Scattering of Electromagnetic Plane Waves from Inhomogeneous Spherically Symmetric Objects*†

PHILIP J. WYATT
Plasmadyne Corporation, Santa Ana, California
(Received March 19, 1962)

Unlike its nuclear optical model counterpart, the problem of the scattering of light from spherically symmetric inhomogeneous absorbing particles cannot in general be predicted directly from solutions of the usual wave equation. An exact theory is derived from Maxwell's equations and the resultant scattering coefficients are found to depend upon the solutions of two differential equations, one of which is of the familiar Schrödinger form. Specific calculations based on the Green-Wyatt form factor illustrate the dependence of the polarization and scattered intensities on surface diffuseness. The results are compared with similar nuclear calculations.

I. INTRODUCTION

SINCE the advent of high-speed digital computers, the effects upon particle scattering of a great variety of nuclear potential wells have been examined in detail. Before the development of these machines the calculations involved were often prohibitive and various approximation techniques were necessarily devised. In the field of electromagnetic scattering phenomena, on the other hand, only homogeneous objects of sharp boundary have been heretofore thoroughly examined.[1] Despite this "simple" geometry, the calculational aspects are in general quite complicated and extensive analyses[2,3] still require the use of high-speed computers.

Insofar as inhomogeneous objects are concerned, Montroll, Hart, and Greenberg[4–6] developed various approximations which proved quite interesting and useful in the interpretation of the scattering of light from diffuse, or "soft," particles. Schiff[7] subsequently re-examined the problem from a very formal point of view (especially in the "high-energy" limit) and has, more recently,[8] concluded that there are many practical applications relating to the study of the interaction of diffuse particles with electromagnetic waves.

The present paper is concerned with the derivation, and reduction for numerical solution, of an exact theory on which basis the characteristic scattering of plane polarized light from arbitrary spherically symmetric scatterers may be deduced. For this initial investigation the scatterers are assumed to be composed of media whose optical properties, though otherwise quite general, are assumed to vary continuously with the possible exception of their outermost surfaces (which may be discontinuous with respect to the surrounding medium).

Section II describes the Maxwell formalism to be solved and the scattering coefficients are derived. The notation and methods used are quite similar to Born's[9] treatment. Section III discusses the calculation of the scattering coefficients obtained in Sec. II and summarizes the usual scattering quantities expressible in terms of them. In Sec. IV a specific example of a diffuse surface is considered and both the scattered differential intensities and polarization are examined as a function of surface thickness (diffuseness) and compared with the similar nuclear problem. Conclusions and applications are discussed in Sec. V.

II. SOLUTIONS OF MAXWELL'S EQUATIONS AND DERIVATION OF THE SCATTERING COEFFICIENTS

If the time dependence of the steady state fields be described as usual by the factor $e^{-i\omega t}$, Maxwell's equations reduce to the forms

$$\operatorname{div}(\epsilon\mathbf{E})=0, \qquad \operatorname{div}\mathbf{H}=0, \tag{1}$$

$$\operatorname{curl}\mathbf{E}=k_2\mathbf{H}, \quad \operatorname{curl}\mathbf{H}=-k_1\mathbf{E}, \tag{2}$$

where $k_1=(i\omega/c)(\epsilon+i4\pi\sigma/\omega)$ and $k_2=i\omega/c$ are related to the propagation constant, k, by $k^2=-k_1k_2$. Equations (1) and (2) are valid both inside and outside the spherically symmetric scatterer, which is assumed to have a magnetic permeability of unity.

At the particle surface the tangential components of $\mathbf{E}$ and $\mathbf{H}$ must be continuous. Thus, if the superscripts I and II refer to the surrounding medium and particle medium, respectively, in a spherical coordinate system the boundary conditions reduce to the simple form,

$$\begin{aligned} E_\theta^{\mathrm{I}}&=E_\theta^{\mathrm{II}}, \quad H_\theta^{\mathrm{I}}=H_\theta^{\mathrm{II}},\\ E_\phi^{\mathrm{I}}&=E_\phi^{\mathrm{II}}, \quad H_\phi^{\mathrm{I}}=H_\phi^{\mathrm{II}}, \end{aligned} \tag{3}$$

---

* Some of the features of this paper were presented at the Spring meeting of the American Physical Society [Bull. Am. Phys. Soc. 7, 358 (1962)].

† Most of this work was performed while the author was at the Aeronutronic Division of Ford Motor Company, Newport Beach, California.

[1] See H. C. Van de Hulst, *Light Scattering by Small Particles* (John Wiley & Sons, Inc., New York, 1957).

[2] P. J. Wyatt and V. R. Stull (unpublished, 1960).

[3] D. Deirmendjian, R. Clasen, and W. Viezee, J. Opt. Soc. Am. **51**, 620 (1961).

[4] R. W. Hart and E. W. Montroll, J. Appl. Phys. **22**, 376 (1951).

[5] R. W. Hart and E. W. Montroll, J. Acoust. Soc. Am. **23**, 373 (1951).

[6] E. W. Montroll and J. M. Greenberg, Phys. Rev. **86**, 889 (1952).

[7] L. I. Schiff, Phys. Rev. **104**, 1481 (1906).

[8] L. I. Schiff, J. Opt. Soc. Am. **52**, 140 (1962).

[9] M. Born, *Optik* (Verlag Julius Springer, Berlin, 1933). Chap. VI, pp. 274–298. See also M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, New York, 1959).

at $r=b$, where b is the distance of the interface from the particle center. In the following it will be assumed that the conductivity in region I is zero; i.e., $\sigma^{\mathrm{I}}=0$.

