Singularities of the Five-Point Function in Perturbation Theory

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A rational parametrization of the leading portion of the Landau manifold for the five-point function is developed. The parameters chosen are the integration parameters. If these parameters satisfy a normalization condition, then the physical-sheet singularities are found when all the signs of the real part of all the parameters are the same.

ONE of the chief handicaps in present dispersion-theoretical calculations is the inability to treat three-particle intermediate states. In addition, production processes are beyond the scope of present techniques. As a first step in the incorporation of these states in dispersion calculations, some information is required about the singularities of the five-point function. In this note the location of the singularities of the simplest five-point diagram, shown in Fig. 1, are determined. The reasons for expecting more complicated diagrams to have a similar analytic structure are summarized by Eden.[1]

The integral representing this function has been given as a multiple integral by Cook and Tarski[2] and their set of five independent scalar variables called x_{13}, x_{14}, x_{24}, x_{25}, and x_{35} will be used. They also use quantities $x_{i\,i+1}$ that are simply related to the masses and x_{ii} is defined to be unity; this notation will also be maintained.

The singularities of the amplitude under consideration lie on the Landau surface given by $\det(x_{ij})=0$. The problem is to determine which points of the surface are singular on the physical sheet. As a first step a rational parameterization is introduced for the dynamical variables that satisfy $\det(x_{ij})=0$. If $\det(x_{ij})=0$, then the matrix with elements x_{ij} must have a characteristic vector belonging to characteristic value zero.

The components of this vector are denoted by u_i.[3] The equation that this vector solves, $x_{ij}u_j=0$, may be rewritten in the following way: Only those terms that involve a u and an x_{ij} that is a dynamical variable are retained on the left-hand side, all others are transposed to the right-hand side. Finally, the five dynamical variables may be regarded as components of an unknown vector, and a matrix of coefficients may be constructed from the u's. This inhomogeneous system of equations may be solved for the dynamical variables in terms of the u's and the masses. The solution is given by

$$x_{13}=(-u_1^2-u_2^2-u_3^2+u_4^2+u_5^2-2u_1u_2x_{12} -2u_2u_3x_{23}+2u_4u_5x_{45})/2u_1u_3, \quad (1)$$

and the four other permutations. If each u is multiplied by the same number z the value of x is unchanged. A unique choice of u's for each x will be specified later.

[1] R. J. Eden, Phys. Rev. **119**, 1763 (1960).
[2] L. Cook and J. Tarski, J. Math. Phys. **3**, 1 (1962).
[3] The u_i are the integration parameters in the multiple integral defining the amplitude.

The same treatment may be carried out for the three-point function and it yields

$$x_{12}=(-u_1^2-u_2^2+2u_3^2)/2u_1u_2,$$

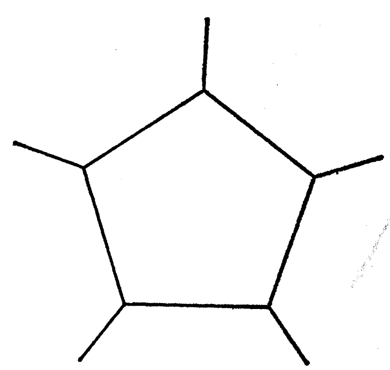
which is essentially the law of cosines and suggests the usual trigonometric parameterization $x_{ij}=\cos\gamma_{ij}$. For the four-point function there appears to be no natural way to choose four of the invariants as dynamical variables and so the present procedure does not appear applicable.

