

# Dispersion Relations for Electron Scattering from Atomic Helium\*

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Dispersion relations are deduced appropriate to the scattering of electrons by helium atoms either in their ground state or in their first excited metastable state. The analysis extends earlier work on dispersion relations for potential scattering and electron-hydrogen scattering and points out that in the case of scattering from excited He the dispersion relation generally involves an integral over unphysical energies, as well as residue contributions from bound states of  $\text{He}^-$  embedded in the continuum. The symmetries of those  $\text{He}^-$  bound states making nonvanishing residue contributions, and the form of the optical theorem when the exclusion principle is taken into account, are also discussed.

## I. INTRODUCTION

DISPERSION relations—expressions which relate the real and imaginary parts of a scattering amplitude—provide a novel approach to atomic scattering problems. Previous papers<sup>1,2</sup> have derived the appropriate relations for scattering of electrons by hydrogen atoms in their ground states, and have used these relations to decide between two conflicting experiments. More recently, application of the dispersion relations to electron scattering by atomic helium has been suggested<sup>3</sup>; specifically it is thought that measured  $e\text{-He}$  differential cross sections<sup>4</sup> peaking sharply in the forward direction at incident energies as low as 30 eV might be confirmed or ruled out by the dispersion relations.

The following sections derive the dispersion relations for scattering of electrons by atomic helium in its ground state and in its first excited state. Our treatment follows the lines of our previous paper, and the form of our final result is not very different from that previously obtained. Nevertheless, we consider it worthwhile to give the derivation in detail, because a number of facets of our  $e\text{-He}$  analysis, though characteristic of scattering from all many-electron atoms, were absent in the especially simple case previously examined.

The dispersion relation for scattering from the first excited state has some complicating features, namely, (a) It involves an integral of the scattering amplitude over a range of negative kinetic energies, and (b) it may contain residue contributions from negative ion

(electron plus neutral atom) bound states embedded in the continuum, i.e., from quadratically integrable states of  $\text{He}^-$  having positive energies relative to neutral He in its ground state. The derivation makes it obvious that the above complicating features, which were not remarked in the especially simple case previously examined,<sup>1</sup> are characteristic of the dispersion relations for electron scattering from any excited state of any atom.

From our procedures in helium, it should be clear how to derive the dispersion relations for any given electron-atom elastic scattering collision. On physical grounds we expect, though our manipulations do not make it *obvious*, that the following features of our  $e\text{-He}$  results also should be characteristic of the dispersion relations for electron scattering by an arbitrary atomic state: (i) The ordinary and exchange amplitudes may have poles (in the complex energy plane) at real energies corresponding to nonphysical exclusion-principle-forbidden states of the negative ion, but the residue contributions made by these poles disappear from dispersion relations involving actually measurable cross sections, as is assumed in the dispersion relations commonly employed in elementary particle physics, e.g., the Mandelstam representation. (ii) Similarly, neglecting spin-dependent forces, if the atom is initially in an  $S$  state, the dispersion relation appropriate to a scattering channel of given total spin contains no residue contributions from negative ion states of different spin, e.g., quartet bound states of  $\text{He}^-$  make no residue contributions to the dispersion relations for doublet scattering from  $\text{He } ^3S_1$  states. (iii) When the exclusion principle is taken into account, the “forward scattering amplitude” appearing in the optical or cross-section theorem need not be simply related to the differential cross section in the forward direction, even when the initial atom and electron beams each are polarized in spin states of definite magnetic quantum number. (iv) In a scattering channel of given total spin, on the other hand, the forward scattering amplitude appearing in the cross-section theorem is the

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<sup>1</sup> E. Gerjuoy and N. A. Krall, *Phys. Rev.* **119**, 705 (1960).

<sup>2</sup> N. A. Krall and E. Gerjuoy, *Phys. Rev.* **120**, 143 (1960).

<sup>3</sup> H. S. W. Massey, at the *Second International Conference on Physics of Electron and Atomic Collisions*, Boulder, Colorado, June, 1961 (W. A. Benjamin, New York, 1961).

<sup>4</sup> H. S. W. Massey and E. H. S. Burhop, *Electronic and Ionic Impact Phenomena* (Clarendon Press, Oxford, England, 1956), p. 92.

same as the amplitude whose square yields the elastic differential cross section, again assuming the atom is initially in an  $S$  state, and neglecting spin-orbit forces but taking the exclusion principle into account. The proviso that the atom is initially in an  $S$  state is required in (ii) and (iv) because the channel spin, compounded from the initial electron spin and the total angular momentum of the initial atom, otherwise would not necessarily be conserved in the absence of spin-dependent forces.

The above feature (i) is to be expected because the properties of a completely antisymmetric (under space and spin interchange) solution of Schrödinger's equation should be quite unrelated to the properties of solutions belonging to symmetric or other even more complicated representations of the permutation group (see Appendix B). Similarly, when the Hamiltonian is spin independent, the properties of a solution with given total spin should be quite unrelated to the properties of solutions with different total spin, as found in feature (ii). Nevertheless, because the dispersion relations involve analytic continuation of the scattering amplitudes to complex energies, where solutions to Schrödinger's equation satisfying the boundary conditions do not exist, we have deemed it desirable to explicitly demonstrate features (i) and (ii) in the  $e$ -He scattering case. In fact, the demonstration does bring out some points of interest, notably (see Appendix B) the form of the residue contributions from degenerate space-dependent eigenfunctions of the total Hamiltonian.

Features (iii) and (iv) are understood on much the same grounds as are (i) and (ii). Moreover, questions raised by analytic continuation have no bearing on features (iii) and (iv), i.e., these expected features have little need for an explicit  $e$ -He verification. Our motivation for detailing the demonstration of features (iii) and (iv) in  $e$ -He scattering is mainly pedagogical; we have seen several recent scattering calculations whose attempts to incorporate the requirements of the exclusion principle were rather confused.

We caution that our procedures, as described in the following sections, probably are not valid for electron-ion scattering, wherein the long-range Coulomb force between the electron and ion produces formal complications, namely, an infinite number of bound states whose energies are everywhere dense<sup>5</sup> in the neighborhood of the energy at which continuum (unbound) states appear.

## II. SCATTERING BY He GROUND STATE

In light atoms, e.g., He, it is customary to neglect spin-dependent and other non-Coulomb interactions. Thus, with the origin at the (supposed infinitely massive) atomic nucleus, the Hamiltonian for the

electron-He scattering system is

$$H = T_1 + T_2 + T_3 - 2e^2/r_1 - 2e^2/r_2 - 2e^2/r_3 + e^2/r_{12} + e^2/r_{13} + e^2/r_{23}, \quad (1)$$

where  $T$  represents the kinetic energy operator and the subscripts 1, 2, 3 refer to the three electrons in the system. With a spin-independent Hamiltonian the ground state of He is pure  $^1S_0$ , and the differential cross section for elastic scattering of electrons by He in its ground state is<sup>6</sup>

$$(d\sigma/d\Omega)_\theta = |f - g|^2. \quad (2)$$

Here,  $f(\theta)$  is the ordinary amplitude for scattering of the incident electron from its initial direction  $\mathbf{n}_i$  into a final direction  $\mathbf{n}_f$ , with  $\theta$  the angle between  $\mathbf{n}_i$  and  $\mathbf{n}_f$ ;  $g(\theta)$  is the corresponding exchange amplitude.

