Flux Quantization and Time-Reversal Degeneracy*

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It is shown on the basis of very general considerations that the observed quantization of the magnetic flux enclosed by a ring superconductor in units of $\pi\hbar c/e$ implies and is implied by time-reversal degeneracy of the states of the superconductor. This degeneracy results in vanishing electric current density at thermodynamic equilibrium, which is required for consistency with the Meissner effect. The quantization of flux is also used to shed some light on the applicability of the Bardeen-Cooper-Schrieffer approximation in the presence of an enclosed magnetic field.

I. INTRODUCTION

SUPERCONDUCTORS are characterized by the Meissner effect, the vanishing of the magnetic field and consequently of the electric current density, in the interior of the superconductor. It has recently been observed[1,2] that ring superconductors have a second striking characteristic. The magnetic flux threading the ring is quantized in units of

$$\mathcal{L}=\pi\hbar c/e. \tag{1.1}$$

The existence of two distinct phenomena characteristic of the superconducting state raises the question of how these phenomena are related to each other.

London[3] anticipated the quantization of the surrounded flux. He concluded that the Meissner effect requires the flux to be a multiple of $2\mathcal{L}$. More recently, Onsager[4] and others[5] have derived the quantization of the surrounded flux in multiples of $\mathcal{L}$. These authors assume that the electrons may be replaced by particles of charge $2e$. From the fact that the current vanishes within the superconductor, they arrive at the observed flux quantization.

Byers and Yang[6] have related the observed flux quantization to the theory of Bardeen, Cooper, and Schrieffer[7] (BCS). They point out that it is only at the observed flux values that there is the necessary degeneracy among pairs of single-electron states in a superconductor. They show that the system is stable against small changes away from the observed flux values. Their results are based on the assumptions of independent electron motion in a cylindrically symmetric region.[7a]

We shall show that the observed quantization of flux implies (and is implied by) a time-reversal degeneracy of the states of a superconductor. This result in turn leads to zero current density at thermodynamic equilibrium. Our result, which is model independent, shows that the observed quantization of flux implies the zero current density which is needed for consistency with the Meissner effect.

The converse, that vanishing current *requires* the observed flux values, is false. Counter examples are easily given.

Methods developed to derive the connection between flux quantization and time-reversal degeneracy are also used to prove that the BCS approximation is as valid when the surrounded flux is a multiple of $2\mathcal{L}$ as it is in the absence of magnetic fields.

2. SYMMETRY THEOREM

Consider a system of electrons and positive ions confined to a region which is topologically like the interior of a loop of wire. An externally fixed magnetic field of total flux F threads the loop, but vanishes in the region where the particles move. The vector potential $\mathbf{A}(\mathbf{x})$ in the field-free region containing the particles can be written in the form

$$\mathbf{A}(\mathbf{x})=\operatorname{grad} S(\mathbf{x}), \tag{2.1}$$

where $S(\mathbf{x})$ is a differentiable but multiple-valued scalar function. It changes by the amount

$$\Delta S=\oint_C \mathbf{A}(\mathbf{x})\cdot d\mathbf{x}=F \tag{2.2}$$

when $\mathbf{x}$ makes one circuit of the flux along a closed path C within the allowed region.

We shall use only two properties of the system described.

Property 1. The Hamiltonian $\mathcal{H}(S)$ for the interacting

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6 N. Byers and C. N. Yang, Phys. Rev. Letters **7**, 46 (1961).
7 J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
7a *Note added in proof.* J. M. Blatt [Progr. Theoret. Phys. (Kyoto) **26**, 721 (1961)] predicted the quantization of flux in the observed units. His approach is based on the Bogoliubov version of the theory of superconductivity.

electrons and ions is a function of the particle coordinates $\mathbf{x}_k$, spins $\mathbf{s}_k$, and velocities

$$\mathbf{v}_k(S)=\frac{1}{m_k}\left[\frac{\hbar}{i}\boldsymbol{\nabla}_k-\frac{e_k}{c}[\boldsymbol{\nabla}_k S(\mathbf{x}_k)]\right]. \tag{2.3}$$

The Hamiltonian does not involve the external magnetic field except through the velocity (2.3). The quantities e_k and m_k stand for the charge and mass of the kth particle. Each e_k is of course an integral multiple of the electron charge $-e$.

Property 2. The dynamical laws for the entire system, including the sources of the external magnetic field, are invariant under time reversal. Consequently, $\mathcal{H}(S)$ is transformed into $\mathcal{H}(-S)$ under time reversal combined with a reversal in sign of the external magnetic field. It follows that if $\psi(S)$ is an eigenfunction of $\mathcal{H}(S)$, then $\mathcal{H}(-S)$ has an eigenfunction $\psi'(-S)$ which has the same energy as does $\psi(S)$, and which is related to $\psi(S)$ by the condition

$$\langle\psi'(-S)|f[\mathbf{x},\mathbf{v}(-S),\mathbf{s}]|\psi'(-S)\rangle =\langle\psi(S)|f[\mathbf{x},-\mathbf{v}(S),-\mathbf{s}]|\psi(S)\rangle. \tag{2.4}$$

Equation (2.4) is understood to hold for every real function f of all the particle coordinates, velocities, and spins.

