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Ground State of the Charged Bose Gas*

M. GIRARDEAU
The Enrico Fermi Institute for Nuclear Studies, The University of Chicago, Chicago, Illinois
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The ground-state energy of the charged Bose gas is calculated by the pair-correlation variational method of Girardeau and Arnowitt. The method is exact in the high-density limit ($r_s \ll 1$) and provides a variational extrapolation to intermediate densities. The leading terms of the high-density expansion, obtained by iteration of the variational integral equation, are $u_0 = -0.804 r_s^{-3/4} - (1/8)\ln r_s + O(1)$, where u_0 is the ground-state energy per particle in Rydbergs and r_s is the ratio of the mean interparticle spacing to the Bohr radius. The first term was obtained previously by Foldy, but the logarithmic term is new; it is related to screening of the long-range correlations at a distance $r_0 \sim r_s^{-1/2}\rho^{-1/3}$, in analogy with the logarithmic term in the correlation energy of the electron gas. Results of numerical solutions for the intermediate-density region are presented, ranging up to $r_s = 10$. On the basis of a comparison with the energy calculated from the known low-density expansion, it is estimated that the transition into Wigner's electron crystal (here a boson crystal) should occur at $r_s \sim 5$.

1. INTRODUCTION

THE high-density charged Bose gas has recently been studied by Foldy[1] using the canonical transformation method of Bogoliubov[2]; the reader is referred to Foldy's paper for a discussion of background and motivation. The Bogoliubov method is a weak-coupling treatment which takes advantage of the nearly complete Bose condensation into the state $\mathbf{k}=0$. That it is applicable to the charged Bose gas at high density follows from the fact that the only dimensionless number which can be constructed from the parameters available is $r_s \sim e^2\rho^{-1/3}$ in units with $\hbar = m = 1$; thus, weak coupling (small e^2) for given density is equivalent to high density (large ρ) for given e^2. This argument is of course identical with that used in the electron-gas problem, e.g., in the paper by Gell-Mann and Brueckner.[3]

As a method of extending Foldy's high-density results to intermediate density, the pair-correlation variational method of Girardeau and Arnowitt[4] immediately suggests itself; that method was, in fact, derived as a variational extension of the Bogoliubov method to intermediate coupling strengths. The present paper consists of a straightforward application of the general formulas derived in I to the special case of the Coulomb interaction. However, the results obtained in the case of high density bear no resemblance to those previously given[5] for the case of weak repulsive interactions; the latter results were limited to finite-range interactions, whereas the infinite range of the Coulomb interaction is decisive in the present problem.

The mathematical formulation is given in Sec. 2. In Section 3 the leading terms in the weak-coupling expansion for the ground-state energy are obtained by analytical interation of the variational integral equation starting with the Bogoliubov approximation; the essential difference from the result of Foldy is the occurrence of a logarithmic term related to the screening of the Coulomb interaction. The results of numerical solution of the integral equation are presented in Sec. 4; these extend up to $r_s=10$, whereas the high-density expansion is only useful for $r_s \ll 1$.

2. FORMULATION

In units with $\hbar = m = 1$, the Hamiltonian is

$$H = \sum_{\mathbf{k}} \tfrac{1}{2}k^2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \tfrac{1}{2}\Omega^{-1} \sum_{\mathbf{q}\mathbf{k}\mathbf{k}'}{}' (4\pi e^2/q^2) a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}'-\mathbf{q}}^\dagger a_{\mathbf{k}'} a_{\mathbf{k}}, \quad (1)$$

where the prime implies omission of the terms with $\mathbf{q}=0$, corresponding to cancellation of the boson charge by a uniform background charge so as to preserve over-all charge neutrality. Equation (1) corresponds to Eq. (I.2) with

$$\nu(\mathbf{k}) = 4\pi e^2/k^2, \quad \mathbf{k}\neq 0; \quad \nu(0)=0. \quad (2)$$

The variational trial ground state is given by Eqs. (I.16), (I.19), and (I.20), and its energy is given by (I.21). Assuming ϕ spherically symmetric, $\phi(\mathbf{k}) = \phi(k)$, one finds

$$E_0/n = -(\rho_0/\rho)I_3 + (4\pi^2\rho)^{-1}\int_0^\infty \left\{[k^2+I_2(k)] \times \frac{\phi^2(k)}{1-\phi^2(k)} + I_1(k)\frac{\phi(k)}{1-\phi^2(k)}\right\} k^2 dk, \quad (3)$$

where ρ_0, I_1, and I_2 are given by (I.22), and[6]

$$I_3 \equiv I_1(0) - I_2(0). \quad (4)$$

It is convenient, following Gell-Mann and Brueckner[3] and Foldy,[1] to define dimensionless quantities

$$\begin{aligned} u_0 &\equiv (2/e^4)(E_0/n), \\ r_s &\equiv (3/4\pi\rho)^{1/3} e^2, \\ p &\equiv (4\pi\rho e^2)^{-1/4} k, \\ g(p) &\equiv \phi(k). \end{aligned} \quad (5)$$

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[1] L. L. Foldy, Phys. Rev. **124**, 649 (1961).
[2] N. N. Bogoliubov, J. Phys. (U.S.S.R.) **11**, 23 (1947).
[3] M. Gell-Mann and K. A. Brueckner, Phys. Rev. **106**, 364 (1957).
[4] M. Girardeau and R. Arnowitt, Phys. Rev. **113**, 755 (1959), denoted herein by I; Eq. (n) of this paper will be denoted by (I.n).
[5] M. Girardeau, Phys. Rev. **115**, 1090 (1959); Eq. (n) of this paper will be denoted by (II.n).

[6] We shall find that the integrals $I_1(0)$ and $I_2(0)$ are both divergent in lowest order, but that their difference I_3 is convergent.

The parameter r_s is the ratio of the mean interparticle spacing to the Bohr radius, u_0 is the ground-state energy per particle in Rydbergs, and p is a dimensionless momentum defined so that in the high-density case ($r_s \ll 1$) one has, as will presently be shown,

$$g(p) \approx p^{-4}; \quad p \gg 1, \quad r_s \ll 1. \tag{6}$$

Then

$$u_0 = -2\times 3^{1/2} r_s^{-3/2} f i_3 + 2\times 3^{1/4}\pi^{-1} r_s^{-3/4} \times \int_0^\infty \left\{ [p^4 + i_2(p)] \frac{g^2(p)}{1-g^2(p)} + i_1(p) \frac{g(p)}{1-g^2(p)} \right\} dp, \tag{7}$$

where one finds after performing the angular integrations in (I.22)[7]

$$\begin{aligned}
f &\equiv \rho_0/\rho = 1 - 2\times 3^{-1/4}\pi^{-1} r_s^{3/4} \int_0^\infty \frac{g^2(p)}{1-g^2(p)} p^2 dp, \\
i_1(p) &\equiv (4\pi\rho e^2)^{-1} k^2 I_1(k) = 3^{-1/4}\pi^{-1} r_s^{3/4} p \int_0^\infty p' \ln\left|\frac{p+p'}{p-p'}\right| \frac{g(p')}{1-g^2(p')} dp', \\
i_2(p) &\equiv (4\pi\rho e^2)^{-1} k^2 I_2(k) = 3^{-1/4}\pi^{-1} r_s^{3/4} p \int_0^\infty p' \ln\left|\frac{p+p'}{p-p'}\right| \frac{g^2(p')}{1-g^2(p')} dp', \\
i_3 &\equiv (4\pi\rho e^2)^{-1/2} I_3 = 2\times 3^{-1/4}\pi^{-1} r_s^{3/4} \int_0^\infty \frac{g(p)}{1+g(p)} dp.
\end{aligned} \tag{8}$$

