Integral Representation for a Scattering Amplitude with Complex Singularities*

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A double integral representation for a scattering amplitude, which includes the contribution of a box diagram with complex singularities, is obtained. The integrations extend over real values of energy and momentum transfer only.

I

ATTEMPTS to apply the Mandelstam representation to a variety of special problems have been frustrated by the existence of complex singularities of scattering amplitudes.[1] Although it is in principle possible to generalize the representation by integrating over complex surfaces, it is questionable that this would be very useful, since the spectral functions would have to be calculated for complex values of their arguments.

In this note we show, in a very special case, how a representation that involves integration over real surfaces only may be found, even when complex singularities are present. The basic idea is to apply the Bergman-Oka-Weil (BOW) representation.[2]

It is well known that a straightforward application of the Mandelstam representation to $\Sigma-\Sigma$ scattering does violence to the diagram of Fig. 1. Let M, m, and μ be the masses of the Σ, Λ, and π, respectively. If (wrongly) we take $M^2<m^2+\mu^2$, then the amplitude associated with that particular diagram is

$$A(s,t)=(2\pi)^6(g^2/4\pi)^2(16m^2\mu^2)^{-1}F(x,y). \tag{1}$$

[Our amplitude is precisely a matrix element of the T matrix, defined by $S=1-2\pi iT$. We ignore pin and isotopic spin and use the normalization $\langle \mathbf{p}|\mathbf{p}'\rangle=2E_p(\mathbf{p}-\mathbf{p}')$.] Here, g is the $\Sigma\Lambda\pi$ coupling constant, $s=(p_1+p_2)^2$, $t=(p_1+p_3)^2$, $x=s/4m^2$, $y=t/4\mu^2$. The function $F(x,y)$ has the Mandelstam representation

$$F(x,y)=(\pi i)^{-2}\int_1^\infty dx'\int_{y_k'}^\infty dy'\ (x'-x)^{-1}(y'-y)^{-1} \times[-x'y'f(x',y')]^{-1/2}, \tag{2}$$

where

$$f(x,y)=a-(x-1)(y-1),\quad y_k=1+a/(x-1), \tag{3}$$

$$a=[(M^2-m^2-\mu^2)/2m\mu]^2. \tag{4}$$

The spectral function

$$\rho(x,y)=[-xyf(x,y)]^{-1/2} \tag{5}$$

has singularities on the manifolds

$$(1)\ x=0,\quad (2)\ y=0,\quad (3)\ f(x,y)=0, \tag{6}$$

and none of these surfaces have intersections with the region of integration in (2) if $a\neq 0$.

The physical amplitude is defined as the analytic continuation of (1), (2) from $M^2<m^2+\mu^2$ to the actual value.[3]

II

Let M^2 have a small positive imaginary part, so that just after M^2 has passed through the value $m^2+\mu^2$, a has the direction of $(1+i\epsilon)$ with ϵ small. The endpoint of the y' integration, at $y'=y_k'$, has now crossed the real axis just below $y'=1$, and is in the upper half of the complex y' plane. As a reaches its final value, at the physical value of M^2, the y' contour follows the path of the y_k'. It cannot be straightened out, for then the physical boundary value of $F(x,y)$ would suffer a discontinuous change. If now x' were integrated all the way down to $x'=1$, the point y_k' would move to infinity, and the physical boundary value of $F(x,y)$ would not be reached without crossing the y' contour. Therefore, the x' contour must be deformed. A possible x' contour is shown in Fig. 2(a); x must be taken large enough to lie outside.

For fixed x' consider the y' contour. It starts at $y'=y_k'$, passes through $y'=1$, and follows the real axis

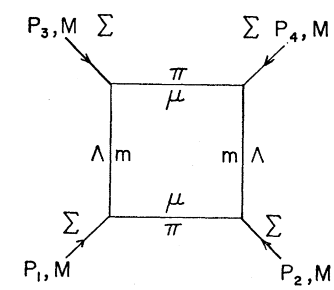

FIG. 1. Fourth-order contributions to $\Sigma-\Sigma$ scattering include this diagram.