We now prove the following symmetry theorem: *If the value of the magnetic flux F is an integral multiple of $\mathfrak{L}$, then the eigenfunctions of $\mathcal{H}(S)$ are degenerate under time reversal.* That is, if $\psi(S)$ is an eigenfunction of $\mathcal{H}(S)$, then $\mathcal{H}(S)$ has an eigenfunction $\bar{\psi}(S)$, at the same energy, which is the time-reversal transform of $\psi(S)$. The time-reversal transform $\bar{\psi}(S)$ is defined by

$$\langle\bar{\psi}(S)|f[\mathbf{x},\mathbf{v}(S),\mathbf{s}]|\bar{\psi}(S)\rangle =\langle\psi(S)|f[\mathbf{x},-\mathbf{v}(S),-\mathbf{s}]|\psi(S)\rangle. \tag{2.5}$$

The states represented by $\bar{\psi}(S)$ and by $\psi(S)$ need not be distinct.

The time-reversal transform $\bar{\psi}(S)$ defined by Eq. (2.5) must be distinguished from the time-and-flux-reversal transform $\psi'(-S)$ defined by Eq. (2.4). The existence of $\psi'(-S)$ is the substance of property 2. The existence of the degenerate eigenfunction $\bar{\psi}(S)$, for particular flux values, is the assertion of the symmetry theorem.

The proof of the symmetry theorem proceeds by construction. The time-reversal transform $\bar{\psi}(S)$ is constructed from the postulated $\psi'(-S)$ by the unitary transformation

$$\bar{\psi}(S)=U(S)\psi'(-S), \tag{2.6}$$

where

$$U(S)=\exp\{(2i/\hbar c)\textstyle\sum_k e_k S(\mathbf{x}_k)\}. \tag{2.7}$$

Under this unitary operation, the velocity transforms according to

$$\mathbf{v}(S)=U(S)\mathbf{v}(-S)U(S)^{-1}. \tag{2.8}$$

By property 1, this implies

$$\mathcal{H}(S)=U(S)\mathcal{H}(-S)U(S)^{-1}. \tag{2.9}$$

Property 2 gives

$$\mathcal{H}(-S)\psi'(-S)=E\psi'(-S) \tag{2.10}$$

as a consequence of

$$\mathcal{H}(S)\psi(S)=E\psi(S). \tag{2.11}$$

From Eqs. (2.6), (2.9), and (2.10), it follows that

$$\mathcal{H}(S)\bar{\psi}(S)=E\bar{\psi}(S). \tag{2.12}$$

That $\bar{\psi}(S)$ is the time-reversal transform of $\psi(S)$ is verified by substituting Eqs. (2.6) and (2.8) into (2.4) to get (2.5).

Since all wave functions must be single-valued functions[8] of the coordinates $\mathbf{x}_k$, the unitary transformation $U(S)$ must itself be single valued. When $\mathbf{x}_k$ is carried about the closed path C of Eq. (2.2), $U(S)$ is multiplied by the factor

$$g=\exp\{(2i/\hbar c)e_k F\}. \tag{2.13}$$

For g to be equal to unity, F must be an integral multiple of $\pi\hbar c/e_k$. As all the e_k are integral multiples of the electron charge, the limitation of the symmetry theorem to flux values which are integral multiples of $\mathfrak{L}$ assures the single-valuedness of $U(S)$. This completes the proof.

The symmetry theorem states that particular flux values F lead to time-reversal degeneracy. That this degeneracy does not occur for all flux values may be verified by considering the case of a single electron.[9] The symmetry theorem does not of itself require the quantization of flux, since there is no reason *a priori* for time-reversal degeneracy in the presence of an external magnetic field.

3. CONVERSE OF THE SYMMETRY THEOREM

It will now be shown that time-reversal degeneracy requires quantization of the surrounded flux in units of $\mathfrak{L}$. Thus, quantization of the surrounded flux in these units is necessary and sufficient for time-reversal degeneracy.

The time-reversal transormation and the time-and-flux-reversal transformation are both represented by anti-unitary operators. Therefore the two must be related to each other by a unitary transformation V.

$$\bar{\psi}(S)=V\psi'(-S), \tag{3.1}$$

$$V^{\dagger}V=1. \tag{3.2}$$

It is clear from the definitions (2.4) and (2.5) of $\psi'(-S)$ and $\bar{\psi}(S)$ that the sole effect of V must be to

[8] The physical reasons for requiring single-valuedness are discussed on the basis of symmetry theory by L. J. Tassie and M. Peshkin, Ann. Phys. (New York) **16**, 177 (1961).

