Analyticity of the Positions and Residues of Regge Poles

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The amplitude $a(l,k)$ for scattering by a superposition of Yukawa potentials is considered as a meromorphic function of angular momentum l and linear momentum k. An N/D representation with the usual properties is explicitly given, and is then used to show that the position, $\alpha(k)$, and the residue, $\beta(k)$, of its poles in l—the so-called Regge poles—are holomorphic functions of k, under a certain assumption. Further considered as functions of energy, α and $\beta/k^{2\alpha}$ are shown to be real-analytic with no left-hand cut.

I. INTRODUCTION

THE work of Regge *et al.*[1–3] in which is defined a continuation $S(l,k)$ of the physical partial wave S matrix for scattering by a Yukawa potential, into l and k complex, is well known; also familiar is the significance of the poles of $S(l,k)$ as corresponding to bound states and resonances of the potential. It is of great interest to have a "dynamics" of Regge poles, and as a first step we examine the analyticity properties of their positions $\alpha(k)$ and residues $\beta(k)$.

In Sec. II we sketch the definition of $S(l,k)$ given by Bottino *et al.*,[3] and discuss the choice of "physical" Riemann sheets. In Sec. III we give a representation of the amplitude as N/D where N and D have all the usual properties. This representation is used in Sec. IV to prove that $\alpha(k)$ is holomorphic: it is found that as a function of energy, α is real-analytic with no left-hand cut. Some difficulty is encountered with the possibility of unwanted branch points. Finally, in Sec. V we establish exactly the same properties for $\beta(k)/k^{2\alpha(k)}$ as for $\alpha(k)$.

It will be seen that the results concerning α and β are the same as those recently obtained by Barut and Zwanziger on the basis of the Mandelstam representation.[4] In Sec. VI we make a brief comparison of the methods.

II. DEFINITION OF $S(l,k)$ AS A MEROMORPHIC FUNCTION OF l AND k

Our starting point is the Schrödinger radial wave equation for general complex energy and angular momentum:

$$D\Psi(r)\equiv[d^2/dr^2-V(r)-l(l+1)/r^2+E]\Psi(r)=0,$$

where we use units with $\hbar=2m=1$ and $V(r)$ is a short-range potential. One may hope, at least in certain regions of l and k, $(k^2=E)$, to find two types of solutions to this equation, characterized by their boundary conditions: firstly, the *physical scattering solution*, $\varphi(l,k,r)$:

$$\varphi(l,k,r)\underset{r\to 0}{\sim} r^{l+1},\quad \varphi(l,k,r)=\varphi(l,-k,r),$$

and secondly the *ingoing and outgoing waves*, $f(l,\pm k,r)$:

$$f(l,k,r)\underset{r\to\infty}{\sim} e^{-ikr},\quad f(l,k,r)=f(-l-1,k,r).$$

Whenever both types of solution exist we shall be able to expand the physical solution, $\varphi(l,k,r)$, in terms of the ingoing and outgoing waves, f, giving a measure of the amount of scattering in the state (l,k):

$$\varphi(l,k,r)=-\frac{\mathfrak{f}(l,-k)f(l,k,r)-\mathfrak{f}(l,k)f(l,-k,r)}{2ik},\quad (2.1)$$

$$\varphi(l,k,r)\underset{r\to\infty}{\sim}-\frac{\mathfrak{f}(l,-k)e^{-ikr}-\mathfrak{f}(l,k)e^{ikr}}{2ik}.\quad (2.2)$$