As a consequence of the rational parameterization of the Landau surface, the 4×4 principal minors of $\det(x_{ij})$ have a simple structure on this surface that will now be exhibited. The equation $\det(x_{ij})=0$ is quadratic in any of its variables and may be solved by the quadratic formula. A theorem of Tarski[4] states that this solution takes the form

$$x_{ij}=[-B_{ij}+(L_i)^{\frac{1}{2}}(L_j)^{\frac{1}{2}}]/A_{ij}, \quad (2)$$

where A_{ij} and B_{ij} are minors and L_i and L_j are the 4×4 principal minors formed by leaving out the ith row and column and the jth row and column, respectively. If the rational parameterization (1) found above is substituted in (2) on both sides of the equation, $(L_i)^{\frac{1}{2}}(L_j)^{\frac{1}{2}}$ must also be rational. This implies that L_i has the form f_i^2g when it is expressed in terms of the u's and masses. Detailed calculation confirms this and f_i is simply u_i. The function g may be calculated in a straightforward way and is a rational function of the u's and masses. It is given in the appendix. Its one significant property is that the numerator is of deg 8 in the u's and the de-

FIG. 1. The graph for the five-point function.



[4] J. Taraki, J. Math. Phys. **1**, 149 (1960).

nominator is of deg 10. That the numerator and denominator differ in degree by 2 follows since the numerator of L_i itself must be of equal degree and a factor u_i^2 has to be extracted from the numerator. The freedom of choice for the u_i may now be exploited to choose a standard set of u's such that $g=1$. This amounts to solving an equation $Pz^2=Q$, where P is the denominator of g and Q the numerator at the point u, and replacing u by $u'=zu$. In the three-point function the trigonometric parameterization specifies that $g=1$. In the ensuing it is assumed that only those points u for which $g=1$ are chosen; these of course give all distinct x that come from (1).

Under the assumption $g=1$, (2) takes the form

$$x_{ij}=[-B_{ij}+(u_i^2)^{\frac{1}{2}}(u_j^2)^{\frac{1}{2}}]/A_{ij}. \tag{3}$$

The expression $(u_i)^{\frac{1}{2}}$ now occurs in (n-1) places if all possible equations of the form (2) are written down. It has been shown that it is possible to choose this radical in the same way each time it occurs.[5] That is, $(u_i^2)^{\frac{1}{2}}$ may always be chosen to be $+u_i$ or $-u_i$ in each place it occurs but not a mixture of both conventions. From symmetry it can be shown that this sign must be taken in the same way for all i; that is, if $(u_1^2)^{\frac{1}{2}}$ is taken to be $-u_1$ then $(u_2^2)^{\frac{1}{2}}=-u_2$, etc. For convenience, $(u_i^2)^{\frac{1}{2}}$ is chosen to be u_i. It has also been shown that those points of the five-point function that are singular on the physical sheet are just those for which the real parts of all the radicals have the same sign.[5] Thus, those points of the Landau surface generated by parameter u all having positive real parts are the physical-sheet singular points. There are some points u that will lie in cuts and will be singular in some limits and not in others. These are points such that the real part of some u vanishes. There are also some points x on the leading portion of the Landau curve not parameterized by (1). These points are common to the four- and five-point Landau surface and must be treated separately.

APPENDIX

If the rational parameterization (1) is substituted in the minor L_5, the following expression is found:

