and two initial spinors separately, whereas usually, in nucleon-nucleon scattering, one takes the matrix element between initial and final spinors for each of the two particles. So let us reorder the legs of the Pauli spinors from $(\;)_{\alpha\alpha'}$, $(\;)_{\beta\beta'}$, to $(\;)_{\alpha\beta}(\;)_{\alpha'\beta'}$. The right-hand side of Eq. (A13) becomes

$$\mathcal{P}_S[a^2P_S-ab(P_{SD}+P_{DS})+b^2P_D], \quad (A17)$$

where $\mathcal{P}_S$, P_S, P_{SD}, P_{DS}, and P_D are the projection operators in the charge singlet state for ${}^3S\to{}^3S$, ${}^3S\to{}^3D$, ${}^3D\to{}^3S$, and ${}^3D\to{}^3D$, respectively. On the

other hand, T is given by

$$T=\mathcal{P}_S[\cos^2\epsilon\, P_S+\cos\epsilon\,\sin\epsilon\,(P_{SD}+P_{DS})+\sin^2\epsilon\, P_D](e^{2i\delta}-1)/2ip, \quad (A18)$$

where δ is a Blatt-Biedenharn ${}^3S_1$ eigenphase shift and ϵ is their mixing parameter. In the above equation we omitted the other eigenamplitudes, but these give no contribution in Eq. (A13) because in the limit of $(p+p')^2\to M^2$ these have no pole while the retained term in (A18) does have a pole.

Let us substitute Eqs. (A17) and (A18) to (A13) and estimate the values of a and b [related to A and B by Eq. (A16)] from the knowledge of low-energy n-p scattering phase shifts and mixing parameter. To do this we have to extrapolate the phase shift from positive energy to the (negative) binding energy of deuteron. We assume that such a procedure makes sense. This extrapolation can be done by the help of the effective-range formula for the n-p scattering phase shift:

$$p\cot\delta=-\gamma+\tfrac{1}{2}\rho(p^2+\gamma^2), \quad (A19)$$

where $\gamma=(mB)^{\frac{1}{2}}$ (B: binding energy of deuteron) and ρ is the triplet n-p effective range. If we put (A19) into (A18), then we find that this has a pole at $p=i\gamma$. This is nothing but the pole of the bound state in S-matrix theory.

Finally, we obtain

$$\begin{aligned}A(-m^2)&\approx-\Gamma(1+\alpha),\\ B(-m^2)&\approx3(m^2/\gamma^2)\alpha\Gamma,\end{aligned} \quad (A20)$$

where

$$\begin{aligned}\Gamma&=\left[\frac{8\pi\gamma/m}{(1+2\alpha^2)(1-\rho\gamma)}\right]^{\frac{1}{2}},\\ \alpha&=\lim_{p^2\to-\gamma^2}2^{-\frac{1}{2}}\tan\epsilon.\end{aligned} \quad (A21)$$

APPENDIX B. CALCULATION IN A SIMPLE FORM

We have actually applied our relativistic formulas only in the nonrelativistic limit. It may be illuminating to see directly a nonrelativistic derivation of some of our results; it will be seen that the calculations are really simple, as simple as the ordinary quantum mechanical calculation.

We need the following vertex factors:

$$\gamma p\xrightarrow{E1}p:\quad e\frac{\mathbf{p}\cdot\boldsymbol{\varepsilon}}{m(2k)^{\frac{1}{2}}}, \quad (B1)$$

where $\mathbf{p}$ is the momentum of the proton, and $\boldsymbol{\varepsilon}$ is the polarization unit vector of the photon.

$$\gamma d\xrightarrow{E1}d:\quad e\frac{\mathbf{d}\cdot\boldsymbol{\varepsilon}}{M(2k)^{\frac{1}{2}}}, \quad (B2)$$

where $\mathbf{d}$ is the momentum of the deuteron.

$$\gamma p\xrightarrow{M_1}\psi:\quad -\frac{e}{2m}\mu_p\frac{i\boldsymbol{\sigma}\cdot\mathbf{k}\times\boldsymbol{\varepsilon}}{(2k)^{\frac{1}{2}}}, \quad (B3)$$

where $\mathbf{k}$ is the momentum of the photon, and $\mu_p\approx2.8$ is the magnetic moment of the proton; the same formula holds of course for the neutron.

$$\gamma d\xrightarrow{M_1}d:\quad -\frac{e}{2m}\mu_d(-i\boldsymbol{\varepsilon}_f\times\boldsymbol{\varepsilon}_i)\cdot\left[\frac{i\mathbf{k}\times\boldsymbol{\varepsilon}}{(2k)^{\frac{1}{2}}}\right], \quad (B4)$$

where $\boldsymbol{\varepsilon}_{i\cdot f\cdot}$ is the initial (final) polarization unit pseudovector of the deuteron.