According to (I.23), the variational trial function $g(p)$ is determined by the nonlinear integral equation

$$[f - i_1(p)][1 + g^2(p)] - 2[\tfrac{1}{2}p^4 + f + i_2(p) + p^2 i_3] g(p) = 0 \tag{9}$$

which, when formally solved for $g(p)$, becomes

$$\begin{aligned}
g(p) &= A^{-1}[B - (B^2 - A^2)^{1/2}], \\
A(p) &\equiv f - i_1(p), \\
B(p) &\equiv \tfrac{1}{2}p^4 + f + i_2(p) + p^2 i_3.
\end{aligned} \tag{10}$$

3. HIGH DENSITY

The leading terms in the high-density expansion of u_0 can be obtained by analytical iteration of the variational integral equation, in analogy to the procedure used previously[5] for the case of finite-range potentials. In constrast to that case, however, the Bogoliubov approximation

$$g^{(0)}(p) = 1 + \tfrac{1}{2}p^4 - p^2(1 + \tfrac{1}{4}p^4)^{1/2}, \tag{11}$$

obtained by neglecting all the integrals in (10) including that occurring in f, cannot now be used as the lowest-order approximation to g in evaluating u_0, because the integrals

$$\int_0^\infty i_1(p) \frac{g(p)}{1-g^2(p)} dp, \quad \int_0^\infty i_2(p) \frac{g^2(p)}{1-g^2(p)} dp \tag{12}$$

in (7) would then diverge logarithmically at their lower limits.[8] As in the case of the charged Fermi gas,[3] this can be taken as an indication of the fact that the correct $g(p)$ is such as to effectively provide a low-momentum cutoff proportional to some positive power of r_s, thereby replacing the logarithmic divergence by a logarithmic dependence on r_s. This conjecture will be verified by the subsequent analysis.

A better approximation to $g(p)$ is obtained by retaining the various integrals in (10), but making the Bogoliubov approximation in evaluating these integrals. When one makes this approximation in the condensed fraction f [Eq. (8)] one obtains the zero-order approximation

$$f^{(0)} = 1 - 3^{-1/4}\pi^{-1} r_s^{3/4} \int_0^\infty \left[\frac{1+\frac{1}{2}p^4}{(1+\frac{1}{4}p^4)^{1/2}} - p^2 \right] dp = 1 - \tfrac{1}{3}(4/3)^{1/4}\pi^{-1} r_s^{3/4} K(2^{-1/2}) = 1 - 0.2114 r_s^{3/4}, \tag{13}$$

where K is the complete elliptic integral of the first kind[9]; this result was obtained already by Foldy. Similarly, the corresponding approximation to i_3 is

$$i_3^{(0)} = 3^{-1/4}\pi^{-1} r_s^{3/4} \int_0^\infty \left[1 - \frac{p^2}{2(1+\frac{1}{4}p^4)^{1/2}} \right] dp = 0.2899 r_s^{3/4}. \tag{14}$$

[7] The ubiquitous term $(4\pi\rho e^2)^{1/2}$ is just the classical plasma frequency.
[8] The functions $i_1^{(0)}(p)$ and $i_2^{(0)}(p)$, obtained by replacing g by $g^{(0)}$ in (8), vary linearly with p as $p \to 0$ [see Eq. (16)], whereas $g^{(0)}/[1-(g^{(0)})^2]$ and $(g^{(0)})^2/[1-(g^{(0)})^2]$ behave like p^{-2}.
[9] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, Inc., New York, 1945), pp. 52ff.

The functions $i_1^{(0)}(p)$ and $i_2^{(0)}(p)$,

$$\begin{aligned} i_1^{(0)}(p) &= 3^{-1/4}\pi^{-1}r_s^{3/4}p\int_0^\infty \ln\left|\frac{p+p'}{p-p'}\right|\frac{dp'}{2p'(1+\frac{1}{4}p'^4)^{1/2}}, \\ i_2^{(0)}(p) &= 3^{-1/4}\pi^{-1}r_s^{3/4}p\int_0^\infty \ln\left|\frac{p+p'}{p-p'}\right|\left[\frac{1+\frac{1}{2}p'^4}{2p'(1+\frac{1}{4}p'^4)^{1/2}}-\tfrac{1}{2}p'\right]dp' \end{aligned} \tag{15}$$

are not expressible in closed form except for small and large p. It is shown in Appendix A that

$$\begin{aligned} &i_1^{(0)}(p)\approx i_2^{(0)}(p)\approx \tfrac{1}{4}\times 3^{-1/4}\pi r_s^{3/4}p = 0.5968 r_s^{3/4}p, \quad p\ll 1; \\ &i_1^{(0)}(p)\xrightarrow[p\to\infty]{} 0.6341 r_s^{3/4}, \quad i_2^{(0)}(p)\xrightarrow[p\to\infty]{} 0.2114 r_s^{3/4}. \end{aligned} \tag{16}$$

The first-order approximation $g^{(1)}$ to the solution g of (10) is obtained by replacing f, i_1, i_2, and i_3 by their zero-order approximations (13) to (15); thus

$$\begin{aligned} g^{(1)}(p) &= (A^{(1)})^{-1}\{B^{(1)}-[(B^{(1)})^2-(A^{(1)})^2]^{1/2}\}, \\ A^{(1)}(p) &= 1-0.2114 r_s^{3/4}-i_1^{(0)}(p), \\ B^{(1)}(p) &= \tfrac{1}{2}p^4+1-0.2114 r_s^{3/4}+i_2^{(0)}(p)+0.2899 r_s^{3/4}p^2. \end{aligned} \tag{17}$$

In the low-momentum limit[10]

$$g^{(1)}(p)\approx 1-1.5452 r_s^{3/8}p^{1/2}, \quad p\ll r_s^{1/4}, \tag{18}$$

whereas $g^{(1)}$ reduces to the Bogoliubov approximation $g^{(0)}$ [Eq. (11)] for $p\gg r_s^{1/4}$. The effective low-momentum cutoff in the integrals (12) thus occurs at $p\sim r_s^{1/4}$; for $p\ll r_s^{1/4}$ one sees from (18) and (16) that the integrands behave like $p^{1/2}$, so that the integrals converge at their lower limits. On the other hand, for $r_s^{1/4}\ll p\ll 1$ the functions $g^{(1)}/[1-(g^{(1)})^2]$ and $(g^{(1)})^2/[1-(g^{(1)})^2]$ behave like p^{-2} and hence the integrands behave like p^{-1}. It is thus clear that the integrals (12) have a logarithmic dependence upon the effective cutoff $r_s^{1/4}$.