The noteworthy feature of Eq. (2) is that the amplitudes  $f$  and  $g$  appearing therein are spin-independent matrix elements, computed as if the electrons were completely distinguishable; all complications engendered by spin and particle indistinguishability are taken into account in the particular form of Eq. (2), e.g., its dependence on the difference rather than the sum of  $f$  and  $g$ . Explicitly, suppose electron 1 is incident along direction  $\mathbf{n}_i$  with kinetic energy  $\hbar^2 k^2/2m$ . Let the total energy  $E$  of the three-electron system be measured relative to the (negative of the) total electronic binding energy<sup>7</sup> (79 eV = 24.6 + 54.4) of He in its ground state, so that  $E = \hbar^2 k^2/2m$ . Then for the spin-independent Hamiltonian Eq. (1), the scattering is described by a spin-independent solution  $\Psi_i^{(+)}$  of the Schrödinger equation

$$(H - E + 79)\Psi_i^{(+)} = 0, \quad (3)$$

and we have for  $f$  and  $g$ ,<sup>6,8</sup>

$$f(\theta, E) = -\frac{m}{2\pi\hbar^2} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \psi_{f_0}^* V_i \Psi_i^{(+)}, \quad (4)$$

$$g(\theta, E) = -\frac{m}{2\pi\hbar^2} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \psi_{f_0}^* V_f \Psi_i^{(+)},$$

where

$$\Psi_i^{(+)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \psi_i + \Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \quad (5)$$

$$\psi_i(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = e^{ik\mathbf{n}_i \cdot \mathbf{r}_1} u_0(\mathbf{r}_2, \mathbf{r}_3),$$

$$\psi_{f_0}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = e^{ik\mathbf{n}_f \cdot \mathbf{r}_1} u_0(\mathbf{r}_2, \mathbf{r}_3), \quad (6)$$

$$\psi_{f_e}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \psi_{f_0}(\mathbf{r}_2, \mathbf{r}_1, \mathbf{r}_3) = e^{ik\mathbf{n}_f \cdot \mathbf{r}_2} u_0(\mathbf{r}_1, \mathbf{r}_3),$$

$$V_i(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = e^2/r_{12} + e^2/r_{13} - 2e^2/r_1, \quad (7a)$$

$$V_f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = V_i(\mathbf{r}_2, \mathbf{r}_1, \mathbf{r}_3) = e^2/r_{21} + e^2/r_{23} - 2e^2/r_2. \quad (7b)$$

The quantity  $u_0$  is the spin-independent part of the

<sup>6</sup> L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), p. 244.

<sup>7</sup> C. W. Allen, *Astrophysical Quantities* (Athlone Press, London, 1955), p. 36.

<sup>8</sup> E. Gerjuoy, *Ann. Phys. (New York)* **5**, 58 (1958).

<sup>5</sup> E. W. Hobson, *The Theory of Functions of a Real Variable* (Dover Publications, New York, 1957), Vol. 1, p. 77.

ground state He wave function, which is known to be symmetric in the space coordinates of particles 2 and 3, i.e.,

$$u_0(\mathbf{r}_2, \mathbf{r}_3) = u_0(\mathbf{r}_3, \mathbf{r}_2). \quad (8)$$

Of course,<sup>6</sup> Eq. (4) is derived using Eqs. (5)–(7) from the fundamental definitions

$$\lim_{r_1 \rightarrow \infty} \int d\mathbf{r}_2 d\mathbf{r}_3 u_0^*(\mathbf{r}_2, \mathbf{r}_3) \Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = f \frac{e^{ikr_1}}{r_1}, \quad (9a)$$

$$\lim_{r_2 \rightarrow \infty} \int d\mathbf{r}_1 d\mathbf{r}_3 u_0^*(\mathbf{r}_1, \mathbf{r}_3) \Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = g \frac{e^{ikr_2}}{r_2}, \quad (9b)$$

of  $f$  and  $g$ .

In Eqs. (9a) and (9b), respectively,  $r_1$ ,  $r_2$  approach infinity along  $\mathbf{n}_f$ , and, of course, the asterisk denotes the complex conjugate in Eqs. (4) and (9).

We now proceed as in the case<sup>1</sup> of electron scattering by hydrogen. Substitute

$$\Psi_i^{(+)} = \psi_i - (H - E + 79 - i\epsilon)^{-1} V_i \psi_i \quad (10)$$

$$R_{oj}(E) = \frac{1}{E - E_j} \frac{m}{2\pi\hbar^2} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 e^{\alpha_j n_i \cdot \mathbf{r}_1} u_0^*(\mathbf{r}_2, \mathbf{r}_3) V_i(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) w_j(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ \times \int d\mathbf{r}_1' d\mathbf{r}_2' d\mathbf{r}_3' w_j^*(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3') V_i(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3') u_0(\mathbf{r}_2', \mathbf{r}_3') e^{-\alpha_j n_i \cdot \mathbf{r}_1'}, \quad (13)$$

$$R_{ej}(E) = \frac{1}{E - E_j} \frac{m}{2\pi\hbar^2} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 e^{\alpha_j n_i \cdot \mathbf{r}_2} u_0^*(\mathbf{r}_1, \mathbf{r}_3) V_j(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) w_j(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ \times \int d\mathbf{r}_1' d\mathbf{r}_2' d\mathbf{r}_3' w_j^*(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3') V_j(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3') u_0(\mathbf{r}_2', \mathbf{r}_3') e^{-\alpha_j n_i \cdot \mathbf{r}_1'}, \quad (14)$$

where

$$\alpha_j = (2m/\hbar^2)^{1/2} |E_j|^{1/2}, \quad (15)$$

and  $w_j$  is the bound-state eigenfunction of  $H$  corresponding to the energy  $E_j$ .