$$d\to np:\quad G\chi_p^*(\sigma\cdot\boldsymbol{\varepsilon}/\sqrt{2})\chi_n{}^C,$$

where

$$\chi_n{}^C=i\sigma_2\chi_n^*. \quad (B5)$$

As an example of the meaning of this formula, consider the case that the deuteron has spin up; then

$$\boldsymbol{\varepsilon}=-(\hat{x}+i\hat{y})/\sqrt{2}$$

and so

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}+i\sigma_2)/\sqrt{2}=-i(\sigma_1+i\sigma_2)\sigma_2/2=(1+\sigma_3)/2,$$

which is unity if the spins of proton and neutron are up, and zero otherwise.

The npd- coupling constant G is most directly determined by comparing the pole term of ${}^3S$ n-p scattering as calculated in two ways, firstly by regarding the deuteron as an intermediate particle, and secondly by evaluating the scattering amplitude at the pole by "effective range" techniques. The first calculation yields

$$f^{\text{pole}}=\frac{im^2G^2}{8\pi\gamma}\frac{1}{p-i\gamma}. \quad (B6)$$

The second calculation evaluates the first factor in

$$f^{\text{pole}}=\left\{\frac{1}{\partial_p[p\cot\delta-ip]}\right\}_{p=i\gamma}\frac{1}{p-i\gamma}, \quad (B7)$$

by the use of the lemma

$$p_2\cot\delta(p_2)-p_1\cot\delta(p_1)=(p_i^2-p_1^2)\int_0^\infty dr\,[\phi_2{}^0\phi_1{}^0-\phi_2\phi_1], \quad (B8)$$

where $\phi_p{}^0(r)=\sin(pr+\delta)/\sin\delta$ and $\phi_p(r)$ is the exact radial wavefunction, normalized so as to equal $\phi_p{}^0$ outside the range of the potential. When $p=i\gamma$ then $\delta=-i\infty$ and so $\phi_{i\gamma}{}^0=e^{-\gamma r}$ (naturally), while $\phi_{i\gamma}=u/\mathfrak{N}$ where u is the normalized bound state radial wave function, $\mathfrak{N}$ being its asymptotic amplitude, i.e., $u=\mathfrak{N}e^{-\gamma r}$ outside the potential (for a zero-range potential, $\mathfrak{N}=(2\gamma)^{\frac{1}{2}}$. Thus, (B8) gives us

$$\{\partial_p p\cot\delta\}_{p=i\gamma}=2i\gamma[(1/2\gamma)-(1/\mathfrak{N}^2)], \quad (B9)$$

and so, according to (B7),

$$f^{\rm pole}=(i\mathfrak{N}^2/2\gamma)[1/(p-i\gamma)]. \tag{B10}$$

Comparing the results (B6) and (B10), we conclude that

$$G=-(4\pi)^{\frac{1}{2}}\mathfrak{N}/m, \tag{B11}$$

where the sign is fixed by considering the calculation of the amplitude of the np component of the deuteron by perturbation theory, which also immediately makes clear that the coupling constant is proportional to the asymptotic bound-state amplitude.

In the effective-range approximation, i.e.,

$$F^{\rm er}\approx[-\gamma-ip+(\gamma^2+p^2)\gamma/2]^{-1}$$

and so

$$f^{\rm pole,er}\approx[i/(1-\gamma r)][1/(p-i\gamma)], \tag{B12}$$

we have immediately, by comparison with (B6),

$$G^{\rm er}\approx(1/m)[8\pi\gamma/(1-\gamma r)]^{\frac{1}{2}}. \tag{B13}$$

We can now proceed to calculate the Born approximation photodisintegration matrix elements. For $E1$, the term in which the photon is absorbed by the deuteron contributes nothing since $\mathbf{k}\cdot\boldsymbol{\varepsilon}=0$; so the only term is that in which the proton absorbes the photon:

$$M_{E1}{}^{\rm B}=e\frac{\mathbf{p}\cdot\boldsymbol{\varepsilon}}{m(2k)^{\frac{1}{2}}}\frac{1}{(-k)}G\chi_p{}^*\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{E}}}{\sqrt{2}}\chi_n{}^C. \tag{B14}$$