The solutions of Eqs. (1) and (2) will now be represented as the superposition of two linearly independent fields; viz., the so-called transverse magnetic and transverse electric fields: (${}^e\mathbf{E}$, ${}^e\mathbf{H}$) and (${}^m\mathbf{E}$, ${}^m\mathbf{H}$), respectively. The transverse magnetic field solution is that field for which $H_r=0$. Since $E_r\neq 0$ in the former case, the field is often referred to as the electric field; hence the superscript "e." Similar remarks apply to the magnetic field (i.e., transverse electric) indicated by the superscript "m."

Consider first the transverse magnetic field (${}^e\mathbf{E}$, ${}^e\mathbf{H}$). The vanishing of the radial component of the curl ${}^e\mathbf{E}$ suggests that ${}^eE_\theta$ and ${}^eE_\phi$ may be represented in terms of the gradient of a scalar function; viz.

$$ {}^eE_\theta=(1/r)\partial U/\partial\theta, \tag{4}$$

$$ {}^eE_\phi=(1/r\sin\theta)\partial U/\partial\phi. \tag{5}$$

The additional requirements that $\operatorname{div}\mathbf{D}=\operatorname{div}\mathbf{H}=0$ may be satisfied by choosing

$$U=(1/k_1)(\partial/\partial r)(r\,{}^e\Omega), \tag{6}$$

where ${}^e\Omega(r,\theta,\phi)$ is a scalar function which must be a solution of

$$\nabla^2\,{}^e\Omega-\frac{1}{k_1r}\frac{\partial k_1}{\partial r}\frac{\partial}{\partial r}(r\,{}^e\Omega)+k^2\,{}^e\Omega=0. \tag{7}$$

Note that were it not for the second term, Eq. (7) would resemble the "usual" wave equation.

Similar considerations for the transverse electric field (${}^m\mathbf{E}$, ${}^m\mathbf{H}$) show that if one chooses

$$ {}^mH_\theta=(1/r)\partial\psi/\partial\theta, \tag{8}$$

$$ {}^mH_\phi=(1/r\sin\theta)\partial\psi/\partial\phi, \tag{9}$$

where

$$\psi=(1/k_2)(\partial/\partial r)(r\,{}^m\Omega), \tag{10}$$

then Eqs. (1) and (2) may all be satisfied provided ${}^m\Omega$ is a solution of

$$\nabla^2\,{}^m\Omega+k^2\,{}^m\Omega=0. \tag{11}$$

Equation (11), of course, is of a form similar to the Schrödinger equation and its solutions are those of interest in the electromagnetic case when the scatterer is homogeneous. Further analogies with nuclear scattering will be discussed in Sec. IV.

In terms of the potential functions ${}^e\Omega$ and ${}^m\Omega$ the fields may now be written in spherical coordinates as

$$E_r={}^eE_r+{}^mE_r=\frac{1}{k_1}\left[\frac{\partial^2}{\partial r^2}-\frac{1}{k_1}\frac{\partial k_1}{\partial r}\frac{\partial}{\partial r}+k^2\right]r\,{}^e\Omega, \tag{12}$$

$$E_\theta={}^eE_\theta+{}^mE_\theta=\frac{1}{k_1r}\frac{\partial^2}{\partial\theta\partial r}(r\,{}^e\Omega)+\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\,{}^m\Omega, \tag{13}$$

$$E_\phi={}^eE_\phi+{}^mE_\phi=\frac{1}{k_1r\sin\theta}\frac{\partial^2}{\partial\phi\partial r}(r\,{}^e\Omega)-\frac{\partial}{\partial\theta}\,{}^m\Omega, \tag{14}$$

$$H_r={}^eH_r+{}^mH_r=\frac{1}{k_2}\left[\frac{\partial^2}{\partial r^2}+k^2\right]r\,{}^m\Omega, \tag{15}$$

$$H_\theta={}^eH_\theta+{}^mH_\theta=\frac{-1}{\sin\theta}\frac{\partial}{\partial\phi}\,{}^e\Omega+\frac{1}{k_2r}\frac{\partial^2}{\partial\theta\partial r}(r\,{}^m\Omega), \tag{16}$$

$$H_\phi={}^eH_\phi+{}^mH_\phi=\frac{\partial}{\partial\theta}\,{}^e\Omega+\frac{1}{k_2r\sin\theta}\frac{\partial^2}{\partial\phi\partial r}(r\,{}^m\Omega). \tag{17}$$

Since k is a function of r only, Eqs. (7) and (11) are readily solved using the method of separation of variables. Thus, letting

$$ {}^{e,m}\Omega={}^{e,m}R(r)\Theta(\theta)\Phi(\phi), \tag{18}$$

the Θ and Φ solutions of Eqs. (7) and (11) are, as expected,

$$\Theta=c_{lm}P_l{}^m(\cos\theta) \tag{19}$$

and

$$\Phi=a_m\cos m\phi+b_m\sin m\phi, \tag{20}$$

where a_m, b_m, and c_{lm} are constants and $P_l{}^m(\cos\theta)$ is the usual associated Legendre polynomial. The radial part of Eq. (7) satisfies the differential equation

$$\frac{d^2W_l}{dr^2}-\frac{2}{k}\frac{dk}{dr}\frac{dW_l}{dr}+\left[k^2-\frac{l(l+1)}{r^2}\right]W_l=0, \tag{21}$$

where $W=r\,{}^eR$, while the radial solution of Eq. (11) satisfies

$$d^2G_l/dr^2+[k^2-l(l+1)/r^2]G_l=0, \tag{22}$$

where $G=r\,{}^mR$.