We are now in a position to evaluate the leading terms in the ground-state energy. Let us begin with the first term, proportional to fi_3, in (7). According to (13),

$$f=1-0.2114 r_s^{3/4}+\cdots, \tag{19}$$

the terms not explicitly indicated being of higher order. Similarly, the leading term in i_3 is $i_3^{(0)}$ [Eq. (14)]; but since the second term will also be needed, we shall evaluate the integral more accurately by using $g^{(1)}$ instead of $g^{(0)}$. Thus, by (8) and (17) one finds after a little algebraic manipulation

$$i_3\approx 3^{-1/4}\pi^{-1}r_s^{3/4}\int_0^\infty\left[1-\left(\frac{B^{(1)}-A^{(1)}}{B^{(1)}+A^{(1)}}\right)^{1/2}\right]dp. \tag{20}$$

In view of (18) it is desirable to consider $\int_0^{r_s^{1/4}}$ and $\int_{r_s^{1/4}}^\infty$ separately. It follows from (17) and (16) that the integrand of (20) is $1+O(r_s^{1/2})$ for $0\leqslant p\leqslant r_s^{1/4}$; hence

$$3^{-1/4}\pi^{-1}r_s^{3/4}\int_0^{r_s^{1/4}}[\quad]dp = 3^{-1/4}\pi^{-1}r_s^{3/4}\int_0^{r_s^{1/4}}dp+O(r_s^{3/2}). \tag{21}$$

To evaluate $\int_{r_s^{1/4}}^\infty$ we first note from (17) that

$$\begin{aligned} B^{(1)}-A^{(1)} &= \tfrac{1}{2}p^4+i_1^{(0)}(p)+i_2^{(0)}(p)+0.2899 r_s^{3/4}p^2, \\ B^{(1)}+A^{(1)} &= 2+\tfrac{1}{2}p^4-0.4228 r_s^{3/4}+i_2^{(0)}(p) \\ &\quad -i^{(0)}(p)+0.2899 r_s^{3/4}p^2. \end{aligned} \tag{22}$$

The leading term in $B^{(1)}-A^{(1)}$ for $p>r_s^{1/4}$ is $\frac{1}{2}p^4$, while the leading term in $B^{(1)}+A^{(1)}$ is $2+\frac{1}{2}p^4$. Thus, expanding in inverse powers of these leading terms, one finds

$$\left(\frac{B^{(1)}-A^{(1)}}{B^{(1)}+A^{(1)}}\right)^{1/2} = \frac{p^2}{2(1+\frac{1}{4}p^4)^{1/2}}\left[1+\frac{i_1^{(0)}(p)+i_2^{(0)}(p)}{p^4}+\frac{0.2899 r_s^{3/4}}{p^2} + \frac{0.4228 r_s^{3/4}+i_1^{(0)}(p)-i_2^{(0)}(p)-0.2899 r_s^{3/4}p^2}{2+\frac{1}{2}p^4}+\cdots\right]. \tag{23}$$

The dominant contribution to $\int_{r_s^{1/4}}^\infty$ in the neighborhood of its lower limit comes from the first two terms in this

[10] Equation (18) is obtained by dropping the term $\frac{1}{2}p^4$ in $A^{(1)}$, inserting the low-momentum approximations (16) to $i_1^{(0)}$ and $i_2^{(0)}$, and expanding the resultant expression for $g^{(1)}$ for small p. Since $(1/2)p^4$ is no longer negligible when it becomes comparable to $0.5968 r_s^{3/4}p$, (18) is only valid for $p\ll r_s^{1/4}$.

expansion; thus we separate these explicitly:

$$
\begin{aligned}
&3^{-1/4}\pi^{-1}r_s^{3/4}\int_{r_s^{1/4}}^{\infty}\left[1-\left(\frac{B^{(1)}-A^{(1)}}{B^{(1)}+A^{(1)}}\right)^{1/2}\right]dp \\
&\quad=3^{-1/4}\pi^{-1}r_s^{3/4}\int_{r_s^{1/4}}^{\infty}\left[1-\frac{p^2}{2(1+\frac{1}{4}p^4)^{1/2}}\right]dp-3^{-1/4}\pi^{-1}r_s^{3/4}\int_{r_s^{1/4}}^{\infty}\frac{i_1^{(0)}(p)+i_2^{(0)}(p)}{2p^2(1+\frac{1}{4}p^4)^{1/2}}dp \\
&\quad\quad+3^{-1/4}\pi^{-1}r_s^{3/4}\int_{r_s^{1/4}}^{\infty}\left[-\left(\frac{B^{(1)}-A^{(1)}}{B^{(1)}+A^{(1)}}\right)^{1/2}+\frac{p^2}{2(1+\frac{1}{4}p^4)^{1/2}}+\frac{i_1^{(0)}(p)+i_2^{(0)}(p)}{2p^2(1+\frac{1}{4}p^4)^{1/2}}\right]dp. \quad (24)
\end{aligned}
$$

The integrand of the first integral on the right side is just that of (14). In view of the low-momentum behavior (16) of $i_1^{(0)}$ and $i_2^{(0)}$, the second integral has a logarithmic contribution from its lower limit,[11] given by

$$-\tfrac{1}{4}\times 3^{-1/2}r_s^{3/2}\int_{r_s^{1/4}}\frac{dp}{p}=(1/16)3^{-1/2}r_s^{3/2}\ln r_s. \quad (25)$$

The integrand of the third integral, when expanded, gives those terms in (23) which have not been separated explicitly in (24); one can show in this way that the integral is only of order $r_s^{3/2}$. Then, substituting (21), (24), and (25) into (20) and using (14),[12] one finds

$$i_3=0.2399r_s^{3/4}+(1/16)3^{-1/2}r_s^{3/2}\ln r_s+O(r_s^{3/2}), \quad (26)$$

so that the first term in the expression (7) for the ground-state energy becomes, with use of (19),

$$-2\times 3^{1/2}r_s^{-3/2}fi_3=-1.0042r_s^{-3/4}-\tfrac{1}{8}\ln r_s+O(1). \quad (27)$$

Application of similar methods to the evaluation of the other integrals in (7) gives the kinetic energy contribution[13]

$$
\begin{aligned}
2\times 3^{1/4}\pi^{-1}r_s^{-3/4}&\int_0^{\infty}p^4\frac{g^2(p)}{1-g^2(p)}dp \\
&=0.2005r_s^{-3/4}+\tfrac{1}{8}\ln r_s+O(1), \quad (28)
\end{aligned}
$$

and the pair-pair scattering and forward scattering contributions[14]

$$
\begin{aligned}
2\times 3^{1/4}\pi^{-1}r_s^{-3/4}&\int_0^{\infty}\left[i_1(p)\frac{g(p)}{1-g^2(p)}\right. \\
&\left.+i_2(p)\frac{g^2(p)}{1-g^2(p)}\right]dp=-\tfrac{1}{8}\ln r_s+O(1). \quad (29)
\end{aligned}
$$

Adding the contributions (27) to (29), one finds

$$u_0=-0.8037r_s^{-3/4}-\tfrac{1}{8}\ln r_s+O(1) \quad (30)$$

as the leading terms in the asymptotic expansion of the ground-state energy for the case of high density. The first term agrees with that found by Foldy,[15] but the logarithmic term is new; it arises from the forward scattering terms $N_{\mathbf{k}}N_{\mathbf{k}'}$ and pair-pair scattering terms $a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}'}a_{\mathbf{k}'}$ in the Hamiltonian, such terms being omitted in the Bogoliubov approximation.[16] It is

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[11] Since, according to (16), $i_1^{(0)}$ and $i_2^{(0)}$ approach a constant value of order $r_s^{3/4}$ for $p\gg 1$, there is no logarithmic contribution from the upper limit, but instead a negligible contribution of order $r_s^{3/2}$.

[12] Changing the integrand of the integral on the right side of (21) from 1 to $1-p^2/2(1+\frac{1}{4}p^4)^{1/2}$ so that it can be combined with the corresponding integral in (24), one only changes the value of the integral by an amount of order $r_s^{3/2}$.