Because the Hamiltonian Eq. (1) is spinless, its eigenfunctions  $w_j$  depend only on the space coordinates  $\mathbf{r}_1$ ,  $\mathbf{r}_2$ ,  $\mathbf{r}_3$ . Because Eq. (1) and the definitions of  $f$  and  $g$  treat the electrons as distinguishable, the eigenfunctions  $w_j$  are not required to have any particular symmetry. Thus, states  $w_j$  which are ruled out by the Pauli exclusion principle may contribute to  $R_{oj}$  and/or  $R_{ej}$ . In the relatively simple three-electron system under present consideration the only eigenfunctions of Eq. (1) ruled out by the exclusion principle are those  $w_j$  which are completely symmetric in the coordinates  $\mathbf{r}_1$ ,  $\mathbf{r}_2$ ,  $\mathbf{r}_3$ . These  $w_j$  are ruled out by the exclusion principle because to yield a physically allowed  $\text{He}^-$  state they have to be multiplied by a completely antisymmetric spin function, and there is no combination of three spin 1/2 functions which is antisymmetric in the spin coordinates of all three particles. In scattering by more complicated atoms ( $Z > 2$ ), the exclusion principle rules out a larger variety of eigen-

into (4); define

$$Q_o(E) = f(0, E) - f_{\text{Born}}(0, E), \\ Q_e(E) = g(0, E) - g_{\text{Born}}(0, E); \quad (11)$$

suppose that in the complex energy plane the functions  $Q(E)$  have the analytic properties assumed previously<sup>1</sup>; and apply Cauchy's theorem. We obtain again, as previously,<sup>1</sup>

$$\text{Re} f(0, E) = f_{\text{Born}}(0, E) \\ + \frac{1}{\pi} \int_0^\infty dE' \frac{\text{Im} f(0, E')}{E' - E} - \sum_i R_{oj}(E), \quad (12)$$

$$\text{Re} g(0, E) = g_{\text{Born}}(0, E)$$

$$+ \frac{1}{\pi} \int_0^\infty dE' \frac{\text{Im} g(0, E')}{E' - E} - \sum_i R_{ej}(E),$$

summed over the quadratically integrable (and energy  $E_j < 0$ ) bound states of the Hamiltonian (1). The ordinary and exchange residues,  $R_{oj}$  and  $R_{ej}$ , respectively, are (see Appendix A)

functions of the spin-independent Hamiltonian,

$$H = \sum_{i=1}^{Z+1} T_i - \sum_{i=1}^{Z+1} \frac{Ze^2}{r_i} + \sum_{i > j} \frac{e^2}{r_{ij}}, \quad (16)$$

describing the entire system of  $Z+1$  electrons.

Returning to  $e$ -He scattering, the average potential energy of an electron in the field of an undistorted ground-state He atom is negative.<sup>9</sup> It is believable, therefore, that the Hamiltonian Eq. (1) has bound states of negative energy. In particular, it is possible that there is only one such bound state, whose wave function  $w_0(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$  is completely symmetric in  $\mathbf{r}_1$ ,  $\mathbf{r}_2$ ,  $\mathbf{r}_3$ , i.e., of the type forbidden by the exclusion principle. This possibility is consistent with the fact that no actual bound states of  $\text{He}^-$  have been observed. In addition, there are the following theoretical grounds for expecting the ground state of Eq. (1) to be completely symmetric. For the pure central field obtained when the last three terms in Eq. (1) are dropped, the ground-state eigenfunction  $w_0$  obviously belongs to the

<sup>9</sup> N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), 2nd ed., p. 184.

configuration  $(1s)^3$ , and therefore is completely symmetric. As the noncentral last three terms of Eq. (1) are slowly and continuously brought up from zero to their actual values, the ground-state eigenfunction cannot change its symmetry (since such a change would be a discontinuous modification of the eigenfunction) except in the unlikely event that another state of different symmetry becomes degenerate with the completely symmetric state corresponding to  $(1s)^3$  in the pure central field limit. Note that because H is symmetric in  $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ , its eigenfunctions should belong to some irreducible representation of  $\pi_3$ , the three-particle symmetric permutation group<sup>10</sup> (see Appendix B); this is why any change of symmetry of  $w_0$  necessarily is discontinuous, e.g., from completely symmetric to completely antisymmetric. Note also that the long-lived state of  $\text{He}^-$  which has been observed, presumably a  $^4P_{5/2}$  state,<sup>11</sup> does not contribute to Eqs. (13), (14) because it is not a state of energy  $E < 0$  (see Appendix A).

Using Eq. (7b) and the symmetry of  $w_0$ ,

$$w_0(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = w_0(\mathbf{r}_2, \mathbf{r}_1, \mathbf{r}_3). \quad (17)$$

After interchanging the dummy variables  $\mathbf{r}_1, \mathbf{r}_2$  in Eq. (13), it is seen that  $R_{oj} = R_{ej}$  for the symmetric  $j=0$  ground state of  $\text{He}^-$ . Consequently, since the cross-section theorem has the expected form (as can be shown by methods similar to those used previously<sup>1,8</sup>)

$$\text{Im}(f-g)_{\theta=0} = (k/4\pi)\sigma(E), \quad (18)$$

the dispersion relation for ground-state  $e$ -He scattering should be

$$\text{Re}[f-g] = (f-g)_{\text{Born}} + \frac{1}{4\pi^2} \int_0^\infty dE' \frac{k'\sigma(E')}{E'-E}. \quad (19)$$

In Eq. (19),  $f$  and  $g$  are evaluated at  $\theta=0$ ;  $\sigma$  is the total cross section; and the residues  $R_{oj} = R_{ej}$  cancel for the combination  $f-g$ . Thus, the dispersion relation for electron scattering from He in its ground state contains no unphysical contributions from spatially symmetric [satisfying Eq. (17)] exclusion principle forbidden states of  $\text{He}^-$ .<sup>11a</sup>

In ground-state  $e$ -H scattering it was found<sup>1</sup> that no nonsinglet bound states of  $\text{H}^-$  contribute residues to the dispersion relation for the singlet scattering amplitude; similarly no nontriplet bound states of  $\text{H}^-$  contribute to the dispersion relation for the triplet

amplitude. We shall show, correspondingly, that in ground-state  $e$ -He scattering, which necessarily is doublet scattering since the ground state of He is  $^1S_0$ , there can be no residue contributions from quartet bound states of  $\text{He}^-$ . Suppose, for instance, there actually did exist a  $\text{He}^-$  negative energy quartet bound state, e.g., a  $^4S_{3/2}$  state. Because quartet states are completely symmetric in spin coordinates, the spatially dependent part of this quartet  $\text{He}^-$  state must be a completely antisymmetric eigenfunction  $w_1(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$  of the Hamiltonian Eq. (1). Consequently, we now find from Eqs. (13) and (14) that  $R_{oj} = -R_{ej}$  for this bound state, implying the ordinary and exchange residue contributions would not cancel for the combination  $f-g$ . This lack of cancellation is inconsequential, however, because for this quartet state  $R_{oj} = R_{ej} = 0$ , a result which follows from interchange of the dummy variables  $\mathbf{r}_2, \mathbf{r}_3$ , in the first integral of Eq. (13); performing this interchange, noting that  $V_i$  is symmetric in  $\mathbf{r}_2, \mathbf{r}_3$  by (7a), and recalling Eq. (8), it is at once seen that  $R_{oj} = -R_{oj} = 0$  for any antisymmetric  $w_j(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ .

### III. SCATTERING BY $\text{He } ^3S_1$ METASTABLE STATE

We now consider electron scattering from the  $\text{He } ^3S_1$  metastable state whose energy lies 19.8 eV above the ground  $^1S_0$  state, so that the total initial electron binding energy is  $79 - 19.8 = 59.2$  eV. Then with the zero of energy now chosen at  $-59.2$  eV, the total energy  $E = \hbar^2 k^2 / 2m$  as before; consequently, Eqs. (4)-(7) remain unchanged, except that  $u_0(\mathbf{r}_2, \mathbf{r}_3)$  is replaced by  $u_1(\mathbf{r}_2, \mathbf{r}_3)$ , the spin-independent part of the  $^3S_1$  metastable wave function. Since the triplet spin function is symmetric,  $u_1$  must be antisymmetric, i.e.,

$$u_1(\mathbf{r}_2, \mathbf{r}_3) = -u_1(\mathbf{r}_3, \mathbf{r}_2). \quad (20)$$

With  $u_0$  replaced by  $u_1$  in Eqs. (9),  $f$  and  $g$  of Eq. (4) remain identical with  $f$  and  $g$  defined by Eqs. (9), and continue to represent, respectively, the ordinary and exchange amplitudes. Because of Eq. (20), the order of  $\mathbf{r}_2, \mathbf{r}_3$  in  $u_1(\mathbf{r}_2, \mathbf{r}_3)$  now must be carefully watched. That Eqs. (4) and (9) are consistent when the subscript 0 on  $u_0$  is replaced by 1 is obvious from the explicit coordinate representations<sup>8</sup> of the Green's functions used in the time-independent derivations of Eq. (4). In computing the dispersion relation for scattering from the  $^3S_1$  level, the contour in the complex energy plane is the same as for scattering from the ground state. However our shift of the energy scale, so as to keep  $E$  equal to the laboratory kinetic energy of the incident electron, means that in  $^3S_1$  scattering the continuum begins at  $E = -19.8$  eV. As a result the integrals appearing in the dispersion relations now run from  $E' = -19.8$  to  $\infty$ ; except for this change in the lower limit of integration, the dispersion relations for  $^3S_1$  scattering have precisely the form Eq. (12). The residue contributions to the dispersion relations for  $f$

<sup>10</sup> H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1943), Chap. 15.

<sup>11</sup> E. Holg  n, Proc. Phys. Soc. (London) **71**, 357 (1958).

<sup>11a</sup> Note added in proof. Although it seems clear that starting from the Pauli Principle there should be no contribution from the nonphysical states, we know of no simple direct means of taking the exclusion principle into account in a many-particle scattering problem involving particle exchange except for the awkward methods used above. In addition, when the scattering amplitudes are analytically continued to complex energies it is not formally evident that nonphysical states cannot make contributions, although on physical grounds we do not expect them to do so.

and  $g$  are given by Eq. (13) and Eq. (14), respectively, except that for  ${}^3S_1$  scattering  $u_1$  replaces  $u_0$ .

The appropriate modification of Eq. (18) for  ${}^3S_1$  scattering will relate only the positive energy  $E' > 0$  amplitudes  $f(0, E')$ ,  $g(0, E')$  to measured cross sections. Consequently, the dispersion relation for  ${}^3S_1$  scattering includes an integral over a nonphysical negative energy range; obviously an integral over negative energies always will appear, in scattering from hydrogen as well as from atoms with  $Z > 2$ , whenever the atom is initially in an excited state. Such nonphysical integrals are common in applications of dispersion relations at nonforward scattering angles<sup>12</sup> and high energies,<sup>13</sup> but have not been noted heretofore at zero scattering angle and nonrelativistic energies. For  ${}^3S_1$  scattering the eigenfunction  $w_j$  of Eq. (1) corresponding to the  $\text{He}^- {}^4P_{5/2}$  state may make a residue contribution to Eqs. (12), although it did not contribute to the  ${}^1S_0$  dispersion relation; its contribution is necessarily zero, independent of its symmetry under interchange of  $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ , only if its energy  $E_j > 0$  on the scale wherein  $E$  equals the laboratory kinetic energy (see Appendix A).

As in  ${}^1S_0$  scattering, Eqs. (7b) and (17) imply that the residue contributions of the exclusion principle forbidden  $j=0$  ground state of  $\text{He}^-$  satisfy  $R_{oj}=R_{ej}$  for  ${}^3S_1$  scattering; the fact that Eq. (20) differs from Eq. (8) is irrelevant. Consequently, these ordinary and exchange residue contributions do not cancel from the combinations  $f-2g$  and  $f+g$ , which are, respectively, (see below) the quartet and doublet elastic scattering amplitudes for electrons incident on  ${}^3S_1$  He atoms. Nevertheless, unphysical  $w_0(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$  residue contributions again do not appear, because when  $w$  is symmetric in  $\mathbf{r}_2, \mathbf{r}_3$ , Eq. (17), and  $u$  antisymmetric, Eq. (20),  $R_{oj}=-R_{ej}$  and  $R_{oj}=R_{ej}=0$  (just as in the case of quartet contributions to the ground-state doublet scattering). The ordinary and exchange residue contributions to  ${}^3S_1$  scattering from a spatially antisymmetric  ${}^4P$  or  ${}^4S$  need not vanish, since interchange of  $\mathbf{r}_2$  and  $\mathbf{r}_3$  in the first integral of Eq. (13) yields  $R_{oj}=R_{ej}$  when  $u$  and  $w$  are each antisymmetric. But, as in He ground-state scattering, there are no quartet contributions to  ${}^3S_1$  doublet scattering, because residues  $R_{oj}=-R_{ej}$  cancel for the combination  $f+g$ . That  $R_{oj}=-R_{ej}$  for quartet  $\text{He}^-$  bound states follows as usual from comparison of Eqs. (13) and (14) after interchange of  $\mathbf{r}_1, \mathbf{r}_2$  in Eq. (13); the symmetry of the initial atomic wave function  $u$  again is quite irrelevant. It also can be shown (see Appendix B) that the residue contributions from doublet  $\text{He}^-$  bound states satisfy  $R_{oj}=2R_{ej}$ , implying these doublet residue contributions cancel from the  ${}^3S_1$  quartet amplitude combination  $f-2g$ . We remark that the elastic scattering differential cross section is

$$(d\sigma/d\Omega)_\theta = \frac{2}{3} |f-2g|^2 + \frac{1}{3} |f+g|^2, \quad (21a)$$

<sup>12</sup> A. Klein and C. Zemach, Ann. Phys. (New York) **7**, 365 (1959).

<sup>13</sup> N. N. Khuri and S. B. Treiman, Phys. Rev. **109**, 198 (1958).

so that the cross-section theorem for unpolarized particles is

$$(k/4\pi)\sigma = \text{Im}[\frac{2}{3}(f-2g) + \frac{1}{3}(f+g)] = \text{Im}(f-g). \quad (21b)$$

In other words, the effective amplitude for the scattering of *unpolarized* incident electrons and  ${}^3S_1$  He atoms, whose real part appears in the dispersion relation, is  $f-g$ , as in ground-state He scattering.