It is of interest to note that Eqs. (21) and (22) were derived in a different manner by Nomura and Takaku[10] who were concerned with the propagation of dipole radiation in an inhomogeneous atmosphere at wavelengths small compared with the size of the earth.

Returning now to the problem of the scattering of a plane wave from an inhomogeneous spherically symmetric scatterer, the incident plane wave propagating in the positive z direction may be represented by its six components in spherical coordinates:

$$\left.\begin{aligned}
E_r{}^i&=e^{ik^{\mathrm{I}}r\cos\theta}\sin\theta\cos\phi,\\
H_r{}^i&=(ik^{\mathrm{I}}/k_2{}^{\mathrm{I}})e^{ik^{\mathrm{I}}r\cos\theta}\sin\theta\sin\phi,\\
E_\theta{}^i&=e^{ik^{\mathrm{I}}r\cos\theta}\cos\theta\cos\phi,\\
H_\theta{}^i&=(ik^{\mathrm{I}}/k_2{}^{\mathrm{I}})e^{ik^{\mathrm{I}}r\cos\theta}\cos\theta\sin\phi,\\
E_\phi{}^i&=-e^{ik^{\mathrm{I}}r\cos\theta}\sin\phi,\\
H_\phi{}^i&=(ik^{\mathrm{I}}/k_2{}^{\mathrm{I}})e^{ik^{\mathrm{I}}r\cos\theta}\cos\phi.
\end{aligned}\right\} \tag{23}$$

The electric vector has been chosen to lie in the x direction and the magnetic vector in the y direction.

Since

$$e^{ik^{\mathrm{I}}r\cos\theta}=\sum_{l=0}^{\infty}i^l\frac{(2l+1)}{k^{\mathrm{I}}r}\psi_l(k^{\mathrm{I}}r)P_l(\cos\theta), \tag{24}$$

[10] Y. Nomura and K. Takaku, Tohoku Research Institutes, Res. Inst. Elec. Comm. **7B**, 107 (1955).

[where $\psi_l(\rho)$ is the Riccati-Bessel function, $(\pi\rho/2)^{\frac{1}{2}} \times J_{l+\frac{1}{2}}(\rho)$], the transverse magnetic potential contribution to the incident field is readily shown to be given by

$$r\,{}^e\Omega^i = \frac{k_1^{\mathrm{I}}}{(k^{\mathrm{I}})^2} \sum_{l=1}^{\infty} \frac{i^{l-1}(2l+1)}{l(l+1)} \psi_l(k^{\mathrm{I}}r) P_l^1(\cos\theta)\cos\phi. \quad (25)$$

The transverse electric part, on the other hand, is representable by

$$r\,{}^m\Omega^i = \frac{1}{k^{\mathrm{I}}} \sum_{l=1}^{\infty} \frac{i^{l}(2l+1)}{l(l+1)} \psi_l(k^{\mathrm{I}}r) P_l^1(\cos\theta)\sin\phi. \quad (26)$$

The components of the fields, Eqs. (23), may easily be deduced from Eqs. (25) and (26) by using Eqs. (12) to (17) and noting that k is constant in region I.

Two other waves are required to completely describe the interaction between the incident plane wave and the scatterer. These are the so-called transmitted wave (i.e., the electromagnetic field within the scatterer) and the scattered wave; the latter is needed to calculate most of the quantities of physical interest.

Since k is a function of r in region II, the potentials of the transmitted waves are representable by

$$r\,{}^e\Omega^t = \frac{k_1^{\mathrm{II}}}{(k^{\mathrm{II}})^2} \sum_{l=1}^{\infty} \frac{i^{l-1}(2l+1)}{l(l+1)}\, {}^eA_l W_l(r) P_l^1(\cos\theta)\cos\phi \quad (27)$$

and

$$r\,{}^m\Omega^t = \frac{i}{k^{\mathrm{II}}} \sum_{l=1}^{\infty} \frac{i^{l-1}(2l+1)}{l(l+1)}\, {}^mA_l G_l(r) P_l^1(\cos\theta)\sin\phi, \quad (28)$$

where $W_l(r)$ and $G_l(r)$ are those solutions of Eqs. (21) and (22), respectively, which are regular at the origin and ${}^eA_l$ and ${}^mA_l$ are constants.

At large distances from the scatterer the scattered wave must be of the form e^{ikr}/r which suggests that the potentials of the scattered wave be expanded as

$$r\,{}^e\Omega^s = \frac{-1}{k_2^{\mathrm{I}}} \sum_{l=1}^{\infty} {}^eB_l \frac{i^{l-1}(2l+1)}{l(l+1)} \zeta_l^{(1)}(k^{\mathrm{I}}r) P_l^1(\cos\theta)\cos\phi \quad (29)$$

and

$$r\,{}^m\Omega^s = \frac{i}{k^{\mathrm{I}}} \sum_{l=1}^{\infty} {}^mB_l \frac{i^{l-1}(2l+1)}{l(l+1)} \zeta_l^{(1)}(k^{\mathrm{I}}r) P_l^1(\cos\theta)\sin\phi, \quad (30)$$

where $\zeta_l^{(1)}(\rho)$ is the Riccati-Hankel function of the first kind given by

$$\zeta_l^{(1)}(\rho) = (\pi\rho/2)^{\frac{1}{2}} H_{l+\frac{1}{2}}^{(1)}(\rho) \quad (31)$$

and ${}^eB_l$ and ${}^mB_l$ are constants. The ζ_l's are, of course, acceptable solutions of either Eq. (21) or Eq. (22) in region I.