This expansion defines the *Jost function*, $\mathfrak{f}(l,k)$, which with some simple algebra can be written

$$\mathfrak{f}(l,k)=W[f(l,k,r),\varphi(l,k,r)],\quad (2.3)$$

where W is the Wronskian

$$W[g(r),h(r)]=g(r)h'(r)-g'(r)h(r).$$

Comparing the behavior (2.2) of the physical solution φ at infinity with that of the free solution φ_0,

$$\varphi_0(l,k,r)\underset{r\to\infty}{\sim}\text{const}\times(e^{-ikr}-e^{-i\pi l}e^{ikr}),$$

one is led to define as (diagonal) S matrix for general l and k:

$$S(l,k)=[\mathfrak{f}(l,k)/\mathfrak{f}(l,-k)]e^{i\pi l}.\quad (2.4)$$

Since the Jost function $\mathfrak{f}(l,k)$ is the Wronskian of f and φ, (2.3), it will be holomorphic[5] wherever f, φ, f', and φ' are. Then by (2.4), $S(l,k)$ will be meromorphic[5] in the intersection of the domains in which $\mathfrak{f}(l,\pm k)$ are holomorphic. The holomorphy of the wave functions φ and f and their derivatives has been investigated by Bottino *et al.*[3] and Squires[6] subject to three condi-

[1] T. Regge, Nuovo cimento **14**, 951 (1959).
[2] T. Regge, Nuovo cimento **18**, 947 (1960).
[3] A. Bottino, A. M. Longoni, and T. Regge, Nuovo cimento **23**, 954 (1962).
[4] A. O. Barut and D. E. Zwanziger, Phys. Rev. **127**, 974 (1962).
[5] For functions of one or more complex variables, we use "holomorphic" to imply the existence of a power series expansion at every point. The traditional terms, "analytic" or "regular," have unfortunately lost meaning through imprecise use. A "meromorphic" function is locally the quotient of two holomorphic functions; in one variable this means just "analytic except for poles."
[6] E. J. Squires, University of California Radiation Laboratory Report UCRL-10033, 1962 (unpublished).

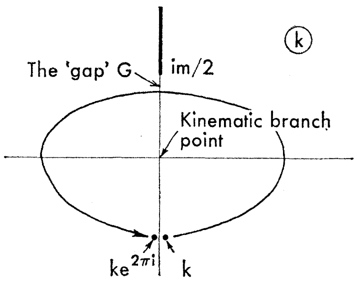

FIG. 1. The path for continuation of $S(l,k)$ through the "gap," $G=\{k|k=iK, 0<K<m/2\}$.

tions[7] on $V(r)$: (i) $V(r)=\int_{m>0}^{\infty} d\mu\, \sigma(\mu)e^{-\mu r}/r$, (ii) $\int_0^\infty d\rho\, \rho|V(\rho e^{i\theta})|<\infty$ for all $|\theta|<\pi/2$, and (iii) $rV(r)$ regular at $r=0$. Their results are that $\varphi(l,k,r)$, $\varphi'(l,k,r)$ are holomorphic in (l plane)$\times$(k plane) and $f(l,k,r)$, $f'(l,k,r)$ are holomorphic in (l plane)$\times$(k plane cut on positive imaginary axis). It follows that $\mathfrak{f}(l,k)$ is holomorphic in the latter domain and $\mathfrak{f}(l,-k)$ in a similar domain, but with the cut on the negative imaginary k axis.

Since now $\mathfrak{f}(l,\pm k)$ together have cuts on the whole imaginary k axis, we must *a priori* regard Eq. (2.4) as defining two separate meromorphic functions on disjoint domains; the two domains are of course the products of the whole l plane with the right and left half k planes. We adopt (2.4) as definition of $S(l,k)$ in the right half k plane, and show that it can be continued through the imaginary k axis. To do this we must examine the singularities of $\mathfrak{f}(l,k)$ in its cut. These arise from those of $f(l,k,r)$, and by examining the integral equation for $f(l,k,r)$, Bottino *et al.*[3] have shown that $\mathfrak{f}(l,k)$ has a potential-independent branch point at $k=0$, and then a "potential cut" starting at $k=im/2$. In fact, the circuit relation for a tour once around the origin going through the gap, $G=\{k|k=iK, 0<K<m/2\}$ is just

$$\mathfrak{f}(l,ke^{2\pi i})=\mathfrak{f}(l,k)-2i\sin\pi l\,\mathfrak{f}(l,ke^{\pi i}) \tag{2.5}$$

(see Fig. 1).