For pedagogical reasons, as explained in Sec. I, we shall explicitly derive the quartet and doublet  ${}^3S_1$  amplitudes used so crucially in the above discussion; although the procedures admittedly are standard,<sup>6</sup> we know of no reference wherein the derivation may be found. As we have previously asserted, and can be verified from tables of Clebsch-Gordan coefficients,<sup>14</sup> the quartet spin states are completely symmetric in the spins of the three electrons. Thus, for quartet scattering a wave function completely antisymmetric in space and spin is the product of the quartet spin function and a completely antisymmetric space function. This antisymmetrized space function is

$$Z_{3/2}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = (1 - P_{12} - P_{13})\Psi_i^{(+)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \quad (22)$$

where  $\Psi_i^{(+)}$  is given by Eq. (5), with  $\Phi$  everywhere outgoing and  $u_1$  replacing  $u_0$  in Eq. (6);  $P_{ij}$  permutes the coordinates of electrons  $i$  and  $j$ . The function  $Z_{3/2}$  is completely antisymmetric because

$$\Psi_i^{(+)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = -\Psi_i^{(+)}(\mathbf{r}_1, \mathbf{r}_3, \mathbf{r}_2). \quad (23)$$

Equation (23), in turn, holds because of Eq. (20) and because  $\psi_i$ , together with the requirement that  $\Phi$  be everywhere outgoing, completely specifies  $\Psi_i^{(+)}$ ; consequently, noting that  $\Phi(\mathbf{r}_1, \mathbf{r}_3, \mathbf{r}_2)$  necessarily is everywhere outgoing if  $\Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$  is outgoing,  $\Psi_i^{(+)}$  must change sign under interchange of  $\mathbf{r}_2, \mathbf{r}_3$  if its source  $\psi_i$  changes sign under the interchange. Equation (9b) with  $u_1$  replacing  $u_0$  then implies

$$\lim_{r_3 \rightarrow \infty} \int d\mathbf{r}_1 d\mathbf{r}_2 u_1^*(\mathbf{r}_1, \mathbf{r}_2) \Phi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = -g \frac{e^{ikr_3}}{r_3}. \quad (24)$$

Hence,

$$\begin{aligned} \lim_{r_1 \rightarrow \infty} \int d\mathbf{r}_2 d\mathbf{r}_3 u_1^*(\mathbf{r}_2, \mathbf{r}_3) Z_{3/2}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ = e^{ik\mathbf{n}_i \cdot \mathbf{r}_1} + (f-2g) \frac{e^{ikr_1}}{r_1}. \end{aligned} \quad (25)$$

Equation (25) means the quartet scattering amplitude is  $f-2g$ .

For doublet scattering the procedure is slightly more complicated because the spin 1/2 states compounded from the incident electron 1 and the  ${}^3S_1$  spin function of electrons 2, 3 have no simple symmetry. For total spin magnetic quantum number equal to 1/2, this spin

<sup>14</sup> E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, New York, 1959), p. 76.

function is<sup>14</sup>

$$\chi_i^{\frac{1}{2}}(1,2,3) = \left(\frac{2}{3}\right)^{\frac{1}{2}}\chi_1^-\chi_2^+\chi_3^+ - \left(\frac{1}{6}\right)^{\frac{1}{2}}\chi_1^+\chi_2^-\chi_3^+ - \left(\frac{1}{6}\right)^{\frac{1}{2}}\chi_1^+\chi_2^+\chi_3^-, \quad (26)$$

where  $\chi_i^{\pm}$  represents the spin functions of electron  $i$ . As in Eq. (22), the completely antisymmetrized space and spin function is

$$Z_i^{\frac{1}{2}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \chi_1, \chi_2, \chi_3) = [1 - P_{12} - P_{13}] \Psi_i^{(+)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \chi_i^{\frac{1}{2}}(1,2,3), \quad (27)$$

where  $P_{ij}$  now permutes the space and spin coordinates of electrons  $i$  and  $j$ ;  $\Psi_i^{(+)}$  is the same function as in Eqs. (22)–(25). Using these equations and Eq. (9) it is seen that

$$\lim_{r_1 \rightarrow \infty} \int d\mathbf{r}_2 d\mathbf{r}_3 u_1^*(\mathbf{r}_2, \mathbf{r}_3) Z_i^{\frac{1}{2}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \chi_1, \chi_2, \chi_3) = \left[ e^{ikn_i \cdot \mathbf{r}_1} + (f+g) \frac{e^{ikr_1}}{r_1} \right] \chi_i^{\frac{1}{2}}(1,2,3), \quad (28)$$

which means the doublet amplitude is  $f+g$ .

Polarized beams in the laboratory generally are not eigenstates of the total (electron plus atom) spin operator. Suppose the incident beams are polarized along the  $z$  direction; let  $M=1, 0, -1$  denote the atomic magnetic quantum number; let  $+$ ,  $-$  represent the spin orientation of the incident electron. Then it can be shown that

$+$ ,  $M=1$  and  $-$ ,  $M=1$ :

$$k\sigma/4\pi = \text{Im}(f-2g)_{\theta=0};$$

$$(d\sigma/d\Omega)_{\theta=0} = |f-2g|^2,$$

$+$ ,  $M=0$  and  $-$ ,  $M=0$ :

$$k\sigma/4\pi = \text{Im}(f-g)_{\theta=0}; \quad (29)$$

$$(d\sigma/d\Omega)_{\theta=0} = |f-g|^2 + 2|g|^2,$$

$+$ ,  $M=-1$  and  $-$ ,  $M=-1$ :

$$k\sigma/4\pi = \text{Im}(f)_{\theta=0};$$

$$(d\sigma/d\Omega)_{\theta=0} = |f|^2 + 2|g|^2.$$

In each of Eqs. (29) we first quote the cross-section theorem and then give the elastic differential cross section (summed over all possible final spin states), for each of the six possible initial spin state combinations of the colliding particles;  $\sigma$  is as usual the *total* cross section. From Eq. (29) we observe that the sum of the two amplitudes  $+$ ,  $M$  and  $-$ ,  $M$  for each fixed  $M$  is proportional to  $f-g$ , as is the sum over  $M$  of the three amplitudes for fixed electron orientation, [see Eq. (32b) below]. In other words, He scattering is so simple that to obtain a  $^3S_1$  effective amplitude different from  $f-g$  the electrons and atoms must both be polarized; at any rate this is the conclusion for the only cases we consider, namely, those in which the incident electrons and atoms are oriented parallel, antiparallel, or perpendicular to the same direction, which we have taken to be the direction of positive  $z$ .

