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Complex Angular Momentum in Two-Channel Problems

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Analytic properties of the two-channel S matrix are investigated, and the threshold behavior of each channel is also studied. We conclude that a certain class of potentials leads to an S matrix which has branch points in linear momentum space and is an entire meromorphic function in the complex angular momentum plane. A formula is given which displays the discontinuity across the threshold cut as an explicit function of the angular momentum.

1. INTRODUCTION

CURRENT conjectures on the role of Regge poles in elementary particle reactions are largely based on Regge's work which treats elastic scattering by Yukawa-type potentials.[1] However, we must also consider problems in which the sum of initial masses differs from the sum of final masses. In this note, a nonrelativistic analog of this problem is studied. We investigate multichannel potential scattering, the channels differing in their masses and thresholds. For simplicity we work out the two-channel case, from which the generalization to N channels is obvious.

In Sec. 2, we formulate the S-matrix theory using Regge's normalization convention. In Sec. 3 it is shown that Froissart's recent work can be generalized to two channels. In Sec. 4 the threshold behavior is studied for each channel, and in Sec. 5, domains of holomorphy of the S matrix are considered.

2. THE TWO-CHANNEL S MATRIX

In this section we present the S-matrix formalism which appears especially convenient for investigating complex angular momentum. In nonrelativistic potential scattering, the channels differ in their masses and threshold energies. However, one can suppress the explicit mass dependence in the normalization.[2] Thus, we write the radial wave equation as

$$\frac{d^2}{dx^2}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}+\begin{pmatrix}k_1^2 & 0\\ 0 & k_2^2\end{pmatrix}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}-\left(\frac{\lambda^2-\frac{1}{4}}{x^2}\right)\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}=\begin{pmatrix}V_{11}(x) & V_{12}(x)\\ V_{21}(x) & V_{22}(x)\end{pmatrix}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}, \quad (1)$$

where $\psi_1(x)$ and $\psi_2(x)$ are the wave functions for channels 1 and 2, with linear momenta k_1 and k_2. Also λ is 1/2 plus the conventional orbital quantum number and will be taken as complex variable. The off-diagonal elements of $V_{ij}(x)$ are responsible for the interaction between channels, and by time-reversal invariance, the matrix $V(x)$ is real and symmetric. We shall assume that the potential is spherically symmetric and of Yukawa type:

$$V_{ij}(x)=x^{-1}\int_{\mu_0>0}^{\infty} d\mu\, \sigma_{ij}(\mu)e^{-\mu x}, \quad (2)$$

where

$$\sigma_{ij}(\mu)=O(\mu^{-\epsilon_0}) \quad \text{for} \quad \epsilon_0>0 \quad \text{as} \quad \mu\to\infty.$$

For simplicity, the lower limits of integration will be taken the same for all elements. This condition implies that

$$\lim_{x\to 0} x^{2-\epsilon}V(x)=0 \quad \text{for} \quad \epsilon<\epsilon_0, \quad (3a)$$

and

$$\lim_{x\to\infty} e^{ax}V(x)=0 \quad \text{for} \quad a<\mu_0. \quad (3b)$$

In the two-channel problem, there are four independent solutions, which are ordinarily chosen to satisfy convenient boundary conditions. We form two 2×2 matrices $F(\lambda,K,x)$ and $\Phi(\lambda,K,c)$ whose columns are these independent solutions. Using the two properties of the potential matrix given in Eq. (3), we can choose F and Φ in such a way that

$$F(\lambda,K,x)\to F^0(\lambda,K,x)=\left(\frac{\pi}{2}\right)^{1/2}\exp[-i\pi\tfrac{1}{2}(\lambda+\tfrac{1}{2})]\begin{bmatrix}(k_1x)^{1/2}H_\lambda^{(2)}(k_1x) & 0\\ 0 & (k_2x)^{1/2}H_\lambda^{(2)}(k_2x)\end{bmatrix}\to\begin{bmatrix}\exp(-ik_1x) & 0\\ 0 & \exp(-ik_2x)\end{bmatrix} \quad \text{as} \quad x\to\infty,$$

and

$$\Phi(\lambda,K,x)\to\Phi^0(\lambda,K,x)=2^\lambda\Gamma(\lambda+1)x^{1/2}\begin{bmatrix}k_1^{-\lambda}J_\lambda(k_1x) & 0\\ 0 & k_2^{-\lambda}J_\lambda(k_2x)\end{bmatrix}\to\begin{bmatrix}x^{\lambda+1/2} & 0\\ 0 & x^{\lambda+1/2}\end{bmatrix} \quad \text{as} \quad x\to 0. \quad (5)$$