In general, we shall try to define the functions $f(l,k,r)$, $\mathfrak{f}(l,k)$, etc., as single-valued functions, holomorphic on cut planes—or their first Riemann sheets. In this way k^{2l} will be defined as the holomorphic function which is real for k and l real and is single-valued on the k plane cut on its negative imaginary axis.

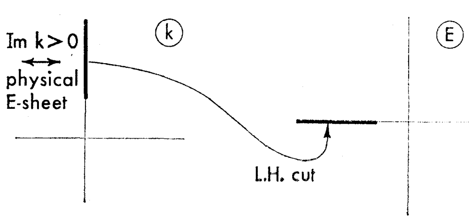

FIG. 2. The correspondence between k and E planes.

The physical E sheet ($E=k^2$), in which the Mandelstam representation is valid, corresponds to the half-plane $\mathrm{Im}k>0$. Thus, a cut on the positive imaginary k axis becomes a left-hand (L.H.) cut in the E plane (see Fig. 2). For brevity we could call such a cut the L.H. cut in either plane.

A function such as $\mathfrak{f}(l,-k)$ which is holomorphic in $\mathrm{Im}k>0$ has no L.H. cut (in E). Further, if such a function is real on the positive imaginary k axis, it is real-analytic in E.[8]

On the other hand, the right-hand (R.H.) cut arises purely from the identification of the positive and negative real k axes in the positive real E axis. We shall encounter functions even in k, $N(k)=N(-k)$, which therefore have no R.H. cut.

Finally, before writing down an explicit form for $S(l,k)$ continued through the gap, we make a note of the properties of φ, f, and $\mathfrak{f}$ under complex conjugation. These follow from the Schrödinger equation and boundary conditions, and are

$$\varphi^*(l,k,r)=\varphi(l^*,k^*,r),$$
$$f^*(l,k,r)=f(l^*,k^*e^{-\pi i},r);$$

hence

$$\mathfrak{f}^*(l,k)=\mathfrak{f}(l^*,k^*e^{-\pi i}), \tag{2.6}$$

where we use

$$g^*(x)\equiv[g(x)]^*.$$

III. REPRESENTATION OF $a(l,k)/k^{2l}$ AS N/D

Using the absence of singularities of $\mathfrak{f}(l,k)$ on the gap, $G=\{k|k=iK,\ 0<K<m/2\}$, we may continue $S(l,k)=\mathfrak{f}(l,k)e^{i\pi l}/\mathfrak{f}(l,-k)$, or equally the *amplitude*,

$$a(l,k)=[S(l,k)-1]/2ik,$$

from the right half k plane to the left. It turns out that a/k^{2l} is real-analytic in E and can be written as N/D; so we define

$$\tilde{a}(l,k)\equiv\frac{a(l,k)}{k^{2l}}\equiv\frac{S(l,k)-1}{2ik^{2l+1}}$$
$$=\frac{\mathfrak{f}(l,k)e^{i\pi l}-\mathfrak{f}(l,ke^{-\pi i})}{2ik^{2l+1}\mathfrak{f}(l,ke^{-\pi i})}$$
$$=N(l,k)/D(l,k),$$

say, where

$$D(l,k)=(-ik)^l\mathfrak{f}(l,ke^{-\pi i}) \tag{3.1}$$

and

$$N(l,k)=[\mathfrak{f}(l,k)e^{i\pi l}-\mathfrak{f}(l,ke^{-\pi i})]/2(ik)^{l+1}. \tag{3.2}$$

These N and D functions have the usual properties: both are real-analytic in l and E and they have only the

[7] These three conditions on $V(r)$ are much more than is necessary, as has been shown by Froissart. [M. Froissart, Princeton, 1962 (to be published)]. The third condition was added by Squires to those used by Bottino *et al.* to ensure holomorphy of $\varphi(l,k,r)/\Gamma(2l+1)$ in the whole l plane. The factor $\Gamma(2l+1)$ obviously cancels out of $S(l,k)$; we can ignore it by redefining $\varphi(l,k,r)$ to include it.