Equations (29), like Eqs. (21), show that the amplitude appearing in the cross-section theorem need not be simply related to the elastic differential cross section. In general, a simple relation exists only when the incident beams are eigenfunctions of the total spin operator [as in the first line of Eq. (29), which necessarily is an eigenstate of total spin  $3/2$ ] whether or not the incident beams are polarized. Of course the amplitude appearing in the dispersion relation is always the combination of  $f$  and  $g$  appearing in the cross section theorem. We shall sketch the derivation of the third of Eqs. (29). In the case  $+$ ,  $M=-1$ , the completely antisymmetrized wave function is, as in Eq. (27),

$$Z_{\frac{1}{2}, -1}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \chi_1, \chi_2, \chi_3) = [1 - P_{12} - P_{13}] \Psi_i^{(+)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \chi_1^+\chi_2^-\chi_3^-. \quad (30)$$

From Eq. (30), again using Eqs. (9) and (23)–(25), we have

$$\lim_{r_1 \rightarrow \infty} \int d\mathbf{r}_2 d\mathbf{r}_3 u_1^*(\mathbf{r}_2, \mathbf{r}_3) Z_{\frac{1}{2}, -1} = \left[ e^{ikn_i \cdot \mathbf{r}_1} + f \frac{e^{ikr_1}}{r_1} \right] \chi_1^+\chi_2^-\chi_3^- - g \frac{e^{ikr_1}}{r_1} \times [\chi_1^-\chi_2^+\chi_3^- + \chi_1^-\chi_2^-\chi_3^+]. \quad (31)$$

Equation (31) shows that the amplitude coherent with the incident plane wave is  $f$ , but that the total differential cross section is  $|f|^2 + 2|g|^2$ , as in Eq. (29). Of course, with unpolarized beams

$$(d\sigma/d\Omega)_{\theta} = \frac{1}{3} [|f-2g|^2 + |f-g|^2 + 2|g|^2 + |f|^2 + 2|g|^2]. \quad (32a)$$

Equation (32a) agrees with Eq. (21a). Similarly, the amplitude satisfies the cross-section theorem for unpolarized particles, from Eq. (29), is

$$\frac{1}{3}(f-2g+f-g+f) = f-g, \quad (32b)$$

in agreement with Eq. (21b).

## APPENDIX A

We suppose that there exists a bound state of Eq. (1) embedded in the continuum, i.e., having energy exceeding  $-79$  eV, the energy of He in its ground state. Let us examine rather more carefully than heretofore<sup>1</sup> the residue contribution to the dispersion relations made by such a state.

For each bound state if lying below  $-79$  eV, the dispersion relation contains a term,

$$R_j(E) = \frac{1}{2\pi i} \oint dE' \frac{Q(E')}{E' - E}, \quad (33)$$

integrated in the positive direction around an infinitesimal circle centered at the bound-state energy. When the bound state is embedded in the continuum the form Eq. (33) is not quite correct, because the con-

tinuum energies coincide with a cut along the real  $E'$  axis, which can cause  $Q(E')$  to be discontinuous across the cut. Examining the contour deformations leading to the dispersion relations, it is seen that when the bound state lies in the continuum its residue contribution is the real part of  $\frac{1}{2}[R_j^+(E)+R_j^-(E)]$ , where  $R_j^+$  and  $R_j^-$  are, respectively, the residues evaluated above and below the cut.

In the neighborhood of the pole at  $E_j$  the Green's function  $(H-E-i\epsilon)^{-1}$  behaves like

$$(E-E_j)^{-1}w_j(\mathbf{r})w_j^*(\mathbf{r}'),$$

which is single valued. Thus, for purposes of computing the residue, the only many-valued functions we need consider are the plane wave factors appearing in Eq. (6). To keep  $Q(E')$  an analytic function of  $E'$ , the plane-wave parts of  $\psi_{fo}^*$  and  $\psi_{fe}^*$  in Eq. (4) must be functions of  $k$ , not of  $k^*$ ; moreover<sup>1</sup>  $\arg k = \arg(E^{1/2})$  is zero for real positive  $E$  above the cut. Consequently the plane-wave part of  $\psi_{fo}^*$  for complex  $K$  is  $\exp(-ik\mathbf{n}_f \cdot \mathbf{r}_1)$ , the analytic continuation of its form for  $\arg E=0$ , and

similarly for  $\psi_{fe}^*$ . We now see that whenever  $E_j < 0$ , i.e., whenever the energy of this  $j$ th bound state of  $\text{He}^-$  is less than the energy of the atom in its initial state,  $Q(E')$  is single valued at  $E'=E_j$  and Eqs. (13)–(15) are valid, whether or not  $E_j$  is embedded in the continuum. We remark that since all the bound-state wave functions appearing in Eqs. (13) and (14) have definite parity, replacing  $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$  by their reflected values  $-\mathbf{r}_1, -\mathbf{r}_2, -\mathbf{r}_3$  merely changes  $\exp(\alpha_j\mathbf{n}_i \cdot \mathbf{r}_1)$ ,  $\exp(\alpha_j\mathbf{n}_i \cdot \mathbf{r}_2)$  in Eqs. (13), (14) to  $\exp(-\alpha_j\mathbf{n}_i \cdot \mathbf{r}_1)$ ,  $\exp(-\alpha_j\mathbf{n}_i \cdot \mathbf{r}_2)$ , respectively, after multiplying Eqs. (13), (14) by  $\pm 1$ . With this alternative form, used in the previous hydrogen calculations,<sup>1</sup> it is obvious that  $R_{oj}(E)$  is the absolute square of an integral over all  $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ , and therefore necessarily is real. However, because the multiplicative factor  $\pm 1$  depends on the parity of  $w_j$ , it is not true that  $R_{oj}$  is always positive, as was claimed previously.<sup>1</sup>

When  $E_j > 0$ ,  $k$  changes sign crossing the cut, and  $Q(E')$  is correspondingly non-single valued. In place of Eqs. (13)–(15) we have

$$R_{oj}^+(E) = \frac{1}{E-E_j} \frac{m}{2\pi\hbar^2} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 e^{-ik_j\mathbf{n}_i \cdot \mathbf{r}_1} u_0^*(\mathbf{r}_2, \mathbf{r}_3) V_i(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) w_j(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ \times \int d\mathbf{r}_1' d\mathbf{r}_2' d\mathbf{r}_3' w_j^*(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3') V_i(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3') u_0(\mathbf{r}_2', \mathbf{r}_3') e^{ik_j\mathbf{n}_i \cdot \mathbf{r}_1'}, \quad (34)$$

$$R_{oj}^-(E) = \frac{1}{E-E_j} \frac{m}{2\pi\hbar^2} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 e^{-ik_j\mathbf{n}_i \cdot \mathbf{r}_2} u_0^*(\mathbf{r}_1, \mathbf{r}_3) V_f(\mathbf{r}_1, \mathbf{r}_3, \mathbf{r}_2) w_j(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ \times \int d\mathbf{r}_1' d\mathbf{r}_2' d\mathbf{r}_3' w_j^*(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3') V_i(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3') u_0(\mathbf{r}_2', \mathbf{r}_3') e^{ik_j\mathbf{n}_i \cdot \mathbf{r}_1'}, \quad (35)$$

$$R_{oj}^-(E) = [R_{oj}^+(E)]^*, \quad R_{ej}^-(E) = [R_{ej}^+(E)]^*, \quad (36)$$

$$k_j = (2m/\hbar^2)^{1/2} E_j^{1/2}. \quad (37)$$

The relation Eq. (36) presumes, as we have been presuming, that  $w_j(\mathbf{r})w_j^*(\mathbf{r}')$  is real. Since Eq. (1) is real, it always is possible to choose the eigenfunctions consistent with this presumption, whether or not the level  $E_j$  is degenerate.