For $M1$ we have all three terms contributing, one of which requires the use of

$$\textstyle\sum_{n'}\chi_n{}^*\boldsymbol{\sigma}_{\rm I}\chi_{n'}\cdot\chi_p{}^*\boldsymbol{\sigma}_{\rm II}\chi_{n'}{}^C=-\chi_p{}^*\boldsymbol{\sigma}_{\rm II}\cdot\boldsymbol{\sigma}_{\rm I}\chi_n{}^C, \tag{B16}$$

giving

$$M_{M1}{}^{\rm B}=\frac{ieG}{2mk\sqrt{2}(2k)^{\frac{1}{2}}}\chi_p{}^*[(\mu_p-\mu_n)\mathbf{k}\times\boldsymbol{\varepsilon}\cdot\boldsymbol{\mathcal{E}}$$
$$+i(\mu_p+\mu_n-\mu_d)\boldsymbol{\sigma}\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})\times\boldsymbol{\mathcal{E}}]\chi_n{}^C$$
$$\approx\frac{ie(\mu_p-\mu_n)}{2m(2k)^{\frac{1}{2}}}\frac{\chi_p{}^*\chi_n{}^C}{\sqrt{2}}(\mathbf{k}\times\boldsymbol{\varepsilon}\cdot\boldsymbol{\mathcal{E}}), \tag{B17}$$

since $\mu_p+\mu_n-\mu_d\approx0$. Our results (B14) and (B17) are, of course, identical with the result of the usual calculations of these matrix elements.

With the assumption that each matrix element is an analytic function of the energy except for Born poles and for the cut on the positive real axis due to final-state rescattering, we can write for each a simple dispersion representation, of the form of (3.16). The worst approximation here is the neglect of the nearest out which is the anomalous cut starting at

$$k=(\mu^2+2\mu\gamma)/m$$

due to the diagrams of Fig. 2. Actually, in this approximation it is unnecessary to write down the dispersion representation, for we can work directly with the analytic properties of the matrix element.

For definiteness, let us work with the $M1$ matrix element, which we shall call $M^{\rm er}$ in our rescattering approximation. It has the following properties:

(a) $M^{\rm rs}=M^{\rm B}$ at $k=0$ (threshold theorem), because all $M1$ matrix elements contain a factor k, and only the Born term has an energy denominator which vanishes at $k=0$.

(b) $M^{\rm rs}(k)$ is analytic in the complex k plane [equivalently the p^2 plane, since $mk=p^2+\gamma^2$] except for the "elastic cut" $k>B=\gamma^2/m$, where its phase is the scattering phase shift of the ^1S n-p state. It follows that

$$M^{\rm rs}(k)=[D(0)/D(p)]M^{\rm B}(k), \tag{B18}$$

where $D(p)$ is the so-called denominator function of the n-p scattering amplitude; it is analytic in the energy plane except that on the elastic cut it has the phase $-\delta$. Omnes has given the formula for $D(k)$:

$$D(p)=\exp\left\{-\frac{1}{\pi}\int_0^\infty\frac{dp'^2}{p'^2-p^2-i\epsilon}\delta(p')\right\}, \tag{B19}$$

which obviously satisfies the requirements.

If we make the further approximation of using the effective range approximation for the n-p scattering amplitude, then the determination of $D(k)$ and hence $M^{\rm rs}$ becomes trivial algebra. In the effective-range form, $p\cot\delta$ is a polynomial in p^2, so that the negative-energy singularities of the scattering amplitude f are isolated poles, and so to find $D(p)$, which has the same phase as f but not the left-hand singularities, the following procedure suffices: For each pole of f on the physical energy sheet, i.e., the upper half momentum plane, say, at $p=i\kappa$, replace in f the factor $(p-i\kappa)^{-1}$ by $p+i\kappa$, thus, removing the singularity but not changing the phase, for real p.

Thus, if $p\cot\delta=-a^{-1}+\frac{1}{2}rp^2$, we have

$$f=[-a^{-1}-ip+\tfrac{1}{2}rp^2]^{-1}=C/(p+i\alpha_-)(p-i\alpha_+), \tag{B20}$$

where $C=2/r$ and $\alpha_\pm=[(1-2r/a)^{\frac{1}{2}}\pm1]/r$. For the case of ^1S scattering we have $a<0$, and so from (B20) we have

$$1/D=C(p+i\alpha_+)/(p+i\alpha_-) \tag{B21}$$

and hence,

$$M^{\rm rs}=\frac{\gamma+\alpha_-}{\gamma+\alpha_+}\frac{p+i\alpha_+}{p+i\alpha_-}\frac{e^{i\delta}\sin\delta}{p}\frac{\gamma+\alpha_-}{\gamma+\alpha_+}\frac{p^2+\alpha_+{}^2}{C}M^{\rm B}$$
$$\equiv e^{i\delta}F(p)M^{\rm B}, \tag{B22}$$

where this is the $F(p)$ of (4.8) of the text.