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[1] The difficulties have been lamented by G. Barton and C. Kacser, Nuovo cimento **21**, 988 (1961), and by many others.

[2] The representation was invented by A. Weil, Math. Ann. **111**, 178 (1935), and justified by the work of K. Oka, J. Sci. Hiroshima Univ. Ser. A, **6**, 245 (1936); **7**, 115 (1937); and **9**, 8 (1939); and by the work of S. Bergman, Mém. Sci. Math. **106** (1947); and **108** (1948). Other useful references are F. Sommer, Math. Ann. **125**, 172 (1952), and A. S. Wightman, Lectures at the 1960 Summer School at Les Houches (unpublished). The first application to field theory was made by G. Källén and J. Toll, Helv. Phys. Acta **33**, 753 (1960).

[3] This technique was first used by S. Mandelstam, Phys. Rev. Letters **4**, 84 (1960), for a partial wave. It was applied to the case treated here by V. N. Gribov, M. V. Terent'ev, and K. A. Ter-Martirosyan, Soviet Phys.—JETP **13**, 229 (1961). In both these references the analytic continuation was carried out with use of the dispersion relation, rather than the Mandelstam representation.

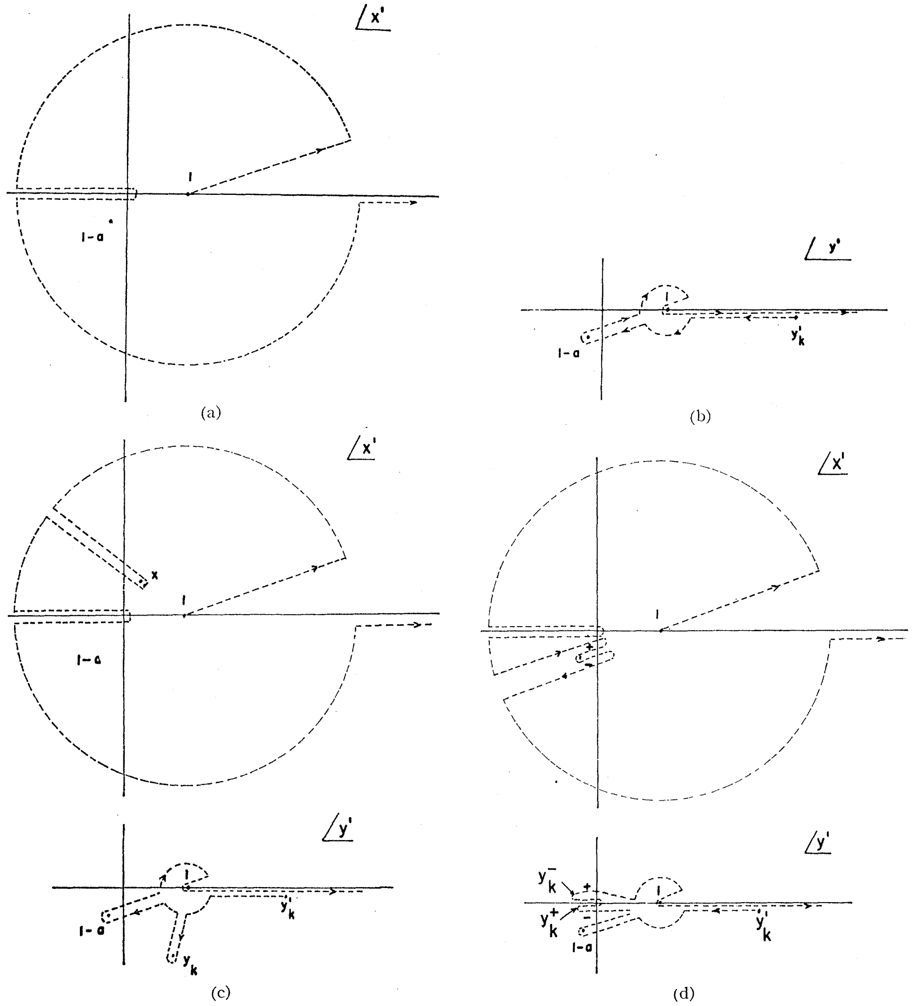

FIG. 2. (a) Deformation of the x' contour, when x is such that the point $x'=x$ lies outside, e.g., on the negative real axis. (b) The locus of the points $y'=y_k'$ as x' moves all around the contour of Fig. 2(a). The y' integration starts at y_k' and follows the arrows. (c) Deformation of the x' and y' contours necessary to obtain $F(x,y)$ for a value of x such that the point $x'=x$ lies inside the contour of Fig. 2(a). (d) The deformed contour in the x' and y' planes when x, y have values near the distinguished boundary (3,5).