Using Eqs. (1) and (7a), we observe that

$$V_i u_0 e^{ik_j\mathbf{n}_i \cdot \mathbf{r}_1} = (H - \epsilon_0 - E_j) u_0 e^{ik_j\mathbf{n}_i \cdot \mathbf{r}_1}, \quad (38)$$

in Eqs. (34) and (35), where  $\epsilon_0$  is the energy of the state  $u_0$  (equal to  $-79$  eV for the ground state of  $\text{He}$ ). Integrating by parts in these integrals, permissible because  $w_j$  is a bound state and therefore quadratically integrable, Eqs. (34) and (35) are seen to vanish because, for the bound states  $E_j > 0$  under present consideration,

$$(H - \epsilon_0 - E_j)w_j = 0. \quad (39)$$

When  $E_j < 0$  there are points at infinity in the integrals of Eqs. (13) and (14) where the exponential factors involving  $\alpha_j$  are large and the integrands not necessarily small, even though  $w_j$  and  $u_0$  are bound states; for example  $\exp(-\alpha_j\mathbf{n}_i \cdot \mathbf{r}_1')$  is large at points  $\mathbf{r}_1'$  anti-

parallel to  $\mathbf{n}_i$ . Consequently, the above integration by parts is not permissible when  $E_j < 0$ , and we cannot conclude that the residues Eqs. (13) and (14) are invariably zero. That these residues do not vanish in general is most immediately obvious by supposing that  $V_i$ ,  $u_0$ , and  $w_j$  have no zeros, a supposition which certainly is conceivable for some force laws even if it is not necessarily true for Eqs. (1) and (7). With this supposition the integrands in Eqs. (13) and (14) always would have the same sign, so that the integrals in these equations could not equal zero. For  $E > 0$  the integrand necessarily contains an oscillating function,  $\exp(ik\mathbf{n} \cdot \mathbf{r})$ , and therefore can vanish.

## APPENDIX B

The possible symmetries of the eigenfunctions of the Hamiltonian (16), and the types of symmetries which are eliminated when particle spin and the exclusion principle are taken into account, have been elegantly analyzed with the aid of Young's symmetry operators<sup>15</sup>;

<sup>15</sup> H. Weyl, *Theory of Groups and Quantum Mechanics* (Methuen and Company, Ltd., London, 1931), pp. 358–376.

the "patterns" associated with these symmetry operators make it obvious that for large  $Z$  most of the eigenfunctions of (16) cannot be combined with spin functions in a fashion consistent with the exclusion principle. In the three-electron case, with Hamiltonian (1), there are three patterns, corresponding to the three irreducible representations of  $\pi_3$ . It is at once seen that the exclusion principle rules out only one of these, the completely symmetric representation, in which every element of  $\pi_3$  is represented by unity. Of the two allowed representations, one is the completely antisymmetric representation, in which even permutations are represented by unity, odd permutations by  $-1$ ; the eigenfunctions of (1) belonging to this representation form the space-dependent parts of the quartet bound states of  $\text{He}^-$ . Thus, the space-dependent parts of doublet bound states of  $\text{He}^-$  necessarily are those eigenfunctions of (1) belonging to the other allowed representation of  $\pi_3$ , whose structure is more complicated than either complete symmetry or complete antisymmetry.

The six elements of  $\pi_3$  fall into three conjugate classes<sup>10</sup>  $C_1, C_2, C_3$  composed of one, three, two elements respectively, namely,  $1; (12)=(21)\equiv P_{12}, (23), (13); (123)=(231)=(12)(23), (132)=(13)(23)$ . The even permutations are contained in  $C_1$  and  $C_2$ , the odd permutations in  $C_3$ . The square of each element of  $C_3$  is unity; the cube of each element of  $C_2$  is unity. The existence of precisely three conjugate classes confirms<sup>10</sup> the existence of precisely three inequivalent irreducible representations of  $\pi_3$ . Also, since the sum of the squares of the dimensions of the inequivalent representations must equal<sup>10</sup> the number of elements in  $\pi_3$ , the third (not completely symmetric or antisymmetric) irreducible representation must be two-dimensional. This result implies that each doublet bound state of  $\text{He}^-$  is associated with a degenerate pair of space-dependent eigenfunctions of (1). Thus, at each energy  $E_j$  corresponding to a doublet bound state of  $\text{He}^-$  the Green's function behaves like

$$(E - E_j)^{-1} [w_{j1}(\mathbf{r})w_{j1}^*(\mathbf{r}') + w_{j2}(\mathbf{r})w_{j2}^*(\mathbf{r}')],$$

where  $w_{j1}, w_{j2}$  are independent eigenfunctions of energy  $E_j$  and have the properties made use of in Appendix A, e.g.,  $w_{j1}(\mathbf{r})w_{j1}^*(\mathbf{r}')$  is real. It follows that the total ordinary and exchange residue contributions from any given doublet bound state of  $\text{He}^-$  are found by summing the respective Eqs. (13) and (14) over  $w_{j1}, w_{j2}$ . In fact, interchanging  $\mathbf{r}_1, \mathbf{r}_2$  in (14), and recognizing  $w_1, w_2$  have definite parity, we find for  $^3S_1$  scattering

$$R_{oj}(E) = \int d\mathbf{r} d\mathbf{r}' F [w_{j1}(1,2,3)w_{j1}^*(1',2',3') + w_{j2}(1,2,3)w_{j2}^*(1',2',3')], \quad (40a)$$

$$R_{ej}(E) = \int d\mathbf{r} d\mathbf{r}' F [w_{j1}(2,1,3)w_{j1}^*(1',2',3') + w_{j2}(2,1,3)w_{j2}^*(1',2',3')], \quad (40b)$$

where we have introduced the simplified notation  $w_{j1}(1,2,3) \equiv w_{j1}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ , etc., and

$$F = \pm \frac{1}{E - E_j} \frac{m}{2\pi\hbar^2} e^{-\alpha_j n_i \cdot (\mathbf{r}_1 + \mathbf{r}_1')} \times u_1^*(\mathbf{r}_2, \mathbf{r}_3) u_1(\mathbf{r}_2', \mathbf{r}_3') V_i(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) V_i(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3'). \quad (41)$$

The sign of  $F$  depends on the parity of  $w_{j1}, w_{j2}$  and is the same in each of Eqs. (40).