The complex index of refraction is usually defined by the relation

$$n = k^{\mathrm{II}}/k^{\mathrm{I}}. \quad (32)$$

Noting that $k^{\mathrm{I}} = 2\pi/\lambda^{\mathrm{I}}$ is a real quantity, it is convenient to introduce the variable $\rho = k^{\mathrm{I}}r$ into Eqs. (21) and (22). Hence

$$\frac{d^2W_l}{d\rho^2} - \frac{2}{n}\frac{dn}{d\rho}\frac{dW_l}{d\rho} + \left[n^2 - \frac{l(l+1)}{\rho^2}\right] W_l = 0 \quad (33)$$

and

$$d^2G_l/d\rho^2 + [n^2 - l(l+1)/\rho^2] G_l = 0. \quad (34)$$

Next, one introduces the boundary conditions to obtain the coefficients ${}^eA_l$, ${}^mA_l$, ${}^eB_l$, and ${}^mB_l$. Expressed in terms of the potentials ${}^e\Omega$ and ${}^m\Omega$, Eqs. (3) become

$$(1/k_1)(\partial/\partial\rho)(\rho\,{}^e\Omega),\ {}^m\Omega,\ {}^e\Omega,\ (1/k_2)(\partial/\partial\rho)(\rho\,{}^m\Omega) \quad (35)$$

continuous at $\rho = k^{\mathrm{I}}b$. Equations (35) reduce to four simultaneous linear equations involving the four coefficients ${}^eA_l$, ${}^mA_l$, ${}^eB_l$, ${}^mB_l$. Specifically, one finds for the scattering coefficients

$${}^eB_l = -\frac{\psi_l(x)}{\zeta_l^{(1)}(x)} \frac{n^2W_l(x)D_l - W_l'(x)}{W_l'(x) - n^2W_l(x)\Gamma_l} \quad (36)$$

and

$${}^mB_l = -\frac{\psi_l(x)}{\zeta_l^{(1)}(x)} \times \frac{G_l'(x) - (1/n)(dn/d\rho)G_l(x) - G_lD_l}{G_l(x)\Gamma_l - G_l'(x) + (1/n)(dn/d\rho)G_l(x)}, \quad (37)$$

where $x = k^{\mathrm{I}}b$,

$$D_l = \psi_l'(x)/\psi_l(x), \qquad \Gamma_l = \zeta_l^{(1)\prime}(x)/\zeta_l^{(1)}(x), \quad (38)$$

and all quantities involving ρ are evaluated at $\rho = x$. The primes refer to differentiation with respect to the argument of the indicated function.

It is of interest to compare Eqs. (36) and (37) with the usual Mie-Debye[9] coefficients; i.e., those obtained when n is constant throughout the scatterer. In this special case,

$$dn/d\rho = 0, \qquad G_l = W_l = \psi_l(nb), \qquad dG_l/dx = dW_l/dx = n\psi_l'(nb), \quad (39)$$

and Eqs. (36) and (37) reduce immediately to this form.

III. CALCULATION OF SCATTERING QUANTITIES

From the potentials ${}^e\Omega^s$ and ${}^m\Omega^s$ the scattered fields may be obtained directly using Eqs. (12) to (17). Thus, for example, one has

$$E_r^s = \frac{1}{(k^{\mathrm{I}})^2} \frac{\cos\phi}{r^2} \sum_{l=1}^{\infty} \frac{i^{l-1}(2l+1)}{l(l+1)}\, {}^eB_l \zeta_l^{(1)}(k^{\mathrm{I}}r) P_l^1(\cos\theta), \quad (40)$$

$$E_\theta^s = \frac{1}{k^{\mathrm{I}}} \frac{\cos\phi}{r} \sum_{l=1}^{\infty} \frac{i^{l-1}(2l+1)}{l(l+1)} [{}^eB_l \zeta_l^{(1)\prime}(k^{\mathrm{I}}r) P_l^{1\prime}(\cos\theta) \times \sin\theta - i\,{}^mB_l \zeta_l^{(1)}(k^{\mathrm{I}}r) P_l^1(\cos\theta)/\sin\theta], \quad (41)$$

$$E_\phi{}^s=\frac{-1}{k^{\mathrm{I}}}\frac{\sin\phi}{r}\sum_{l=1}^{\infty}\frac{i^{l-1}(2l+1)}{l(l+1)}[{}^eB_l\zeta_l{}^{(1)\prime}(k^{\mathrm{I}}r)P_l{}^1(\cos\theta)/\sin\theta-i\,{}^mB_l\zeta_l{}^{(1)}(k^{\mathrm{I}}r)P_l{}^{1\prime}(\cos\theta)\sin\theta], \quad (42)$$

where similar expressions may be obtained for the **H** components. Insofar as one is usually interested in relative scattered intensities at distances large compared to the scatterer size, Eqs. (41) and (42) will be sufficient in their asymptotic forms. The longitudinal component, Eq. (40), may of course be neglected at large r because of its rapid fall off compared to the transverse components. If one introduces the asymptotic forms of $\zeta_l{}^{(1)}$ and $\zeta_l{}^{(1)\prime}$

$$\begin{aligned}\zeta_l{}^{(1)}(x)&\approx(-i)^{l+1}e^{ix},\\ \zeta_l{}^{(1)\prime}(x)&\approx(-i)^{l}e^{ix},\end{aligned} \quad (43)$$

into these equations, the scattered fields may be written

$$E_\theta{}^s=\frac{i\cos\phi}{k^{\mathrm{I}}r}e^{ik^{\mathrm{I}}r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\times[{}^eB_lP_l{}^{1\prime}(\cos\theta)\sin\theta-{}^mB_lP_l{}^1(\cos\theta)/\sin\theta], \quad (44)$$

$$E_\phi{}^s=\frac{i\sin\phi}{k^{\mathrm{I}}r}e^{ik^{\mathrm{I}}r}\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}\times[{}^eB_lP_l{}^1(\cos\theta)/\sin\theta-{}^mB_lP_l{}^{1\prime}(\cos\theta)\sin\theta]. \quad (45)$$