[9] M. Peshkin, I. Talmi, and L. J. Tassie, Ann. Phys. (New York) **12**, 426 (1961).

reverse the sign of S in the velocity operators. Specifically,

$$V\mathbf{x}_k V^{-1} = \mathbf{x}_k, \tag{3.3}$$

$$V\mathbf{s}_k V^{-1} = \mathbf{s}_k, \tag{3.4}$$

$$V\mathbf{v}_k(S)V^{-1} = \mathbf{v}_k(-S). \tag{3.5}$$

From the definition (2.3) of $\mathbf{v}_k(S)$, it then follows that

$$V\mathbf{p}_k V^{-1} = \mathbf{p}_k + (2e_k/c)\nabla_k S(\mathbf{x}_k), \tag{3.6}$$

or

$$[\mathbf{p}_k, V] = -(2e_k/c)[\nabla_k S(\mathbf{x}_k)]V. \tag{3.7}$$

Then V is a solution of the equation

$$V^{-1}(\hbar/i)\nabla_k V = -(2e_k/c)\nabla_k S(\mathbf{x}_k), \tag{3.8}$$

and must have the form

$$V = \textstyle\prod_k \exp\{(2e_k/i\hbar c)S(\mathbf{x}_k)\}, \tag{3.9}$$

in agreement with Eq. (2.7).

As we have seen, V must be single valued to be an acceptable operator. Therefore the postulated existence of V implies that the surrounded flux is an integral multiple of $\mathfrak{L}$.

4. APPLICATION TO SUPERCONDUCTORS

A superconductor has, in fact, vanishing magnetic field strength in its interior, except perhaps for a narrow penetration region at the surface. It surface effects are neglected, the symmetry theorem provides a natural connection between the observed flux quantization and the vanishing current density which is needed for consistency with the Meissner effect. This may be seen simply by observing that the current density is odd under time reversal. Since degenerate states have equal weight in the canonical ensemble, it follows that the statistical expectation of an odd quantity vanishes at thermal equilibrium. Thus the symmetry theorem leads directly to the result that quantization of the enclosed flux at the observed values produces zero current density at thermodynamic equilibrium. These considerations do *not* show that zero current density *requires* the quantization of flux. One can easily construct examples of zero-current states for nonintegral flux values. These states will be found not to be time-reversal degenerate.

5. CONSEQUENCES OF A REPETITION THEOREM

The unitary operation of Eqs. (2.6) and (2.7) has been used earlier[9] to establish a repetition theorem which asserts that the eigenvalue spectrum of $\mathcal{H}(S)$ is periodic[10] in the flux F with period $2\mathfrak{L}$. The proof is obtained by generalizing Eqs. (2.8) and (2.9) to read

$$\mathbf{v}(S+S_0) = U(\tfrac{1}{2}S_0)\mathbf{v}(S)U(\tfrac{1}{2}S_0)^{-1}, \tag{5.1}$$

$$\mathcal{H}(S=S_0) + U(\tfrac{1}{2}S_0)\mathcal{H}(S)U(\tfrac{1}{2}S_0)^{-1}. \tag{5.2}$$

Since $\mathcal{H}(S+S_0)$ is related to $\mathcal{H}(S)$ by a unitary transformation, the two have the same eigenvalue spectrum. The single-valuedness requirement is satisfied when $F_0 = \oint_C \nabla S_0 \cdot d\mathbf{x}$ is an integral multiple of $2\mathfrak{L}$. This completes the proof of the repetition theorem.

This result has also been noted by Byers and Yang,[6] who infer from it that the free energy at thermodynamic equilibrium is periodic in the enclosed flux with period $2\mathfrak{L}$. Their proof, which is independent of their model, shows that if surface effects may be neglected, then superconductivity at zero flux implies superconductivity at integral multiples of flux $2\mathfrak{L}$. Likewise, superconductivity at flux $\mathfrak{L}$ implies the same at odd-integral multiples of $\mathfrak{L}$.

The repetition theorem also yields interesting information about the BCS theory, which provides a very useful description of the superconducting state, at least in the absence of magnetic fields. BCS choose the gauge wherein the vector potential vanishes. They use an approximate Hamiltonian of the form

$$\mathcal{H}_{\mathrm{BCS}} = \mathcal{H}_0 + \mathcal{H}_1, \tag{5.3}$$

where $\mathcal{H}_0$ represents independent motion of the conduction electrons and $\mathcal{H}_1$ is the pairing interaction between electrons in degenerate states which are related by time reversal. This pairing lowers the energy and produces a gap in the excitation spectrum of the system.

When a magnetic field threads the system, the use of the BCS Hamiltonian (5.3) becomes suspect even if it is made formally gauge invariant by using the velocity operators of Eq. (2.3). The doubts arise because a magnetic field threading the circuit is known to influence the motion of electrons and ions.[11] If surface effects are neglected as usual, this difficulty may be removed, at least for even-integral flux values. Both the exact Hamiltonian for all the electrons and ions, and the BCS reduced Hamiltonian (5.3), obey Eq. (5.2). Therefore, each Hamiltonian for even-integral flux values is related to its zero-flux limit by a unitary transformation. This proves that if $\mathcal{H}_{\mathrm{BCS}}$ is a good approximation for zero flux, it is equally good for all even-integral flux values, until the neglect of the penetration region fails.

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[10] Time-and-flux-reversal degeneracy implies further that the eigenvalue spectrum is an even function of $\mathcal{H}(S)$.

[11] The physical reasons for this phenomenon are discussed by Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959) and **123**, 1511 (1961), and from a different point of view by references 8 and 9.