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New Approach to Force-Free Magnetic Fields

G. L. HAND AND M. A. LEVINE
Air Force Cambridge Research Laboratories, Bedford, Massachusetts
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A practical force-free magnetic field must consist of two regions—one conducting and one nonconducting. The boundary between these two regions must have a magnetic field whose intensity is inversely proportional to radius in a cylindrical coordinate system. A unique solution to this problem for an azimuthal symmetric case has been found. This solution satisfies the field equations on the boundary of curl $\mathbf{H}=0$ and $H\sim 1/r$.

INTRODUCTION

IN order to produce a force-free magnetic field it is necessary for the current vector $\mathbf{J}$ and the magnetic field vector $\mathbf{H}$ to be always parallel. This can be represented by $\nabla\times\mathbf{H}=\lambda\mathbf{H}$, where λ is a scalar. The case where $\lambda=$constant has been solved[1,2]; however, the solution for this case is impractical for magnet design. In order to have a useful force-free magnet it is necessary to have regions of magnetic field without conductors present. This requires that λ be zero in part of the magnetic field and nonzero elsewhere.

A field configuration has been proposed[3] which provides force-free regions where the magnetic field is strong and non-force-free regions where the field is weak. This configuration was determined using an electrolytic analog tank and is only approximately force-free.

Here the problem is approached mathematically to determine a configuration which is force-free in the strong field region and has space not occupied by conductors. A toroidal-type winding is considered. Cylindrical coordinates are utilized, and the field does not vary in the azimuthal direction.

SINGLE LAYER WINDING

Suppose a torus is wound with a single layer of conductor such that the conductor has components in the radial, axial, and azimuthal directions. The resulting magnetic field can be separated into two components each occupying different regions of space. The field through the center of the torus, the poloidal field, is $H_p=(H_r^2+H_z^2)^{\frac{1}{2}}$. The azimuthal or toroidal field H_ϕ passes around the torus. In order to be force-free at the interface of the two fields, the magnitudes of the fields must be equal across the interface. In the space not occupied by conductors the curl and divergence of $\mathbf{H}$ both vanish. From the curl condition the trivial result is found that the toroidal field must fall off inversely with the radius. Therefore, the poloidal field must also fall off inversely with the radius at the boundary of the two fields. The problem, thus, arises as to what boundary is required which will give a poloidal field satisfying the conditions

$$\nabla\times\mathbf{H}=0,\quad H_p=c/r \tag{1}$$

on the boundary. Here c is a constant. Substituting $H_r=H_p\cos\beta$ and $H_z=H_p\sin\beta$ in the equation $\partial H_r/\partial z-\partial H_z/\partial r=0$, it is found[4] that

$$\partial H_p/\partial n=H_p/R=\mathbf{n}\cdot\nabla H_p, \tag{2}$$

where $1/R=d\beta/ds$ is the radius of curvature of the field streamline. The angle β is the angle between the streamline and the r axis, and

$$\begin{aligned}\partial/\partial n&=-\sin\beta\partial/\partial r+\cos\beta\partial/\partial z,\\ \partial/\partial s&=\cos\beta\partial/\partial r+\sin\beta\partial/\partial z.\end{aligned} \tag{3}$$

The $\mathbf{n}$ is the principal normal to the streamline and s lies along the streamline. The divergence of the magnetic field does not apply on the boundary which is a field streamline. If $F(r,z)=0$ is the equation of the boundary, then $\mathbf{n}=\nabla F/|\nabla F|$. Also, since $dF/ds=0$, then $\partial F/\partial z=-(\partial F/\partial r)(\partial r/dz)$. Equation (2) then becomes

$$\frac{dH_p}{dr}\frac{\partial F}{\partial r}=\frac{-H_p[(\partial F/\partial r)^2+(\partial F/\partial z)^2]^{\frac{1}{2}}}{[1+(dr/dz)^2]^{\frac{3}{2}}}\frac{d^2r}{dz^2}. \tag{4}$$

Noting (1), this reduces to

$$1+(dr/dz)^2=rd^2r/dz^2. \tag{5}$$

If the conditions $dr/dz=0$ and $z=0$ at $r=r_0$ are included, the solution to (5) is

$$r=r_0\cosh(z/r_0). \tag{6}$$

Now if $H_p(r_0,0)=H_0$, the components of the poloidal field along this boundary are

$$H_r=H_0r_0(r^2-r_0^2)^{\frac{1}{2}}/r^2,\quad H_z=r_0^2H_0/r^2. \tag{7}$$

The solution which has been found expresses the fact that along this streamline it is possible to have a magnetic field whose curl is zero and whose magnitude has a $1/r$ dependency. What has been found is that such a field line can exist and that this is an unique solution to the problem. It is now necessary to complete the problem by finding the current elements which are necessary to establish this field line.