[13] Approximating g by $g^{(1)}$ in (28), as in (20) one finds

$$\int_0^{\infty}p^4\frac{g^2(p)}{1-g^2(p)}dp\approx\tfrac{1}{2}\int_0^{\infty}\left\{\frac{B^{(1)}}{[(B^{(1)})^2-(A^{(1)})^2]^{1/2}}-1\right\}p^4dp.$$

The appropriate decomposition of the integrand gives

$$
\begin{aligned}
\tfrac{1}{2}\int_0^{\infty}\{\quad\}p^4dp=&\tfrac{1}{2}\int_0^{\infty}\left[\frac{p^2(1+\frac{1}{2}p^4)}{(1+\frac{1}{4}p^4)^{1/2}}-p^4\right]dp \\
&-\tfrac{1}{2}\int_{r_s^{1/4}}^{\infty}\frac{(1+\frac{1}{2}p^4)[i_1^{(0)}(p)+i_2^{(0)}(p)]}{p^2(1+\frac{1}{4}p^4)^{1/2}}dp+O(r_s^{3/4}).
\end{aligned}
$$

The first integral on the right side is equal to 0.2393, while the second gives a logarithmic contribution from its lower limit analogous to (25), when one takes into account the low-momentum behavior (16).

[14] The result (29) can be found by the same method as used in obtaining (27) and (28), but, since the first nonvanishing term is the logarithmic one, a simpler method suffices. Using (16) and (18) one finds

$$
\begin{aligned}
\int_0^{r_s^{1/4}}i_1^{(0)}(p)\frac{g^{(1)}}{1-(g^{(1)})^2}dp\sim\int_0^{r_s^{1/4}}i_2^{(0)}(p)\frac{(g^{(1)})^2}{1-(g^{(1)})^2}dp& \\
\sim r_s^{3/8}\int_0^{r_s^{1/4}}p^{1/2}dp=O(r_s^{3/4})&,
\end{aligned}
$$

where the symbol $\sim$ denotes order-of-magnitude equality; although (18) is only valid for $p\ll r_s^{1/4}$, it retains order-of-magnitude validity up to $p=r_s^{1/4}$. Noting from (17) and (16) that

$$\frac{g^{(1)}}{1-(g^{(1)})^2}\approx\frac{(g^{(1)})^2}{1-(g^{(1)})^2}\approx\frac{1}{2p^2},\quad r_s^{1/4}\ll p\ll 1,$$

one finds, using (16)

$$
\begin{aligned}
\int_{r_s^{1/4}}^1 i_1^{(0)}(p)\frac{g^{(1)}}{1-(g^{(1)})^2}dp\approx\int_{r_s^{1/4}}^1 i_2^{(0)}(p)\frac{(g^{(1)})^2}{1-(g^{(1)})^2}dp& \\
\approx\tfrac{1}{8}\times 3^{-1/4}\pi r_s^{3/4}\int_{r_s^{1/4}}^1\frac{dp}{p}=-\tfrac{1}{32}\times 3^{-1/4}\pi r_s^{3/4}\ln r_s+O(r_s^{3/4})&;
\end{aligned}
$$

it should be noted that the coefficient of the logarithmic term is independent of the precise value of the lower limit; all that is important is that it is of order $r_s^{1/4}$. The contributions to the integrals from p of order unity and greater are clearly $O(r_s^{3/4})$.

[15] Eq. (22) of reference 1, as corrected in a later erratum [L. L. Foldy, Phys. Rev. **125**, 2208 (1962)]; Foldy's Eq. (22) should be multiplied by a factor 1/2.

[16] The pair-pair scattering contribution is the part of (29) involving i_1, and the forward scattering contribution is the part involving i_2. The other logarithmic terms [those in (27) and (28)] cancel each other.

shown in Appendix B that this logarithmic term is related to a screening of the long-range correlations at a distance

$$r_0 \sim r_s^{-1/2} \rho^{-1/3}; \tag{31}$$

for $r_s^{1/4} r_0 \ll r \ll r_0$ the pair correlation function falls off like r^{-2}, whereas for $r \gg r_0$ it falls off like r^{-4}. It is noteworthy that the correlation length r_0 is the same as that of the high-density electron gas; although (31) can be derived for the electron gas by a simple Thomas-Fermi calculation,[17] such a derivation is not applicable to a Bose system.

As a consistency check on the derivation of (30) one has available the following well-known consequence of the variational theorem for the ground-state energy E_0:

$$E_0 = E_0(0) + \int_0^1 g^{-1} V_0(g) dg, \tag{32}$$

where $E_0(g)$ and $V_0(g)$ are, respectively, the total and potential energies of the ground state when the interaction e^2/r is replaced by ge^2/r. Noting (5), one finds[18]

$$u_0 = \int_0^1 g v_0(g) dg, \tag{33}$$

where $v_0(g)$ is the potential-energy part of $u_0(g)$, and $u_0(g)$ is obtained from u_0 by replacing r_s by gr_s. Subtraction of (28) from (30) gives

$$v_0(g) = -1.0042 g^{-3/4} r_s^{-3/4} - \tfrac{1}{4} \ln g - \tfrac{1}{4} \ln r_s + O(1) \tag{34}$$

and hence

$$\int_0^1 g v_0(g) dg = -0.8034 r_s^{-3/4} - \tfrac{1}{8} \ln r_s + O(1). \tag{35}$$

Aside from a difference of three in the last place in the coefficient of $r_s^{-3/4}$, which represents the error of the numerical integrations, (35) agrees with (30).

It is shown in Appendix C that the lowest-order corrections to u_0 due to "non-pair" processes, the simplest of which is a three-plasmon "vacuum fluctuation" process, are of order unity and higher. Hence, the term of order unity is not significant either in Foldy's theory or in ours.

4. INTERMEDIATE DENSITY

The high-density expansion (30) was derived under the assumption $r_s \ll 1$. Even when this condition is satisfied the expansion is most probably asymptotic rather than convergent, and hence gives no information about the behavior of the ground-state energy at intermediate densities. Our variational method is not subject to this limitation, although one has to resort to a direct numerical solution of the variational integral equation (10) when r_s is not small. The numerical results described below show that (30) is already in error by 4% at $r_s = 0.1$ and by 40% at $r_s = 1$. In the opposite limit of low density ($r_s \gg 1$) the potential energy dominates the kinetic energy and, as was pointed out by Foldy, the ground state is a crystal with energy[19]

$$u_0 = -1.792 r_s^{-1} + 2.65 r_s^{-3/2} + O(r_s^{-2}). \tag{36}$$

The domain of validity of this expansion, estimated by requiring that the second term be less than 10% of the first, is $r_s \gtrsim 200$. There is thus a large intermediate range of r_s for which neither the high- nor the low-density expansion is useful. This range can be at least partially spanned by our variational method, which is exact in the high-density limit and should, therefore, remain accurate up to intermediate densities.

The variational integral equation (10) can be solved numerically by iteration, the new approximation $g^{(j+1)}$ being obtained by substitution of $g^{(j)}$ for g in the evaluation of the integrals $(1-f)$, i_1, i_2, and i_3. The numerical calculations leading to the results presented here were performed on an IBM 1620 computer; for each value of r_s the iteration was continued until two successive approximations to u_0, calculated by Eq. (7), differed by less than 1%. For the smallest values of r_s the iteration was started with a modification[20] of the Bogoliubov approximation $g^{(0)}$ [Eq. (11)] and for larger r_s it was started with an approximate g obtained by extrapolation of the solutions for smaller values of r_s. The logarithmic singularities of the integrands of i_1 and i_2 [Eq. (8)] at $p = p'$ had to be treated carefully in evaluating the integrals numerically; the method employed is described in Appendix D. Difficulties with convergence of the iterative process were encountered for $r_s > 3$, but it was possible to secure convergence up to $r_s = 10$ by a minor modification of the algorithm.[21] Convergence again failed for $r_s > 10$.

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[17] D. J. Thouless, *The Quantum Mechanics of Many-Body Systems* (Academic Press Inc., New York, 1961), pp. 144 ff.

[18] The unperturbed ground-state energy $E_0(0)$ is, of course, zero for the Bose gas.