Because (1) is real, its eigenfunctions  $w_{j1}, w_{j2}$  can be chosen real;  $u_1$ , which satisfies Eq. (20), similarly can be chosen real. Thus, the asterisks can be deleted from Eqs. (40) and (41). Moreover, interchanging  $\mathbf{r}_2, \mathbf{r}_3$  in (40b), we get

$$R_{ej}(E) = - \int d\mathbf{r} d\mathbf{r}' F [w_{j1}(3,1,2)w_{j1}(1',2',3') + w_{j2}(3,1,2)w_{j2}(1',2',3')]. \quad (42)$$

We next observe that since the cube of (123) is unity, in the plane of  $w_{j1}, w_{j2}$  the operation (123) must be a rotation through an angle  $2\pi/3$ . In other words, with  $w_{j1}(1,2,3)$  and  $w_{j2}(1,2,3)$  as a basis,

$$\begin{pmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \quad (43)$$

represents (123). Since  $(123) = (123)^2$ , the matrix correspondingly representing (132) is

$$\frac{1}{4} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}^2 = -\frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}. \quad (44)$$

Consequently, in Eq. (42),

$$w_{j1}(3,1,2) = (132)[w_{j1}(1,2,3)] = -\frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{2}(w_{j1} + \sqrt{3}w_{j2}), \quad (45a)$$

$$w_{j1}(3,1,2) = -\frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}(-\sqrt{3}w_{j1} + w_{j2}). \quad (45b)$$

Substituting Eqs. (45) in (42)

$$R_{ej}(E) = \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' F [w_{j1}w_{j1}' + w_{j2}w_{j2}' + \sqrt{3}(w_{j2}w_{j2}' - w_{j1}w_{j2}')] \quad (46)$$



where now we have further simplified the notation, (23) are, respectively, namely,

$$w_{j1} \equiv w_{j1}(1,2,3), \quad w_{j1}' \equiv w_{j1}(1',2',3'), \quad \text{etc.}$$

Since the asterisks have been dropped, the last two terms in (46) are seen to cancel by virtue of (41). Therefore, comparing (46) and (40a),  $R_{ej} = \frac{1}{2}R_{oj}$ . To seal the proof we should point out that when Eqs. (43) and (44) represent the permutations (123) and (132), respectively, the matrices representing (12), (13), and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad -\frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}, \quad -\frac{1}{2} \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}. \quad (47)$$

The matrices (47) and (44) cannot be brought simultaneously to diagonal form. Thus, the properties ascribed to  $w_{j1}$ ,  $w_{j2}$ , leading ultimately to  $R_{oj} = 2R_{ej}$ , were indeed those appropriate to the irreducible two-dimensional representation of  $\pi_3$ .

## Nuclear Energy Levels of $C^{13}$ ; Two-Stage Reaction $N^{14}(t,n)O^{16*}(\alpha)C^{12\dagger}$

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Alpha spectra from the triton bombardment of natural nitrogen were studied at laboratory angles of 30 and 90° with a double-focusing magnetic spectrometer. Observation of alpha groups from  $N^{14}(t,\alpha)C^{13}$  allowed the positions of energy levels in  $C^{13}$  to be investigated up to an excitation energy of 12.5 MeV. In addition to previously well-determined levels at lower excitations, levels were observed at 10.736, 10.809, 11.000, 11.078, 11.721, and 12.131 MeV. Standard deviations in these values ranged from 0.020 to 0.030 MeV. Previously reported levels at 5.51, 6.10, and 10.47 MeV were not seen. Certain structural features of the alpha continuum underlying  $C^{13}$  groups could be tentatively identified with alpha groups from the two-stage reaction  $N^{14}(t,n)O^{16*}(\alpha)C^{12}$  proceeding through known levels in  $O^{16}$  near 15-MeV excitation. Cross sections are reported for various features of the alpha spectra as well as for corresponding features of the proton spectra.

### I. INTRODUCTION

THE present experiment was primarily concerned with a study of the alpha-particle spectra produced by the bombardment of  $N^{14}$  with 1.8- to 2.6-MeV tritons. These alpha particles could be attributed to five nuclear reactions:

$$N^{14}(t,\alpha)C^{13}, \quad Q_0 = 12.263 \text{ MeV}; \quad (1)$$

$$N^{14}(t,n\alpha)C^{12}, \quad Q_0 = 7.316 \text{ MeV}; \quad (2)$$

$$N^{14}(t,n)O^{16*}(\alpha)C^{12}, \quad Q_0 = 7.316 \text{ MeV}; \quad (3)$$

$$N^{14}(t,C^{12})He^5(\alpha)n, \quad Q_0 = 7.316 \text{ MeV}; \quad (4)$$

$$N^{14}(t,2\alpha)Be^9, \quad Q_0 = 1.612 \text{ MeV}. \quad (5)$$

The energetic relations between the nuclei involved are summarized in Fig. 1.

Alphas from reaction (1) were emitted with discrete energies corresponding to the energy levels of the residual nucleus,  $C^{13}$ . This experiment provided an opportunity of studying  $C^{13}$  levels with good resolution up to an excitation of 12.5 MeV. Energy levels in this region of excitation are discussed in Sec. III A.

Underlying the particle groups from reaction (1) was a continuous energy distribution of alpha particles from

low energies to a maximum energy determined by the  $Q$  value of reaction (2). These alphas were associated with the three-body breakup reaction in which an alpha particle, a neutron, and a  $C^{12}$  nucleus were produced "simultaneously." In this case, the alpha spectrum at a given laboratory angle was determined by the angular correlations of the three emerging particles.

Another possible source of alpha particles was reaction (3), which differed from reaction (2) in that it involved two, two-body breakup reactions in series,

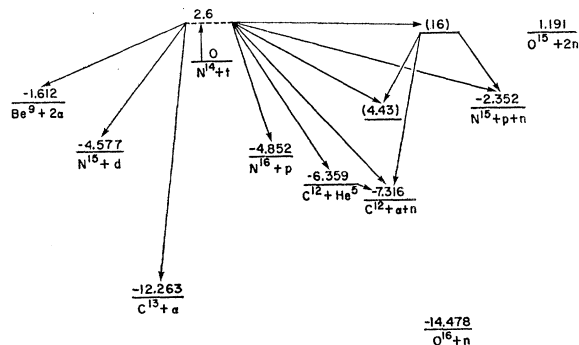


FIG. 1. Some energetic relations in the  $N^{14}+t$  system. Energies are in MeV. Those without parentheses are relative to  $N^{14}+t$ ; those in parentheses are energies of excitation in the nuclei with which they are associated. Arrows represent transitions of concern in the present experiment.

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