[8] A function $g(E)$, meromorphic in some neighborhood of the real E axis, is real-analytic (or Hermitian) in E if $g(E^*)=g^*(E)$.

L.H. and R.H. cuts, respectively. These follow directly from the circuit relation (2.5) for $\mathfrak{f}(l,ke^{2\pi i})$ and the conjugation relation (2.6) for $\mathfrak{f}^*(l,k)$, as below.

D. From the definition (3.1) it is clear that $D(l,k)$ is holomorphic in the product of the l plane with the k plane cut on its negative imaginary axis (see Fig. 3). Further, from (2.6), $\mathfrak{f}(l,k)$ is clearly real for l real, k negative imaginary. Thus, $D(l,k)$ is real for l real, k positive imaginary; that is, D is real-analytic in l and E with no L.H. cut.

N. Since $\tilde{a}(l,k)$ was defined on the whole k plane by continuing through the gap G, we see that $N(l,k)$ is certainly holomorphic in the product of the l plane with the k plane cut on its imaginary axis from $k=im/2$ upwards and $k=0$ downwards. However, using the circuit relation (2.5) in the definition (3.2) of N, one can easily verify that N is even in k, $N(l,ke^{\pi i})=N(l,k)$. Thus, as anticipated earlier, N has no R.H. cut in E; similarly it has no branch point at $k=0$. (See Fig. 4.)

Finally, using both circuit and conjugation relations (2.5) and (2.6) we find that $N(l,k)$ is real for l real, $k=iK$, $0<K<m/2$; thus $N(l,k)$ is real-analytic in l and E with no R.H. cut.

The amplitude $\tilde{a}(l,k)=N/D$ is real-analytic in l and E, since both N and D are also. Its domain of definition as a meromorphic function is simply the intersection of the domains in which $N(l,k)$ and $D(l,k)$ are holomorphic (Figs. 3 and 4). In particular, it has the usual right and left cuts in E which we associate with partial wave amplitudes.

IV. ANALYTICITY OF THE POSITION $\alpha(k)$ OF REGGE POLES

If the S matrix, as a function of l, has a pole at $l=\alpha$, then the physical solution satisfies [see (2.2)]:

$$\varphi(\alpha,k,r) \underset{r\to 0}{\sim} r^{\alpha+1}$$

$$\underset{r\to\infty}{\sim} \text{const}e^{ikr},$$

that is, φ is an outgoing wave only, and we have a resonance or, if $k=iK$, $K>0$, a bound state.

Since $\tilde{a}(l,k)=N(l,k)/D(l,k)$, it can have poles only when $D(l,k)=0$: that is, the Regge poles do indeed lie on the analytic surfaces, $D(l,k)=0$. To solve this for l as a function of k we invoke the *implicit function theorem:*

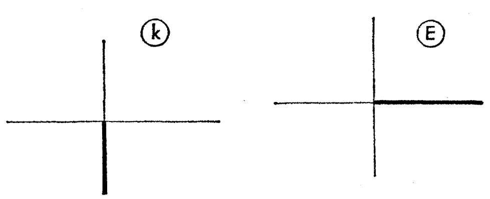

FIG. 3. The domains in which $D(l,k)$, $\alpha(k)$, and $\beta(k)$ are holomorphic as functions of k and of E.

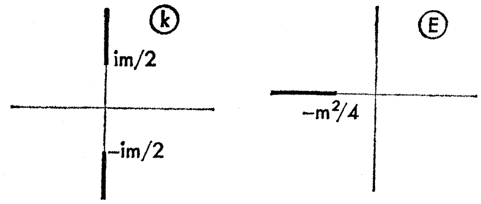

FIG. 4. The domains in which $N(l,k)$ is holomorphic as functions of k and of E.