Hence,

$$|E_\theta{}^s|^2=I_{\shortparallel}\cos^2\phi \quad\text{and}\quad |E_\phi{}^s|^2=I_\perp\sin^2\phi, \quad (46)$$

where

$$I_{\shortparallel}=\frac{1}{(k^{\mathrm{I}}r)^2}\left|\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}[{}^eB_l\pi_l(\cos\theta)+{}^mB_l\tau_l(\cos\theta)]\right|^2, \quad (47)$$

$$I_\perp=\frac{1}{(k^{\mathrm{I}}r)^2}\left|\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}[{}^eB_l\tau_l(\cos\theta)+{}^mB_l\pi_l(\cos\theta)]\right|^2, \quad (48)$$

and the π's and τ's are those used by Van de Hulst[1]; viz.

$$\pi_l(\cos\theta)=P_l{}^1(\cos\theta)/\sin\theta, \quad (49)$$

$$\tau_l(\cos\theta)=-\sin\theta\frac{dP_l{}^1(\cos\theta)}{d(\cos\theta)}=-\sin\theta P_l{}^{1\prime}(\cos\theta). \quad (50)$$

Equation (46) refers to the field intensities at an angle (θ,ϕ) with respect to incident plane polarized wave. As one is often concerned with the scattering of unpolarized light, one must average over all directions of polarization. After averaging over the $\cos^2\phi$ and $\sin^2\phi$ terms, the intensity of light scattered at an angle θ arising from incident unpolarized light would be given by

$$I=\tfrac{1}{2}[I_{\shortparallel}+I_\perp]. \quad (51)$$

The polarization of the scattered light is often denoted by

$$P=(I_{\shortparallel}-I_\perp)/(I_{\shortparallel}+I_\perp). \quad (52)$$

Other important quantities are the extinction (or total), scattering, and absorption cross sections. The first is defined as the rate at which energy is dissipated to the rate at which it is incident per unit area of the scatterer and is readily shown to be given by

$$Q_{\mathrm{ext}}=\sigma_t=\frac{2\pi}{(k^{\mathrm{I}})^2}\operatorname{Re}\sum_{l=1}^{\infty}(2l+1)({}^eB_l+{}^mB_l). \quad (53)$$

The scattering cross section, which represents the ratio of the rate at which energy is scattered to the rate at which it is incident per unit area may be shown to be

$$Q_{\mathrm{scat}}=\sigma_s=\frac{2\pi}{(k^{\mathrm{I}})^2}\sum_{l=1}^{\infty}(2l+1)[|{}^eB_l|^2+|{}^mB_l|^2]. \quad (54)$$

The absorption cross section is found from Eqs. (53) and (54) by

$$Q_{\mathrm{abs}}=Q_{\mathrm{ext}}-Q_{\mathrm{scat}}. \quad (55)$$

Any of the important scattering quantities may be calculated if the coefficients ${}^eB_l$ and ${}^mB_l$ are determined. These in turn are readily found given the solutions of Eqs. (33) and (34) at the outermost surface b of the diffuse scatterer. In general, the index of refraction will be complex. If both its real and imaginary parts and their derivatives are continuous throughout the scatterer, the solutions of Eqs. (33) and (34) may be obtained, for example, using a Runge-Kutta[11] integration procedure. Usually W and G are replaced by W_1+iW_2 and G_1+iG_2, respectively, to reduce these equations to two pairs of coupled equations. Here W_1 and G_1 would represent the real parts of the solutions and W_2 and G_2 the pure imaginary parts. Since, furthermore, only the logarithmic derivatives of the functions W and G are required at the particle interface, under certain circumstances it is convenient to integrate the transformed differential equations expressed in terms of these ratios. In regard to the other important constituents of the scattering coefficients, the D_l's and Γ_l's are easily generated using the Infeld[12] formulas

$$\begin{aligned}D_l(x)&=(l/x-D_{l-1})^{-1}-l/x,\\ D_0(x)&=\cot x,\end{aligned} \quad (56)$$

and

$$\Gamma_l(x)=\Delta_l+is_l,$$

where

$$\begin{aligned}\Delta_l&=\frac{l/x-\Delta_{l-1}}{(l/x-\Delta_{l-1})^2+s_{l-1}{}^2}-l/x,\\ s&=\frac{s_{l-1}}{(l/x-\Delta_{l-1})^2+s_{l-1}{}^2},\\ \Delta_0&=0,\\ s_0&=1.\end{aligned} \quad (57)$$

[11] The classical review of the method may be found in the paper of S. Gill, Proc. Cambridge Phil. Soc. 47, 96 (1951).
[12] L. Infeld, Quart. J. Appl. Math. 5, 113 (1947).

When x itself is complex, Eq. (56) may be suitably divided into real and imaginary parts and recurrence relations similar to Eq. (57) may be deduced.