to infinity. Instead we may integrate y' along the locus of y_k' as x' moves along its contour, and this gives the contour of Fig. 2(b). For fixed x' the integration follows the arrows from $y'=y_k'$. The spike surrounding the point $y'=1-a$ is associated with the tour that the x' contour makes around the origin. This is done to avoid the point $x'=0$, which is a branch point of $\rho(x',y')$.

With the contours as shown in Figs. 2(a) and 2(b), $F(x,y)$ may be evaluated for values of x and y outside the respective contours. Figure 2(b) shows that, independently of the value of x, $F(x,y)$ has a cut from $y=1$ to $y=\infty$ on the real y axis, and another cut along the straight line that joins $y=1$ to $y=1-a$. By symmetry, the same cuts must exist in the x plane, independently of the value of y. To find the other cuts, let x move about in the complex plane, pushing the x' contour in front of it. The most economical way for x to move is on a straight line towards $x=1$. As the x' contour is thus indented, the locus of the point $y'=y_k'$ moves out from $y'=1$ on a straight line to the point $y'=y_k$, and the y' contour must be deformed accordingly. An example is given in Fig. 2(c). Therefore, $F(x,y)$ has a cut on the surface $(x-1)(y-1)=r_3a$, $0\leqslant r_3\leqslant 1$. The cut on the straight line joining $x=1$ to $x=1-a$ arises from the fact that, when x cuts this line, the sling in the y' contour swings over the branch point of $\rho(x',y')$ at $y'=0$.

The possible cuts of $F(x,y)$ are, therefore,[4]

$$
\begin{array}{ll}
(1)\ x>1,\quad (2)\ y>1, & \\
(3)\ x=1-r_1a, & 0\leqslant r_1\leqslant 1,\\
(4)\ y=1-r_2a, & 0\leqslant r_2\leqslant 1,\\
(5)\ (x-1)(y-1)=r_3a, & 0\leqslant r_3\leqslant 1.
\end{array}
$$

III

The distinguished boundary[2] of $F(x,y)$, with the cuts chosen as above, are all the intersections of any two of

[4] This choice of cuts is not quite the same as that of Gribov *et al.* (reference 3). We prefer this because it is symmetrical in x and y, and because it includes the path followed by the singularities of $\rho(x,y)$ as a increases from 0 to its actual value.

the cuts. It is of crucial import that the distinguished boundary consists exclusively of real points. As long as the limit $a \rightarrow$ real is not taken, the two-dimensional intersection of two cuts coincides with a third cut on a one-dimensional manifold only. Let (i,j) denote the intersection of the ith and the jth cut.

On every cut $F(x,y)$ has two boundary values, $F_{\pm}{}^{i}$, say. On every part of the distinguished boundary, $F(x,y)$ has boundary values $F_{\pm\pm}{}^{ij}$. Let $F_{++}{}^{ij}$ be the boundary value as (i,j) is approached from the intersection of the upper half planes, and so on.

The BOW formula is[2]

$$F(x,y)=\sum_{ij}(2\pi i)^{-2}\int dx'\int dy' \times F(x',y')(q_x{}^i q_y{}^j - q_x{}^j q_y{}^i). \quad (7)$$