If $F(l,k)$ is holomorphic in some domain $\mathfrak{D}$ and $(l_0,k_0)\in\mathfrak{D}$ is such that $F(l_0,k_0)=0$, $F_l(l_0,k_0)\neq 0$, then there exists a neighborhood, $N=N_{l_0}\times N_{k_0}\subset\mathfrak{D}$, of (l_0,k_0) such that for each $k\in N_{k_0}$ there is an unique and holomorphic solution $l=\alpha(k)\in N_{l_0}$ of the equation $F(l,k)=0$. (We have written $F_l\equiv\partial F/\partial l$.)

Applying this theorem to $D(l,k)$, which is holomorphic in the product of the l plane with the k plane cut on its negative imaginary axis, we see that the "Regge surfaces" can indeed be written as $l=\alpha(k)$, where $\alpha(k)$ is locally holomorphic except (i) on the negative imaginary k axis and (ii) at a number of *isolated critical points* determined by the condition $D(l,k)=D_l(l,k)=0$. We return to these latter shortly.

First we show that $\alpha(k)$ is real when k is positive imaginary ($E<0$). This was shown by Regge[2] provided $\text{Re}\,\alpha>-1/2$. That it can be shown in general was pointed out by Burke[9]; it has been shown by Mandelstam[10] that the full amplitude $f(E,\cos\theta)$ can be written in such a way that the contribution from a Regge pole goes as $(\cos\theta)^{\alpha(k)}$ as $\cos\theta$ goes to infinity, for any energy. Now this means that the discontinuity across the $\cos\theta$ real axis also goes as $(\cos\theta)^{\alpha(k)}$; but in the region $E<0$ (corresponding roughly to the "crossed channel") this discontinuity is real. It follows that $\alpha(k)$ must be real for $E<0$, unless the poles occur in complex conjugate pairs.

In particular this means that $\alpha(k)$ is real-analytic in E with no L.H. cut, assuming no such conjugate pole-pairs occur.[11]

The Critical Points

To complete the proof that the various surfaces $l=\alpha(k)$ are in fact holomorphic everywhere in the cut k plane, we must show that they are holomorphic at the critical points, which are given by

$$D(l,k)=D_l(l,k)=0, \tag{4.1}$$

or that such points do not exist. The condition (4.1) is precisely the condition for a multiple pole in $\tilde{a}(l,k)$; and

[9] P. Burke (private communication).

[10] S. Mandelstam, University of Birmingham, 1962 (to be published).

[11] At least for the physically interesting poles which emerge into the region $\text{Re}l>-\frac{1}{2}$, this is implied by the assumption below of no multiple poles. However, a recent report by Ahmadzadch, Burke, and Tate at the CERN. conference 1962, shows that double poles and branch points may occur for poles confined to $\text{Re}l<-\frac{1}{2}$.

by generalizing the implicit function theorem we can see that such a multiple pole is a necessary, but not sufficient, condition for a branch point in the Regge surface.

It seems clear from the nature of the eigenvalue problem which determines the Regge poles that no such double solutions will occur, but a theorem sufficiently general to cope with the problem is hard to prove. In any case we have shown above that no branch points occur on the negative real E axis. In what follows, the existence of branch points off the axis is immaterial, since we can confine our attention to a strip near the axis. On the basis of intuition and for aesthetic reasons we shall simply assume that no multiple poles occur.

The domains in which α is holomorphic as functions of k and E respectively are those in Fig. 3.

V. ANALYTICITY OF THE RESIDUE, $\beta(k)$

The residue $\beta(k)$ of $a(l,k)$ at the Regge pole $l=\alpha(k)$ is

$$\beta(k)=\lim_{l\to\alpha(k)}[l-\alpha(k)]a(l,k),$$

or

$$\beta(k)/k^{2\alpha(k)}=N(\alpha(k),k)/D_l(\alpha(k),k),$$

where we recall $D_l\equiv\partial D/\partial l$.

Clearly, $\beta(k)/k^{2\alpha}$ is holomorphic wherever all of $\alpha(k)$, $D(l,k)$ and $N(l,k)$ are. The first two functions are holomorphic on the k plane cut on the negative imaginary axis, while $N(l,k)$ in addition to this cut has a cut running up from $k=im/2$. Thus, $\beta(k)$ is certainly holomorphic in the k plane with both these cuts.