The Riccati-Bessel functions are related to the so-called spherical Bessel functions by

$$\begin{aligned}\psi_l(x)&=xj_l(x),\\ \zeta_l^{(1)}(x)&=xh_l^{(1)}(x)=x[j_l(x)+in_l(x)].\end{aligned}\tag{58}$$

Thus the ratio $\psi_l/\zeta_l^{(1)}$ involves only the spherical functions and may be generated by using the recurrence relations for the spherical functions; viz.,

$$z_{l-1}(x)+z_{l+1}(x)=\frac{2l+1}{x}z_l(x).\tag{59}$$

It is important to note, however, that the j_l's must be generated using a downward recurrence while the n_l's must be generated using an ascending recurrence. This is necessary, especially when using digital computers, because of the behavior of the j_l's and n_l's for large order and the roundoff errors associated therewith.[13]

IV. AN EXAMPLE: THE SCATTERING OF PLANE WAVES FROM A DIFFUSE PARTICLE

As an example of the application of the theory derived in Sec. II, consider the scattering of electromagnetic radiation from a spherical absorbing particle with a diffuse surface. In particular, consider a particle whose index of refraction is given by

$$n=n_1(\rho)+in_2(\rho),\tag{60}$$

where $\rho=2\pi r/\lambda^{\mathrm{I}}$ and

$$n_1=1+(n_{0R}-1)\xi(\rho),\tag{61}$$

$$n_2=n_{0I}\xi(\rho).\tag{62}$$

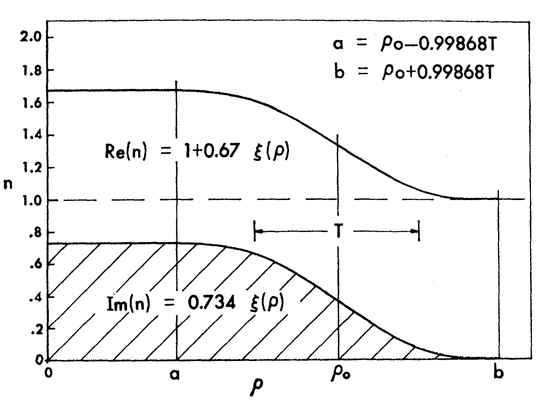

FIG. 1. The complex index of refraction for a particle with a diffuse surface. The form factor is of the Green-Wyatt type and the radial variation is in units of $\lambda^{\mathrm{I}}/2\pi$; i.e., $\rho=2\pi r/\lambda^{\mathrm{I}}$.

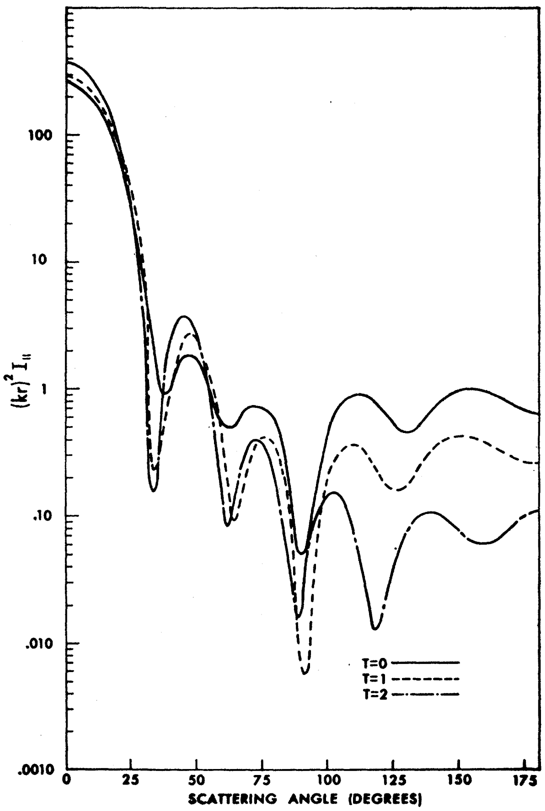

FIG. 2. The differential scattered intensity of light polarized parallel to the plane of scattering in units of $(kr)^2$ and shown for various surface thicknesses. $n_{0R}=1.67$, $n_{0I}=0.734$, and $\rho_0=5.0$.

$\xi(\rho)$ is the Green-Wyatt[14] form factor given by

$$\begin{aligned}\xi(\rho)&=1,\quad \rho<a\\ &=\tfrac{1}{2}-(15/16)Z+(10/16)Z^3-(3/16)Z^5,\quad a<\rho<b\\ &=0,\quad \rho>b\end{aligned}\tag{63}$$

where

$$Z=[\rho-\tfrac{1}{2}(b+a)]/[\tfrac{1}{2}(b-a)],$$

where a and b are related to the surface thickness by

$$\begin{aligned}a&=\rho_0-0.99868T\approx\rho_0-T,\\ b&=\rho_0+0.99868T\approx\rho_0+T.\end{aligned}\tag{64}$$

ρ_0 is the usual half-falloff radius (in units of $\lambda^{\mathrm{I}}/2\pi$) and the surface thickness T is defined as the 0.9 to 0.1 falloff distance. All of these features are illustrated in Fig. 1 with $n_{0R}=1.67$ and $n_{0I}=0.734$. These latter values, which will be used in the example to be presented presently, are quite typical of aerosols in the infrared region.

The particular form factor chosen has several noteworthy features. It provides an excellent representation of the Fermi function

$$\xi(\rho)=\{1+\exp[(\rho-\rho_0)/d]\}^{-1},\tag{65}$$

[13] T. Stegun and M. Abramowitz, Phys. Rev. 98, 1851 (1955).

[14] See talk by A. E. S. Green in *Proceedings of the International Conference on the Nuclear Optical Model, Florida State University Studies, No. 32* (The Florida State University, Tallahassee, 1959).