[19] The leading term is the Madelung energy of Wigner's electron crystal [E. P. Wigner, Phys. Rev. **46**, 1002 (1934)], and the second is the zero-point energy [Rosemary Coldwell-Horsfall and A. A. Maradudin, J. Math. Phys. **1**, 395 (1960); W. J. Carr, Jr., Phys. Rev. **122**, 1437 (1961)]. It was shown by Wigner that the effects of statistics first enter in overlap terms of order $\exp(-\text{const} r_s^{1/2})$.

[20] Since the integrals (12) diverge when g is replaced by $g^{(0)}$, it is necessary to modify (11) at low momentum when using it to start the iterative process. The subroutine for the evaluation of the integrals (12) implicitly assumed the integrands to vanish at $p = 0$, as is indeed the case for the correct solution g, and indeed already for the first approximation $g^{(1)}$ (see Appendix A). This is equivalent to modifying $g^{(0)}$ at low momentum so that the first approximations to (12) converge.

[21] The integral equation (10) can be written symbolically as

$$g = \mathfrak{F}(g),$$

where $\mathfrak{F}$ denotes a certain inhomogeneous and highly nonlinear functional defined by (8) and the right side of (10). The iteration algorithm employed for small values of r_s was

$$g^{(j+1)} = \mathfrak{F}(g^{(j)}).$$

This process converges for $r_s \leq 3$, but for $r_s > 3$ the successive approximations oscillate in an undamped fashion. It was possible to secure converge in the region $3 \leq r_s \leq 10$ using the modified algorithm

$$g^{(j+1)} = (1/2)[g^{(j)} + \mathfrak{F}(g^{(j)})].$$

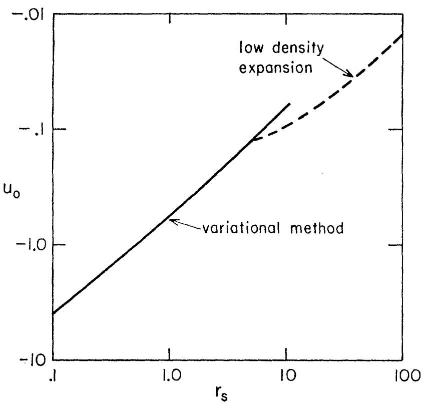

FIG. 1. Ground-state energy.

Although it might have been possible to obtain solutions in this low-density region by further refinements of the iteration algorithm, this was not attempted since our variational method cannot be expected to retain accuracy there; a crystalline wavefunction would be more appropriate for low-density calculations.

The computed values of the ground-state energy u_0 and the condensed fraction f are given in Table I and Figs. 1 and 2. One notes that the energy calculated

TABLE I. Ground-state energy and condensed fraction.

r_s	u_0 (Ry)	f
0.01	−24.6	0.995
0.03	−10.5	0.991
0.10	−4.05	0.983
0.30	−1.65	0.971
1.00	−0.582	0.956
3.00	−0.211	0.945
10.00	−0.0666	0.937

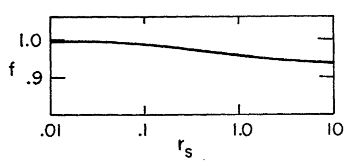

FIG. 2. Condensed fraction.

from the low-density expansion (36) (the dashed line in Fig. 1) lies lower than that given by our variational method when $r_s > 5$; hence the transition into Wigner's electron crystal probably takes place at $r_s \sim 5$.[22] The condensed fraction f displays a marked insensitivity to the density, only falling from 1 to 0.94 as r_s varies from 0 to 10; this is to be contrasted with what (19) would predict were it to remain valid up to intermediate values of r_s. The momentum distribution function[23]

$$n(\mathbf{k}) = \phi^2(k)/[1-\phi^2(k)], \tag{37}$$

the mean number of particles with momentum $\mathbf{k}$, and the pair distribution function $D(r)$ [Eqs. (B1) and (B5)], the relative probability of finding two particles with a separation r, were also calculated numerically for $r_s = 0.1$, 1, and 10, and are plotted in Figs. 3 and 4. The salient feature of the momentum distributions is the increase in the high-momentum components with decreasing density (increasing r_s); the pair distribution functions show the outward displacement of charge responsible for screening of the Coulomb interaction, and the increasing effectiveness of this screening with decreasing density.

5. DISCUSSION

By use of a variational method based on a trial ground state involving pair correlations, we have obtained the leading two terms of the high-density

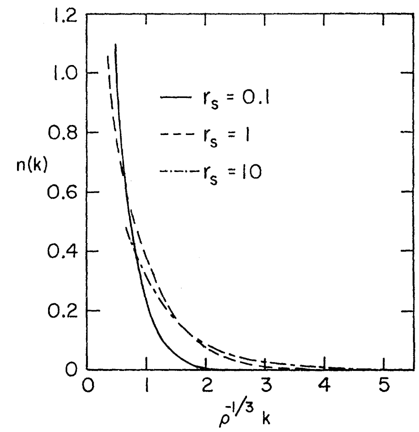

FIG. 3. Momentum distribution functions.

expansion for the ground-state energy [Eq. (30)] as well as a numerical upper bound (Table I and Fig. 1) valid both at high and intermediate densities. This is to be compared with Foldy's application of the Bogoliubov method, which only gives the first term of the high-density expansion correctly and is not applicable at intermediate densities.

We have not examined the excitation spectrum since the excitation energies in the pair theory [Eq. (I.37)] do not satisfy a variational theorem and are hence less accurate than the ground-state energy. Foldy has

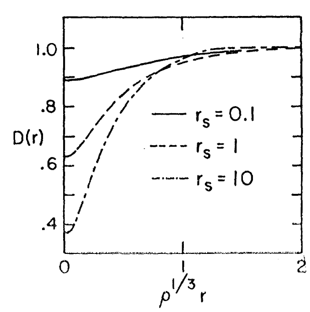

FIG. 4. Pair distribution functions.

[22] Note, however, that this estimate is rather uncertain due to the poor convergence of (36) at $r_s = 5$.

[23] See Eq. (I.32).

already shown that the low excitations at high density are plasmons in the Bologiubov approximation, and we are not able to improve on his expression[24] for the plasmon dispersion relation.

It might be instructive if the high-density expansion (30) could be obtained independently by a diagram-summation method along the lines of the treatment of the high-density electron gas due to Gell-Mann and Brueckner.[3] This might, however, be difficult, since *all* terms (even the first) in the high-density expansion are nontrivial in the Bose case, as was pointed out by Foldy.

ACKNOWLEDGMENT

I am indebted to G. Wentzel for critically reading the manuscript.