$$\begin{aligned}
16u_1^2u_2^2u_3^2u_4^2L_5={}& u_i^8-4u_i^6(u_{i+2}^2+u_{i+3}^2)+6u_i^4u_{1+2}^4-2u_i^4u_{i+1}^4+8u_i^4u_{i+2}^2u_{i+3}^2+4u_i^4u_{i+1}^2u_{i+4}^2-4u_i^4(u_{i+1}^2u_{i+3}^2\\
&+u_{i+2}^2u_{i+4}^2)+u_iu_{i+1}x_{i\ i+1}[4u_i^6+4u_{i+1}^6-4u_{i+2}^6-4u_{i+3}^6-4u_{i+4}^6-4u_i^4(u_{i+1}^2+3u_{i+2}^2+3u_{i+3}^2\\
&-u_{i+4}^2)-4u_{i+1}^4(u_i^2-u_{i+2}^2+3u_{i+3}^2+3u_{i+4}^2)+4u_{i+2}^4(3u_i^2-u_{i+1}^2+u_{i+3}^2+u_{i+4}^2)+4u_{i+3}^4(3u_i^2\\
&+3u_{i+1}^2+u_{i+2}^2+u_{i+4}^2)-4u_{i+4}^4(u_i^2-3u_{i+1}^2-u_{i+2}^2-u_{i+3}^2)+8u_i^2u_{i+1}^2(u_{i+2}^2-u_{i+3}^2+u_{i+4}^2)\\
&+8u_{i+2}^2u_{i+3}^2(u_i^2-u_{i+1}^2-u_{i+4}^2)+8u_{i+4}^2(u_{i+1}^2u_{i+3}^2-u_{i+1}^2u_{i+2}^2-u_i^2u_{i+3}^2-u_i^2u_{i+2}^2)]\\
&+u_i^2u_{i+1}^2x_{i\ i+1}^2[4u_j^4-8u_i^2(u_{i+1}^2+u_{i+2}^2+u_{i+3}^2-u_{i+4}^2)+8u_{i+1}^2(u_{i+2}^2-u_{i+3}^2-u_{i+4}^2)-8u_{i+2}^2(u_{i+3}^2\\
&+u_{i+4}^2)-8u_{i+3}^2u_{i+4}^2]+u_iu_{i+1}^2u_{i+2}x_{i\ i+1}x_{i+1\ i+2}[16(-u_i^4+u_{i+1}^4-u_{i+2}^4+u_{i+3}^4+u_{i+4}^4)\\
&+32(u_i^2u_{i+2}^2-u_{i+1}^2u_{i+3}^2-u_{i+1}^2u_{i+4}^2)]+u_iu_{i+1}u_{i+2}u_{i+3}x_{i\ i+1}x_{i+2\ i+3}[-8(u_i^4+u_{i+1}^4+u_{i+2}^4\\
&+u_{i+3}^4-u_{i+4}^4)+16u_i^2(u_{i+1}^2+u_{i+2}^2+u_{i+3}^2)-16u_{i+1}^2(u_{i+2}^2+u_{i+3}^2)+16u_{i+3}^2u_{i+4}^2]\\
&+u_i^2u_{i+1}^3u_{i+2}x_{i\ i+1}^2x_{i+1\ i+2}16(-u_i^2+u_{i+1}^2+u_{i+2}^2-u_{i+3}^2-u_{i+4}^2)+u_i^3u_{i+1}^2u_{i+4}x_{i\ i+1}^2x_{i\ i+4}16(u_i^2\\
&-u_{i+1}^2-u_{i+2}^2-u_{i+3}^2+u_{i+4}^2)+u_iu_{i+1}^2u_{i+2}^2u_{i+3}x_{i\ i+1}x_{i+1\ i+2}x_{i+2\ i+3}32(u_i^2-u_{i+1}^2-u_{i+2}^2+u_{i+3}^2)\\
&+u_iu_{i+1}^2u_{i+2}u_{i+3}u_{i+4}x_{i\ i+1}x_{i+1\ i+2}x_{i+3\ i+4}16(u_i^2-u_{i+1}^2+u_{i+2}^2+u_{i+3}^2+u_{i+4}^2)\\
&+16u_i^2u_{i+1}^4u_{i+2}^2x_{i\ i+1}^2x_{i+1\ i+2}^2-32u_i^3u_{i+1}^3u_{i+2}u_{i+4}x_{i\ i+1}^2x_{i\ i+4}x_{i+1\ i+2}\\
&+32u_iu_{i+1}^2u_{i+2}^2u_{i+3}^2u_{i+4}x_{i\ i+1}x_{i+1\ i+2}x_{i+2\ i+3}x_{i+3\ i+4},
\end{aligned}$$

where i is summed from one to five *modulo* 5. Since the right-hand side of this equation is symmetric under cyclic permutations of the indices, g may be identified as this expression divided by $u_1^2u_2^2u_3^2u_4^2u_5^2$ and f_i as u_i.

[5] F. R. Halpern (to be published).