TABLE I. Total and scattering cross sections for a diffuse scatterer as a function of ρ_0, the half-falloff value, and the surface thickness T. The complex index of refraction is of the Green-Wyatt form with $n_{0R}=1.67$ and $n_{0I}=0.734$. All results are in units of πr_0^2.

	$\rho_0=1.0$			$\rho_0=5.0$			$\rho_0=10.0$	
T	σ_t	σ_s	T	σ_t	σ_s	T	σ_t	σ_s
0.5	2.319	0.6216	2.0	3.085	1.335	5.0	3.408	1.537
0.1	2.038	0.5664	1.0	2.741	1.256	1.0	2.468	1.239
0.01	1.985	0.5601	0.1	2.592	1.272	0.1	2.401	1.279
0.001	1.980	0.5596	0.01	2.579	1.269	0.01	2.395	1.277
0	1.979	0.5595	0	2.577	1.269	0	2.395	1.277

when the diffuseness distance d is related to the surface thickness T by $T=2d\ \ln 9$. This latter form has been of considerable interest in optical-model studies. The Green-Wyatt form is particularly well suited for numerical integration since its first and second derivatives are continuous throughout the range of definition ($0\leq\rho\leq b$) and the function itself vanishes for finite argument ($\rho=b$). The Fermi function, on the other hand, only vanishes at infinity; this feature occasionally necessitates lengthy numerical integrations.

When ρ_0 is small ($\lesssim 1$) the dependence of the various cross sections upon the surface thickness, T, is found

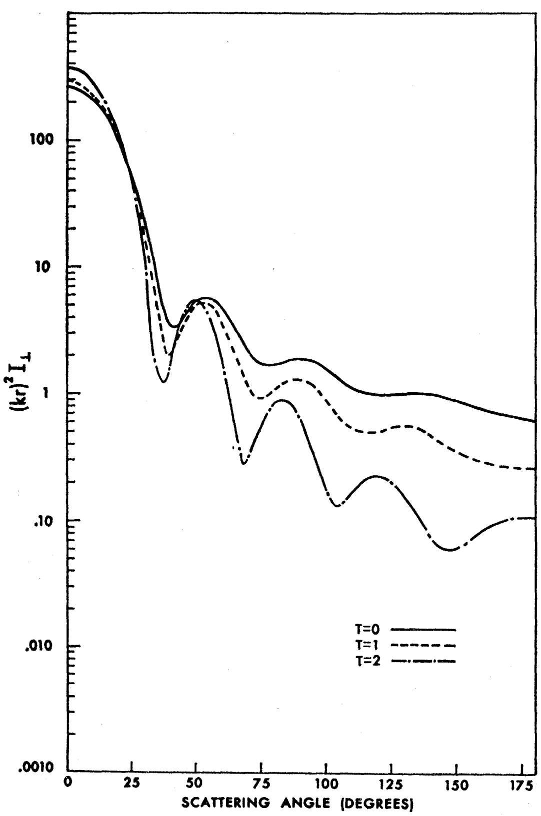

FIG. 3. The differential scattered intensity of light polarized perpendicular to the plane of scattering in units of $(kr)^2$ and shown for various surface thicknesses. $n_{0R}=1.67$, $n_{0I}=0.734$, and $\rho_0=5.0$.

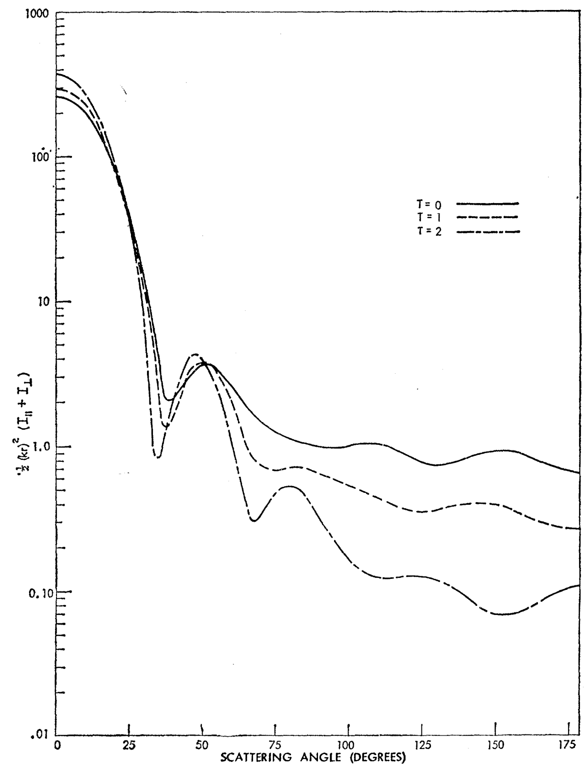

FIG. 4. The differential scattered intensity of unpolarized light in units of $(kr)^2$ and shown for various surface thicknesses. $n_{0R}=1.67$, $n_{0I}=0.734$, and $\rho_0=5.0$.

to be insignificant. This would be expected because in this case one is concerned with the classical Rayleigh scattering region; i.e., the details of the scatterer are not "sensed" by the incident radiation. As ρ_0 becomes larger, however, marked effects become noticeable.

Table I presents the total and scattering cross sections in units of πr_0^2 (r_0=half-falloff radius) as a function of T for various values of ρ_0. Note that although the total cross sections always decrease monotonically with decreasing T, this is not true for the scattering cross sections at the larger values of ρ_0. Nevertheless, just as in the nuclear case, the absorption increases with increasing thickness.[15] Also analogous to the nuclear scattering problem is the increase of total cross section with increasing thickness. This would be expected because of the increased effective radius; i.e., the incident wave begins interacting with the scatterer at larger distances.