APPENDIX A. ASYMPTOTIC BEHAVIOR OF THE INTEGRALS i_1 AND i_2

The integrals $i_1^{(0)}$ and $i_2^{(0)}$ [Eq. (15)] are transformed into the forms

$$\begin{aligned} i_1^{(0)}(p)&=\tfrac{1}{2}\times 3^{-1/4}\pi^{-1}r_s^{3/4}p\int_0^\infty \ln\left|\frac{1+x}{1-x}\right|\frac{dx}{x(1+\frac{1}{4}p^4x^4)^{1/2}},\\ i_2^{(0)}(p)&=\tfrac{1}{2}\times 3^{-1/4}\pi^{-1}r_s^{3/4}p\int_0^\infty \ln\left|\frac{1+x}{1-x}\right|\left[\frac{1+\frac{1}{2}p^4x^4}{x(1+\frac{1}{4}p^4x^4)^{1/2}}-p^2x\right]dx\end{aligned} \tag{A1}$$

by the substitution $p'\to px$. For $p\ll 1$, the terms involving p^4 or p^2 in both integrands may be dropped; the low-momentum approximation (16) then follows from the formula[25]

$$\int_0^\infty x^{s-1}\ln\left|\frac{1+x}{1-x}\right|dx=\pi s^{-1}\tan(\tfrac{1}{2}\pi s),\quad -1<\mathrm{Re}\,s<1 \tag{A2}$$

upon letting $s\to 0$. For $p\gg 1$ the dominant contribution to the integrals comes from $x\lesssim p^{-1}\ll 1$ where the logarithm may be replaced by $2x$. Then the substitution $(1/4)p^4x^4=y^4$ gives

$$\begin{aligned} i_1^{(0)}(p)&\approx(4/3)^{1/4}\pi^{-1}r_s^{3/4}\int_0^\infty\frac{dy}{(1+y^4)^{1/2}}=(4/3)^{1/4}\pi^{-1}r_s^{3/4}K(2^{-1/2}),\\ i_2^{(0)}(p)&\approx(4/3)^{1/4}\pi^{-1}r_s^{3/4}\int_0^\infty\left[\frac{1+2y^4}{(1+y^4)^{1/2}}-2y^2\right]dy=\tfrac{1}{3}(4/3)^{1/4}\pi^{-1}r_s^{3/4}K(2^{-1/2}),\end{aligned} \tag{A3}$$

where K is the complete elliptic integral of the first kind[9]; substitution of the numerical values gives (16).

In order to extend (16) to the general case of i_1 and i_2 [g instead of $g^{(0)}$ in (8)] we note, as in obtaining (A3), that the dominant contribution to the integrals i_1 and i_2 for large p comes from the region $p'\ll p$ where the logarithms in (8) may be replaced by $2p'/p$, so that

$$\begin{aligned} i_1(p)&\xrightarrow[p\to\infty]{} 2\times 3^{-1/4}\pi^{-1}r_s^{3/4}\int_0^\infty p'^2\frac{g(p')}{1-g^2(p')}dp',\\ i_2(p)&\xrightarrow[p\to\infty]{} 2\times 3^{-1/4}\pi^{-1}r_s^{3/4}\int_0^\infty p'^2\frac{g^2(p')}{1-g^2(p')}dp'=1-f.\end{aligned} \tag{A4}$$

The relationship between the condensed fraction f [Eq. (8)] and the asymptotic behavior of i_2 proved useful as a check on the numerical calculations for the intermediate-density region [Sec. 4 and Appendix D].

In the opposite limit of low momentum the dominant contributions come from $p'\gg p$ where the logarithms may be replaced by $2p/p'$, giving

$$\begin{aligned} i_1(p)&\approx 2\times 3^{-1/4}\pi^{-1}r_s^{3/4}p^2\int_0^\infty\frac{g(p')}{1-g^2(p')}dp',\\ i_2(p)&\approx 2\times 3^{-1/4}\pi^{-1}r_s^{3/4}p^2\int_0^\infty\frac{g^2(p')}{1-g^2(p')}dp',\quad p\ll 1.\end{aligned} \tag{A5}$$

Equations (A5) fail when g is replaced by $g^{(0)}$ since the integrals then diverge; the method used above in obtaining the low-momentum behavior (16) of $i_1^{(0)}$ and $i_2^{(0)}$ shows that these functions are linear in p, rather than quadratic, for small p. But $(1-g^{(1)})^{-2}$ diverges only like $(p')^{-1/2}$ as $p'\to 0$ [see Eq. (18)] so that (A5) is correct in first approximation and presumably also in higher approximations.

APPENDIX B. ASYMPTOTIC BEHAVIOR OF THE PAIR CORRELATION FUNCTION

The pair distribution function $D(r)$ for our variational ground state is given by Eq. (I.28); we define

[24] Equation (8) of reference 1.

[25] *Tables of Integral Transforms*, edited by A. Erdélyi (McGraw-Hill Book Company, Inc., New York, 1954), Vol. I, p. 316, Eq. (24).

the pair correlation function $\chi(r)$ by

$$\chi(r)\equiv D(r)-1. \tag{B1}$$

The potential energy per particle of the ground state can be expressed in terms of $\chi(r)$ by

$$V_0/n=\tfrac{1}{2}\rho\int\chi(r)(e^2/r)d^3r. \tag{B2}$$

$\chi(r)$ occurs in (B2) instead of $D(r)$ because of the compensation of the average boson charge by the uniform background; the infinite term with $\mathbf{q}=0$ omitted from (1) can be written in coordinate space in the form $\frac{1}{2}\rho\int(e^2/r)d^3r$ when particle-number conservation is taken into account.

Introducing the dimensionless quantities (5) and a dimensionless distance

$$\xi\equiv(4\pi\rho e^2)^{1/4}r \tag{B3}$$

and performing the angular integrations, one finds

$$v_0\equiv(2/e^4)(V_0/n)=3^{1/2}r_s^{-3/2}\int_0^\infty\chi(\xi)\xi d\xi, \tag{B4}$$

where, according to (B1) and (I.28),

$$\chi(\xi)=-2f[f_1(\xi)-f_2(\xi)]+f_1^2(\xi)+f_2^2(\xi),$$

$$f_1(\xi)=2\times3^{-1/4}\pi^{-1}r_s^{3/4}\xi^{-1}\int_0^\infty\frac{g(p)}{1-g^2(p)}p\sin(\xi p)dp, \tag{B5}$$

$$f_2(\xi)=2\times3^{-1/4}\pi^{-1}r_s^{3/4}\xi^{-1}\int_0^\infty\frac{g^2(p)}{1-g^2(p)}p\sin(\xi p)dp.$$

Because of a near cancellation of the low-momentum contributions of f_1 and f_2 to the term in χ involving f_1-f_2, it is desirable to also separately evaluate the integral

$$f_1(\xi)-f_2(\xi)=2\times3^{-1/4}\pi^{-1}r_s^{3/4}\xi^{-1}\int_0^\infty\frac{g(p)}{1+g(p)}p\sin(\xi p)dp. \tag{B6}$$

Approximating g by $g^{(1)}$ and noting from (17) and (22) that to lowest order in the small parameter r_s

$$\begin{aligned}&A^{(1)}\approx1,\\&B^{(1)}\approx1+(1/2)p^4,\\&B^{(1)}-A^{(1)}\approx(1/2)p^4+i_1^{(0)}(p)+i_2^{(0)}(p),\\&B^{(1)}+A^{(1)}\approx2+(1/2)p^4\end{aligned} \tag{B7}$$

for all values of p, one finds after algebraic reduction

$$f_1(\xi)\approx3^{-1/4}\pi^{-1}r_s^{3/4}\xi^{-1}\int_0^\infty\{(2+\tfrac{1}{2}p^4)[\tfrac{1}{2}p^4+i_1^{(0)}(p)+i_2^{(0)}(p)]\}^{-1/2}p\sin(\xi p)dp,$$

$$f_2(\xi)\approx3^{-1/4}\pi^{-1}r_s^{3/4}\xi^{-1}\int_0^\infty\left\{\frac{1+\frac{1}{2}p^4}{(2+\frac{1}{2}p^4)^{1/2}[\frac{1}{2}p^4+i_1^{(0)}(p)+i_2^{(0)}(p)]^{1/2}}-1\right\}p\sin(\xi p)dp, \tag{B8}$$

$$f_1(\xi)-f_2(\xi)\approx3^{-1/4}\pi^{-1}r_s^{3/4}\xi^{-1}\int_0^\infty\left\{1-\left[\frac{\frac{1}{2}p^4+i_1^{(0)}(p)+i_2^{(0)}(p)}{2+\frac{1}{2}p^4}\right]^{1/2}\right\}p\sin(\xi p)dp.$$