The integration is extended over all four sides of the distinguished boundary. The sense of integration will be defined presently. The $q_x{}^i$, $q_y{}^i$, $i=1, \cdots, 5$, are functions of x, y, x', y' that satisfy two conditions. First, $q_x{}^i$ and $q_y{}^i$ are analytic in x, y when x', y' are on the ith cut, except possibly on the cuts. Second,

$$(x'-x)q_x{}^i+(y'-y)q_y{}^i=1. \quad (8)$$

A possible set is

$$\begin{aligned} &q_x{}^1=q_x{}^3=(x'-x)^{-1}, && q_y{}^1=q_y{}^3=0,\\ &q_x{}^2=q_x{}^4=0, && q_y{}^2=q_y{}^4=(y'-y)^{-1},\\ &q_x{}^5=(y'+y-2)/2N, && q_y{}^5=(x'+x-2)/2N,\\ &N(x,y,x',y')=(x'-1)(y'-1)-(x-1)(y-1). \end{aligned} \quad (9)$$

These functions have the property that they are one-valued near the cuts.

The sense of integration in (7) is opposite on opposite sides of a single cut. Therefore,

$$F(x,y)=\sum_{ij}(\pi i)^{-2}\int dx'\int dy'\,\rho^{ij}(x',y')q_x{}^i q_y{}^j, \quad (10)$$

$$\rho^{ij}=\tfrac{1}{4}[F_{++}{}^{ij}-F_{-+}{}^{ij}-F_{+-}{}^{ij}+F_{--}{}^{ij}]. \quad (11)$$

All the ρ^{ij} may be found from $\rho^{12}=[-x'y'f(x',y')]^{-1/2}$ [when $f(x',y')<0$] by analytic continuation in x' and y'. Since the distinguished boundary is real, the sense of integration in (10) can be taken in the direction of increasing x' and y'. The correctness of this choice is verified by the fact that one of the $q_x{}^i$, $q_y{}^j$ is always $(x'-x)^{-1}$ or $(y'-y)^{-1}$ and the residue of the other one at this pole is $(y'-y)^{-1}$ or $(x'-x)^{-1}$.

IV

As the distinguished boundary (1,2) is approached, the discontinuity function is

$$\begin{aligned} \rho^{12}&=[-xyf(x,y)]^{-1/2} && \text{for } y>y_k\\ &=0 && \text{for } 1<y<y_k. \end{aligned} \quad (12)$$

The discontinuity across (1,4) and (1,5) vanish, because if x' is on (1) then the y' contour around (4) and (5) does not exist. Similarly, ρ^{34} vanishes, because when x' is on (3) then the y' contour around (4) does not exist. By symmetry, $\rho^{23}=\rho^{25}=0$. The intersections (1,3) and (2,4) have too low dimensionalities. There remains to calculate $\rho^{35}=\rho^{45}$. The configuration (3,5) is shown in Fig. 2(d), and the result is

$$\begin{aligned} \rho^{35}(x,y)&=2[-xyf(x,y)]^{-1/2}, && x>0\\ &=2i[xyf(x,y)]^{-1/2}, && x<0. \end{aligned} \quad (13)$$

When these results are inserted into (10), the contribution of ρ^{35}, $x<0$ and ρ^{45}, $y<0$ combine to give a term with ordinary Cauchy denominators. Thus,

$$\begin{aligned} F(x,y)=\frac{-1}{\pi^2}\bigg\{ &\int_1^\infty dx'\int_{y_k'}^\infty dy'\frac{\rho(x',y')}{(x'-x)(y'-y)}\\ &+\int_{1-a}^0 dx'\int_{y_k'}^0 dy'\frac{2i\rho'(x',y')}{(x'-x)(y'-y)}\\ &+\int_{-\infty}^0 dx'\int_{\max(0,y_k')}^1 dy'\frac{(y'+y-2)\rho(x',y')}{(y'-y)N(x,y,x',y')}\\ &+(x\leftrightarrow y)\bigg\}. \end{aligned} \quad (14)$$

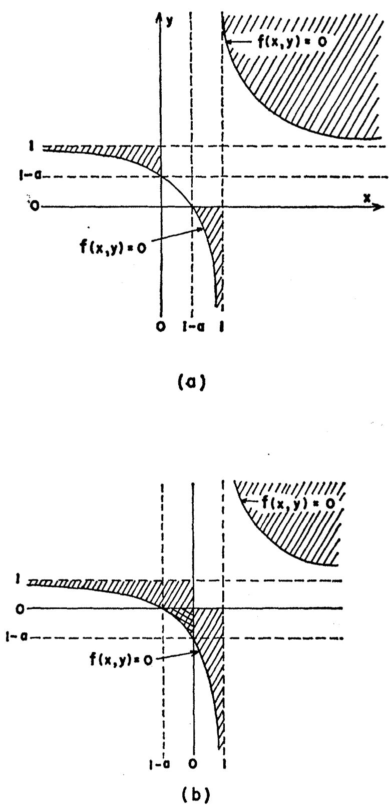

FIG. 3. The region of integration of x', y' in (14) is shown shaded in (a), the stable case, and (b), the unstable case.