Figures 2 and 3, respectively, illustrate the behavior of the scattered intensity for polarizations parallel and perpendicular to the plane of scattering as a function of scattering angle for various surface thicknesses. The half-falloff parameter, ρ_0, was chosen as 5.0 and the index of refraction as previously indicated. These scattered intensities, when measured in units of $(kr)^2$,

[15] A. E. Glassgold, W. B. Cheston, M. L. Stein, S. B. Schuldt, and G. W. Erickson, Phys. Rev. **106**, 1207 (1957).

are somewhat akin to the differential scattering cross sections of nuclear physics. The polarized intensities are usually not calculated in the nuclear problem because of the sparse availability of polarization data. Glassgold *et al.*[15] point out that the least reliable rule of the optical model is that which governs the behavior of the diffraction patterns with changes in surface thickness. The usual effect is a shifting downward of the diffraction pattern while also shifting toward smaller angles. Exceptions are readily seen in Figs. 2 and 3.

Figure 4 represents a combination of Figs. 2 and 3, viz., the scattering intensity of unpolarized light. This result is more akin to the nuclear problem and demonstrates (for the very special case chosen) a behavior quite similar to that expected from optical model results. The results, however, appear far more extreme than the corresponding nuclear situation, especially as regards the behavior of the maxima.

Figure 5 shows the variation of the polarization of the scattered light with varying surface thickness. In general, electromagnetic scattering shows a greater sensitivity of polarization than does nuclear scattering for this variation.

V. CONCLUSIONS AND APPLICATIONS

It is interesting to note that although a comparison of Eq. (34) with the radial part of the Schrödinger equation yields an exact correspondence between the

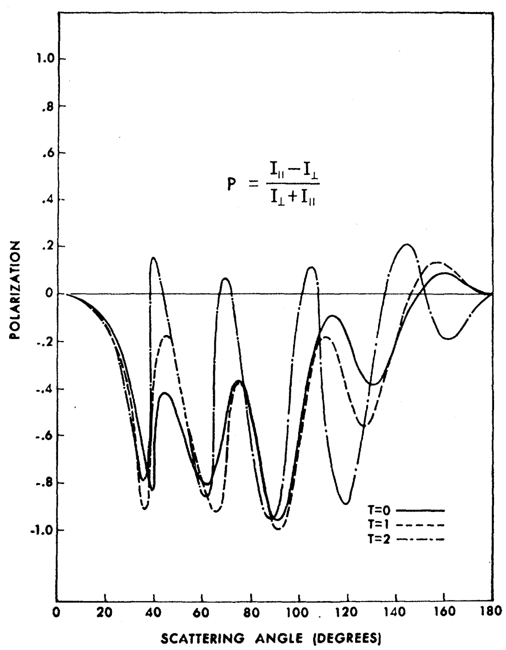

FIG. 5. The polarization of light scattered from a diffuse particle shown for various surface thicknesses, $n_{0R}=1.67$, $n_{0I}=0.734$, and $\rho_0=5.0$.

complex index of refraction and the complex nuclear potential; viz.

$$n^2-1=V/E \tag{66}$$

(where E is the energy of the incident nucleon), Eq. (33) has no such simple analogy. Indeed, terms similar to

$$\frac{1}{n}\frac{dn}{d\rho}\frac{d\psi}{d\rho} \tag{67}$$

are found only in "nonlocal" nuclear models. The Kisslinger[16] model of pion-nucleon scattering and its modifications by Baker, Byfield, and Rainwater[17] display this term. The nonlocal model of Frahn and Lemmer[18] and its modifications by Wyatt, Wills, and Green[19] also manifest terms of this form. Nevertheless, a great variety of other terms also occur; e.g., terms similar to

$$\frac{1}{n^2}\left(\frac{dn}{d\rho}\right)^2, \quad \frac{1}{n^2}\frac{d^2n}{d\rho^2}, \text{ etc.} \tag{68}$$

are found in most of the nonlocal descriptions. It is difficult to reconcile these terms with the single extra term occurring in Eq. (33). Even if it were possible to neglect terms similar to those in Eq. (68) for the nonlocal models, an exact analogy between the nuclear and electromagnetic cases would still not exist. The electromagnetic results would, in this event, represent a hybrid situation in that contributions from "local" [via Eq. (34)] and "nonlocal" [via Eq. (33)] effects would be included. On this basis it seems doubtful that the electromagnetic problem will provide a useful analog to the nuclear problem.[20]

Other, more meaningful, applications of the formalism developed in this paper, however, include the description of the scattering of soft x rays from various macromolecules and viruses which have characteristic diffuse surfaces. The scattering of radiowaves by planets and the scattering of radar by plasma surrounded re-entry vehicles provide other interesting consequences of the theory. Last, but certainly not least, are the applications to the scattering of infrared radiation by small particles. The theory is quite capable of describing the effects of transition regions such as might occur in oxide-coated metallic particles.

---

[16] L. S. Kisslinger, Phys. Rev. **98**, 761 (1955).
[17] W. F. Baker, H. Byfield, and J. Rainwater, Phys. Rev. **112**, 1773 (1958).
[18] W. C. Frahn and R. H. Lemmer, Nuovo cimento **5**, 1564 (1957); **6**, 664 (1957).
[19] P. J. Wyatt, J. G. Wills, and A. E. S. Green, Phys. Rev. **119**, 1031 (1960).
[20] This conclusion is contrary to the hope expressed by L. I. Schiff (reference 8) that the electromagnetic problem might provide an analogy more readily accessible to experiment.