Although these integrals cannot be evaluated analytically for general values of ξ, it is possible to determine their asymptotic behavior both for $1\ll\xi\ll r_s^{-1/4}$ and $\xi\gg r_s^{-1/4}$. In the former case the dominant contribution comes from $p\gg r_s^{1/4}$ where $i_1^{(0)}$ and $i_2^{(0)}$ may be dropped since they are $\ll p^4$ [cf. (16)], so that

$$f_1(\xi)\approx3^{-1/4}\pi^{-1}r_s^{3/4}\xi^{-1}\int_0^\infty\frac{\sin(\xi p)}{p(1+\frac{1}{4}p^4)^{1/2}}dp\approx\tfrac{1}{2}\times3^{-1/4}r_s^{3/4}\xi^{-1},$$

$$f_2(\xi)\approx3^{-1/4}\pi^{-1}r_s^{3/4}\xi^{-1}\int_0^\infty\left[\frac{1+\frac{1}{2}p^4}{p^2(1+\frac{1}{4}p^4)^{1/2}}-1\right]p\sin(\xi p)dp\approx\tfrac{1}{2}\times3^{-1/4}r_s^{3/4}\xi^{-1}. \tag{B9}$$

These asymptotic forms were evaluated by noting that since $\xi\gg1$, the dominant contribution comes from $p\ll1$ where p^4 may be dropped compared to unity, so that the integrals reduce to $\int_0^\infty p^{-1}\sin(\xi p)dp=\pi/2$. The exact asymptotic form of f_1-f_2 in the interval $1\ll\xi\ll r_s^{-1/4}$ cannot be obtained in this way, but one estimates

$$f_1(\xi)-f_2(\xi)\approx3^{-1/4}\pi^{-1}r_s^{3/4}\xi^{-1}\int_0^\infty\left[1-\frac{p^2}{2(1+\frac{1}{4}p^4)^{1/2}}\right]p\sin(\xi p)dp\approx\alpha r_s^{3/4}\xi^{-1}e^{-\beta\xi}, \tag{B10}$$

where α and β are positive constants of order unity, by comparison with the integral $\int_0^\infty(1+\frac{1}{2}p^2)^{-1}p\sin(\xi p)dp=2\pi e^{-2\xi}$ whose integrand is the same as that of (B10) for small p but falls off less rapidly for large p. The

contributions from f_1^2 and f_2^2 dominate the exponentially small contribution from f_1-f_2 in (B5), so that

$$\chi(\xi)\approx\tfrac{1}{2}\times 3^{-1/2}r_s^{3/2}\xi^{-2},\quad 1\ll\xi\ll r_s^{-1/4}. \tag{B11}$$

To obtain the asymptotic behavior for $\xi\gg r_s^{-1/4}$ we note that the dominant contributions then come from $p\ll r_s^{-1/4}$ where $i_1^{(0)}$ and $i_2^{(0)}$ dominate p^4 in (B8); then using the low-momentum approximation (16) for $i_1^{(0)}$ and $i_2^{(0)}$, one finds

$$\begin{aligned} f_1(\xi)\sim f_2(\xi)&\sim r_s^{3/8}\xi^{-1}\int_0^{\alpha r_s^{1/4}}p^{1/2}\sin(\xi p)dp\\ &=r_s^{3/8}\xi^{-2}[-\alpha^{1/2}r_s^{1/8}\cos(\alpha r_s^{1/4}\xi)+\tfrac{1}{2}(2\pi/\xi)^{1/2}C(\alpha r_s^{1/4}\xi)]\\ &\sim -r_s^{1/2}\xi^{-2}\cos(\alpha r_s^{1/4}\xi),\quad \xi\gg r_s^{-1/4}, \end{aligned}\tag{B12}$$

where the symbol $\sim$ denotes order-of-magnitude equality, α is a constant of order unity, and C is the Fresnel integral

$$C(x)=\int_0^x\frac{\cos t}{(2\pi t)^{1/2}}dt\xrightarrow[x\to\infty]{}\tfrac{1}{2}. \tag{B13}$$

On the other hand, an argument analogous to that used in obtaining (B10) shows that f_1-f_2 is again negligible compared to f_1^2 and f_2^2. Thus, by (B5)

$$\chi(\xi)\approx r_s\xi^{-4}\psi^2(r_s^{1/4}\xi),\quad \xi\gg r_s^{-1/4}, \tag{B14}$$

where $\psi(x)$ is some function which is oscillatory with wavelength ~1 and amplitude ~1 for $x\gg1$.

Substituting (B11) into (B4), one sees that v_0 would have a long-range (large ξ) logarithmic divergence if (B11) were to remain valid for arbitrarily large ξ. However, because of (B14) there is an effective cutoff at $\xi\sim r_s^{-1/4}$, hence $r\sim r_0$ [Eq. (31)], so that the logarithmic divergence is replaced by a finite logarithmic contribution to v_0, given by

$$\tfrac{1}{2}\int^{r_s^{-1/4}}\xi^{-1}d\xi=-\tfrac{1}{8}\ln r_s, \tag{B15}$$

the logarithmic term in (30). The correlation function $\chi(\xi)$ is positive for $\xi\gg1$ because the repulsive Coulomb interaction displaces charge outward from a given boson; χ must accordingly be negative for $\xi\lesssim1$, where the term involving f_1-f_2 in (B5) predominates, although our asymptotic calculation does not allow an investigation of this range here. This effect is evident in the numerical results shown in Fig. 4.

APPENDIX C. "NON-PAIR" CORRECTIONS TO THE GROUND-STATE ENERGY

We use a perturbation method analogous to that employed in Appendix B of reference 5. The simplest "non-pair" contribution to E_0/n comes from a three-plasmon "vacuum fluctuation" process; this is given by Eqs. (II.B9) and (II.B10) with $\nu(\mathbf{k})$ given by (2) and $E(\mathbf{k})$ by (I.37). Since we are interested here in the case of high densities, we can estimate the integral (II.B9) by replacing ϕ by the Bogoliubov approximation $\phi^{(0)}$ [corresponding to $g^{(0)}$, Eq. (11)] provided that the integral is convergent in this approximation. The only singularities of the integrand occur at $\mathbf{k}=0$, $\mathbf{k}'=0$, and $\mathbf{k}+\mathbf{k}'=0$; since these are all equivalent,[26] we shall restrict our attention to what happens at $\mathbf{k}=0$. The function $g(\mathbf{kk}')$ can be written in the form

$$\begin{aligned} g(\mathbf{kk}')=-\{&[1-\phi^2(\mathbf{k})][1-\phi^2(\mathbf{k}')][1-\phi^2(\mathbf{k}+\mathbf{k}')]\}^{-1/2}\\ \times\{&\nu(\mathbf{k})[1-\phi(\mathbf{k})][\phi(\mathbf{k}')+\phi(\mathbf{k}+\mathbf{k}')]\\ &+\nu(\mathbf{k}')[1-\phi(\mathbf{k}')][\phi(\mathbf{k})+\phi(\mathbf{k}+\mathbf{k}')]\\ &+\nu(\mathbf{k}+\mathbf{k}')[1-\phi(\mathbf{k}+\mathbf{k}')][\phi(\mathbf{k})+\phi(\mathbf{k}')]\} \end{aligned}\tag{C1}$$

by grouping terms in (II.B10). The interaction (2) is singular at $\mathbf{k}=0$, but this is canceled by the factor $[1-\phi(\mathbf{k})]$ which is proportional to k^2 for small k in the Bogoliubov approximation [cf. (11)]. Thus, the only singularity in $g(\mathbf{kk}')$ at $\mathbf{k}=0$ comes from the factor $[1-\phi^2(\mathbf{k})]^{-1/2}$, which has a k^{-1} singularity in the Bogoliubov approximation; the function $g^2(\mathbf{kk}')$ in (II.B9) thus has, in the same approximation, a k^{-2} singularity. But this is canceled by the phase-space factor k^2 coming from d^3k. Since the energy denominator is nonvanishing everywhere,[27] the integrand is nonsingular at $\mathbf{k}=0$, and thus also at $\mathbf{k}'=0$ and $\mathbf{k}+\mathbf{k}'=0$. Since there is clearly no trouble at high momenta,[28] one concludes that the integral (II.B9) is convergent in the Bogoliubov approximation. Introducing the dimensionless quantities (5), one then sees that the corresponding contribution to u_0 can be written as a convergent dimensionless integral of order unity, the various factors of ρ and e^2 having canceled; thus the correction to u_0 affects only the term $O(1)$ in (30).