The second term should be left out if $a<1$, which is actually the case in $\Sigma-\Sigma$ scattering, since the Σ is stable. In (14),

$$\rho(x,y)=[-xyf(x,y)]^{-1/2},\quad \rho'(x,y)=[xyf(x,y)]^{-1/2}. \tag{15}$$

If these are replaced by arbitrary functions, then (14) is a representation that includes, besides all diagrams included in the Mandelstam representation, the diagram of Fig. 1. The region of integration is shown in Fig. 3.

V

The representation (14) can be generalized to include a larger set of anomalous diagrams. It should also be generalized to the case when all the eight masses in Fig. 1 are arbitrary. A particularly interesting case is the application to decay amplitudes. However, all this is only procatarctic to tackling production amplitudes. These will include terms of the form (14), but also terms involving three and four integrations.

It is essential that the distinguished boundary consist of real points only. This depends in part on a judicious choice of cuts. A necessary condition, in the case of two complex variables, is that the surfaces of singularities intersect on real points only. For the case of the diagram of Fig. 1, this condition is met under rather general conditions.[5] From (14), a dispersion relation can be derived

$$F(x,y)=F^{\text{normal}}+F^{an},$$

$$-\tfrac{1}{2}F^{an}(x,y)=-\frac{1}{\pi}\int_{\max(1-a,\,0)}^{x_k}dx'\frac{\rho'(x',y)}{(x'-x)}-\frac{i}{\pi}\int_{1-a}^{0}dx'\frac{\rho(x',y)}{(x'-x)},\quad y>1$$

$$\frac{i}{\pi}\int_{x_k}^{\min(0,\,1-a)}dx'\frac{\rho(x',y)}{(x'-x)}+\frac{1}{\pi}\int_{0}^{1-a}dx'\frac{\rho'(x',y)}{(x'-x)},\quad 1>y>0,\,1-a$$

$$\frac{1}{\pi}\int_{x_k}^{1-a}dx'\frac{\rho'(x',y)}{(x'-x)},\qquad 1-a>y>0$$

$$\frac{i}{\pi}\int_{1-a}^{x_k}dx'\frac{\rho(x',y)}{(x'-x)},\qquad 0>y>1-a$$

$$\frac{1}{\pi}\int_{\max(0,\,1-a)}^{x_k}dx'\frac{\rho'(x',y)}{(x'-x)}+\frac{i}{\pi}\int_{1-a}^{0}dx'\frac{\rho(x',y)}{(x'-x)},\quad y<0,\,1-a.$$

If the lower limit of any integral exceeds the upper limit, then that integral should be left out. In the stable case $a<1$ this result agrees with that of Gribov *et al.*,[3] except for a minor difference in the choice of the cut in the case $1>y>1-a$. With Gribov's choice of cuts, which is asymmetrical in x, y, only the "imaginary part" $\rho'(x',y)$ appears in the dispersion relation. However, no such choice of cuts exists if $a>1$.

If M^2 is continued with a negative imaginary part, the sense of both contours change. The only change of $\rho^{ij}(x',y')$ is that the sign in front of (13) is reversed when $x<0$. Thus, the second integral in (14) changes sign, and $F(x,y)$ satisfies

$$F^*(x,y,M^2)=F(x^*,y^*,M^{2*});\tag{16}$$

that is, in the unstable case $F(x,y,M^2)$ is a real analytic function of three complex variables. In the stable case $a<1$, the sign of the imaginary part of M^2 is irrelevant, as has already been noted.[3]

[5] This will be discussed by the authors in a paper to appear shortly.