Further, since $\alpha(k)$ is real for k positive imaginary, while $N(l,k)$ and $D(l,k)$ are real for l real and $k=iK$, $0<K<m/2$ (at least), $\beta(k)/k^{2\alpha(k)}$ is real-analytic in E.

Finally, we can show that $\beta(k)$ has no L.H. cut at all (see Fig. 3). It is clearly sufficient to show that $N(\alpha(k),k)$ has none. Now at a Regge pole:

$$N(\alpha(k),k)=\mathfrak{f}(\alpha(k),k)e^{i\pi\alpha}/2(ik)^{\alpha+1},$$

from (3.2); and from (2.1):

$$\varphi(\alpha(k),k,r)=\mathfrak{f}(\alpha(k),k)f(\alpha(k),-k,r)/2ik.$$

Now all three functions in this equation continue through the "gap," $\{k|k=iK,\ 0<K<m/2\}$, and thus so does the equation. But neither φ nor f has any cut on the positive imaginary k axis at all: thus nor does $\mathfrak{f}(\alpha(k),k)$; and finally, $N(\alpha(k),k)$ has no L.H. cut—even though $N(l,k)$ in general does.

We conclude that $\beta(k)/k^{2\alpha(k)}$ is real-analytic in E with no L.H. cut.

VI. COMPARISON WITH THE RELATIVISTIC CASE

As already remarked, the results that α and $\beta/k^{2\alpha}$ are real-analytic in E with no L.H. cut were obtained by Barut and Zwanziger on the basis of the Mandelstam representation. Since the latter is known to be true for potential scattering these results are not new; nonetheless a proof using the N/D representation of Sec. III seems in a sense more direct.

The relativistic case differs from that considered here because of the presence of crossed channels. In the first place, these give rise to an exchange potential; such a potential forces one to introduce two amplitudes $a^{\pm}$ but does not otherwise affect the results. What does make an important difference is the more complicated structure of the absorptive parts. As shown by Barut and Zwanziger, one can write

$$\tilde{a}(l,k)=\int_{t_0}^{\infty}\frac{dt}{t^{l+1}}F(-4E/t)f_t(E,t),\quad\text{for}\quad \mathrm{Re}\,l>\text{some }N,$$

apart from an unimportant factor. Here the hypergeometric function $F(-4E/t)$ is real-analytic in E with only an L.H. cut from $-t/4$ to $-\infty$; $f_t(E,t)$ is the usual t-absorptive part. In potential scattering f_t is real-analytic in E with only an R.H. cut, arising from the double spectral function ρ_{st}. In the relativistic case f_t has both an R.H. cut due to ρ_{st} and an L.H. cut (in k^2) from $-(t+t_0)/4$ to $-\infty$, due to ρ_{tu}.

Using a power series expansion of F, Barut and Zwanziger show that the L.H. discontinuity of F does not appear in the pole terms of $\tilde{a}$. In the potential case this means that the pole terms have no L.H. cut at all, as they point out.

VII. CONCLUSION

Starting from the representation of $S(l,k)$ as the quotient of two Jost functions in the right half k plane, we have pointed out how it can be continued through a gap in the imaginary k axis. The related amplitude $\tilde{a}(l,k)$ has been given an equivalent representation as N/D, and then if we write

$$\tilde{a}(l,k)\equiv a/k^{2l}\equiv(S-1)/2ik^{2l+1}$$

$$=\sum_{\text{dominant poles}}\frac{\beta(k)}{k^{2\alpha(k)}[l-\alpha(k)]}+\tilde{a}_0(l,k),$$

we have shown that the terms $\tilde{a}$, $\sum_{\text{poles}}$, and $\tilde{a}_0$ are each real-analytic in l and E. The fact that α and $\beta/k^{2\alpha}$ have no L.H. cut in E means, quite surprisingly, that the L.H. cut in $\tilde{a}(l,k)$ is confined exclusively to the "background term," $\tilde{a}_0(l,k)$.

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