[1] R. Lüst and A. Schlüter, Z. Astrophys. **34**, 263 (1954).
[2] S. Chandrasekhar and D. C. Kendall, Astrophys. J. **126**, 457 (1957).
[3] H. P. Furth and M. A. Levine, J. Appl. Phys. **33**, 747 (1962).
[4] L. M. Milne-Thomson, *Theoretical Hydrodynamics* (The Macmillan Company, New York, 1955), p. 103.

One approach is to determine another magnetic field boundary that will force the inverse r dependency of H_p along the curve given by (6). In order to determine this boundary an analog electrolytic tank was used which is described in reference 3. The resultant configuration is shown in Fig. 1. Since the winding cannot be continued indefinitely, the force-free configuration is terminated at a point where the magnetic field is sufficiently weak.

MULTIPLE LAYER WINDINGS

Here it is necessary to find a solution to the equations

$$\nabla\times\mathbf{H}=\lambda\mathbf{H}, \quad \nabla\cdot\mathbf{H}=0, \tag{8}$$

where λ is a variable scalar. If the substitution is made

$$r\lambda H_r=-\partial\psi/\partial z, \quad r\lambda H_z=\partial\psi/\partial r, \quad rH_\phi=\psi, \tag{9}$$

then three of the four equations in (8) are satisfied. It is necessary to solve only

$$\psi_{,rr}-\psi_{,r}/r+\psi_{,zz}+\lambda^2\psi-(1/\lambda)(\psi_{,z}\lambda_{,z}+\psi_{,r}\lambda_{,r})=0. \tag{10}$$

The comma indicates partial derivative. Azimuthal symmetry has been assumed here. Equation (10) has been solved for constant λ by Morikawa[5] for a field within a spherical region. Supplementary conditions must also be satisfied. Taking the divergence of the force-free condition (8) gives the requirement that $\mathbf{H}\cdot\nabla\lambda=0$, or λ is constant along streamlines. The definition of the stream function ψ in (9) requires that ψ also be constant along streamlines.

An equation similar to (10) can be written which may be simpler to solve. By substituting $H_r=H_p\cos\beta$ and $H_z=H_p\sin\beta$ in the ϕ component of $\nabla\times\mathbf{H}=\lambda\mathbf{H}$ and noting (3) gives

$$\partial H_p/\partial n-H_p/R=\lambda H_\phi. \tag{11}$$

Furthermore, using (9)

$$\psi_{,n}=-\sin\beta\psi_{,r}+\cos\beta\psi_{,z}=-r\lambda H_p. \tag{12}$$

Substituting (12) in (11) together with the relation

[5] G. K. Morikawa, Controlled Thermonuclear Conference, Washington, D. C., 1958 [Atomic Energy Commission Report TID 7558 (unpublished)], p. 428.

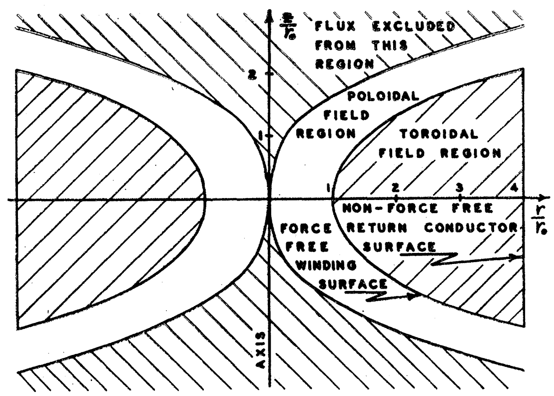

FIG. 1. Configuration determined with analog electrolytic tank.

$\sin\beta=-\partial r/\partial n$ gives

$$\psi_{,nn}-\psi_{,n}(\lambda_{,n}/\lambda+1/R-\sin\beta/r)+\lambda^2\psi=0. \tag{13}$$

The coordinate system (s,n) now replaces (r,z), where s is measured along streamlines and n perpendicular to streamlines. Equation (13) together with the conditions ψ and λ are constant on streamlines is equivalent to (8). In (13), $1/R-\sin\beta/r$ must be a function of n only.

CONCLUSIONS

A field configuration is proposed which is force-free for a single layer winding. The boundary determined by $r=r_0\cosh(z/r_0)$ is unique with the exception of the trivial case of a uniform infinite cylindrical magnet. In order to have equal poloidal and toroidal fields at the boundary,[6] it is only necessary to wind the conductor on the boundary such that the current vector points 45° from the poloidal field vector. The force-free configuration is terminated at a point where the magnetic pressure is within the strength limits of the materials used in the magnet.

Multiple layer windings require the solution of Eqs. (10) or (13). Little can be said about the value of λ to substitute in this equation. Constant λ eliminates the possibility of having space unoccupied by conductors.

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[6] A. A. Kuznetsov, J. Tech. Phys. U.S.S.R. **31**, 650 (1961) [translation: Soviet Phys.—Tech. Phys. **6**, 472 (1961)].