The next-simplest process involves four-plasmon "vacuum fluctuations"; its contribution to E_0/n is given by an expression analogous to (II.B9):

$$-\tfrac{1}{4}\rho^{-1}(2\pi)^{-9}\int\!\!\int\!\!\int'\frac{h^2(\mathbf{kk}'\mathbf{k}'')}{E(\mathbf{k})+E(\mathbf{k}')+E(\mathbf{k}'')+E(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')}d^3kd^3k'd^3k'', \tag{C2}$$

where

$$\begin{aligned} h(\mathbf{kk}'\mathbf{k}'')=\{&[1-\phi^2(\mathbf{k})][1-\phi^2(\mathbf{k}')][1-\phi^2(\mathbf{k}'')][1-\phi^2(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')]\}^{-1/2}\\ &\times\sum_{\mathbf{qq}'\mathbf{q}''}\Delta(\mathbf{qq}'\mathbf{q}''|\mathbf{kk}'\mathbf{k}'')\phi(\mathbf{q})\phi(\mathbf{q}')\nu(\mathbf{q}+\mathbf{q}''), \end{aligned}\tag{C3}$$

[26] The integrand is symmetric under permutations of the set $\{\mathbf{k},\mathbf{k}',-\mathbf{k}-\mathbf{k}'\}$.
[27] In the Bogoliubov approximation, $E(\mathbf{k})$ approaches the plasma frequency $(4\pi\rho e^2)^{1/2}$ as $k\to0$ [see Eqs. (8) and (9) of reference 1].
[28] Because of (6), the integrand falls off very rapidly at high momenta.

the definition of the Δ function and the meaning of the primed integration being the obvious generalizations of their meanings in (II.B9) and (II.B10). Introducing the dimensionless quantities (5) as before, one concludes that the corresponding contribution to u_0 is equal to $r_s^{3/4}$ times a dimensionless integral. However, this integral is in fact divergent, since the term $\nu(\mathbf{q}+\mathbf{q}'')$ introduces a $(\mathbf{q}+\mathbf{q}'')^{-4}$ singularity in h^2 which is not cancelled by any of the factors involving ϕ and is only reduced to a $(\mathbf{q}+\mathbf{q}'')^{-2}$ singularity by inclusion of the phase-space factor. This divergence is an indication of the fact that the "quartet" part of the Hamiltonian (II.B3) actually contributes to u_0 in lower order than the order $r_s^{3/4}$ suggested by a naive dimensional argument. Nevertheless, it seems quite unlikely that the contribution is larger than $O(1)$, or even as large as $O(1)$. If we assume a low-momentum cutoff of order $r_s^{1/4}$ as in (12), then we obtain a contribution to u_0 of order

$$r_s^{3/4}\int_{r_s^{1/4}}^{\infty} p^{-2}dp=O(r_s^{1/2}).$$

Such a cutoff would, no doubt, be introduced if one were to consider not merely the single second-order diagram[29] leading to (C2) but, following Gell-Mann and Brueckner,[3] were to supplement it by an infinite set of higher-order diagrams involving successive interactions all with the same momentum transfer p.

APPENDIX D. NUMERICAL EVALUATION OF INTEGRALS

The numerical integration subroutines used to evaluate the integrals (7) and (8) employed Simpson's rule, the integrands being tabulated at intervals Δp of 0.2 from $p=0$ to $p=2$, of 0.5 from $p=2$ to $p=5$, and of 1 from $p=5$ to $p=13$. Although the logarithmic singularities at $p=p'$ in i_1 and i_2 are integrable, they require special treatment in a numerical evaluation. The method adopted was to represent $g(p')/[1-g^2(p')]$ and $g^2(p')/[1-g^2(p')]$ by quadratics in the vicinity of $p'=p$, and then to use the formula

$$\int_{p-\delta}^{p+\delta} p' \ln\left|\frac{p+p'}{p-p'}\right|[B+C(p'-p)+D(p'-p)^2]dp'$$
$$=\int_{-\delta}^{\delta}(x+p)[\ln(2p)-\ln|x|+(x/2p)-\tfrac{1}{2}(x/2p)^2+\cdots](B+Cx+Dx^2)dx$$
$$=2Bp\delta\ln(2p)+\tfrac{2}{3}[(C+Dp)\ln(2p)+\tfrac{1}{8}p^{-1}(3B+4Cp)]\delta^3-2Bp\delta(\ln\delta-1)-\tfrac{2}{3}(C+Dp)\delta^3(\ln\delta-\tfrac{1}{3})+O(\delta^5,\delta^5\ln\delta), \quad \text{(D1)}$$

the parameter δ being taken equal to the step length used in the numerical integrations. This method permitted $i_1(p)$ and $i_2(p)$ to be calculated at the interior points of the intervals $0\leqslant p\leqslant 2$, $2\leqslant p\leqslant 5$, and $5\leqslant p\leqslant 13$ (see above); the values of i_1 and i_2 at the boundary points were then determined by quadratic interpolation.[30] The error of these numerical calculations was checked for the special case $g=g^{(0)}$ [hence $i_1=i_1^{(0)}$, $i_2=i_2^{(0)}$] by comparison with an independent calculation[31] of $i_1^{(0)}$ and $i_2^{(0)}$, and found to be of the order of 1%; another useful check valid even when r_s is not small was provided by the relationship (A4) between f and the asymptotic behavior of i_2; this relationship was found to be satisfied to within a few percent. Since the accuracy of the numerical integrations involved in calculating f, i_3, and u_0 was even greater,[32] the values of u_0 given in Table I should be rigorous upper bounds to within a few percent.

[30] The accuracy of (D1) was found to be insufficient at the point $p=0.2$ because $g/(1-g^2)$ and $g^2/(1-g^2)$ cannot be represented by quadratics in a neighborhood of $p'=0$; hence, $i_1(0.2)$ and $i_2(0.2)$ were also obtained by quadratic interpolation [note from Appendix A that $i_1(0)=i_2(0)=0$ and that i_1 and i_2 are quadratic in p for small p].

[31] The functions $i_1^{(0)}(p)$ and $i_2^{(0)}(p)$ were calculated from (A1) by numerical integration, the integrations through the singularity at $x=1$ being effected by an analytic formula obtained by integration of the leading terms of the expansions of the integrands about $x=1$. The accuracy of the calculations was checked by the familiar method of doubling the step length and by comparison with the known low- and high-momentum behavior (16).

[29] This diagram is of second order only in the plasmon representation, obtained by the Bogoliubov transformation; in a free-particle representation, the quartet Hamiltonian first contributes in sixth order.

[32] This was checked by doubling the step length.