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[1] A. Bottino, A. M. Longoni, and T. Regge, Istituto di Fisica dell'Univesita, Torino, 1961 (to be published).
[2] See, for instance, R. G. Newton, J. Math. Phys. 2, 188 (1961).

The superscript "0" denotes a solution for zero potential. Equation (1) involves K and λ only as squares, and thus it follows from the boundary conditions that $F(\lambda,K,x)$ is an even function of λ and $\Phi(\lambda,K,x)$ is an even function of K.

Evidently, two possible choices for the complete set of four independent solutions are $F(\lambda,K,x)$ and $F(\lambda,-K,x)$ or $\Phi(\lambda,K,x)$ and $\Phi(-\lambda,K,x)$. Thus, $\Phi(\lambda,K,x)$ can be expressed in the standard way as

$$\Phi(\lambda,K,x)=(1/2i)\{F(\lambda,-K,x)K^{-1}F^T(\lambda,K)-F(\lambda,K,x)K^{-1}F^T(\lambda,-K)\}, \quad (6)$$

where[3]

$$F(\lambda,\pm K)=W[\Phi(\lambda,K,x),F(\lambda,\pm K,x)]. \quad (7)$$

From the asymptotic behavior of Eq. (6) as $x\to\infty$, it follows that one can define the S matrix as

$$S(\lambda,K)=K^{-1/2}F^T(\lambda,K)[F^T(\lambda,-K)]^{-1}K^{1/2}e^{i\pi(\lambda-1/2)}, \quad (8)$$

which is unitary for physical λ and K. By considering the behavior of $\Phi(\lambda,K,x)$ at the origin, we deduce that

$$F(\lambda,-K)K^{-1}F^T(\lambda,K)-F(\lambda,K)K^{-1}F^T(\lambda,-K)=0. \quad (9)$$

From this it follows that $S(\lambda,K)$ is symmetric:

$$S(\lambda,K)=K^{1/2}F^{-1}(\lambda,-K)F(\lambda,K)K^{-1/2}e^{i\pi(\lambda-1/2)}. \quad (8')$$

We will call $F(\lambda,K)$ the Jost matrix and will study its analytic properties in the next section.

3. ANALYTIC PROPERTIES OF THE JOST MATRIX

In order to investigate analytic properties of the Jost matrix, we shall solve the differential equation in Eq. (1) for $\Phi(\lambda,K,x)$ and, using the solution, evaluate $F(\lambda,K)$ by taking the Wronskian of Eq. (7) in the limit $x\to\infty$. For convenience, we regard the matrix as a function of three independent complex variables λ, k_1, and k_2. The interdependence of k_1 and k_2 for physical values will be discussed in a later section.

Let us transform the wave equation into an integral form

$$\Phi(\lambda,K,x)=\Phi^0(\lambda,K,x)+\int_0^x dx'\,G_\lambda(Kx,Kx')V(x')\Phi(\lambda,K,x'), \quad (10)$$

where

$$G_\lambda(Kx,Kx')=-(\pi/2)(xx')^{1/2}\theta(x-x')\begin{bmatrix} J_\lambda(k_1x')N_\lambda(k_1x)-J_\lambda(k_1x)N_\lambda(k_1x') & 0 \\ 0 & J_\lambda(k_2x')N_\lambda(k_2x)-J_\lambda(k_2x)N_\lambda(k_2x') \end{bmatrix}.$$

This equation can be formally solved by iteration.

$$\Phi(\lambda,K,x)=\Phi^0(\lambda,K,x)+\int_0^x dx'\,I_{\lambda K}(x,x')\Phi^0(\lambda,K,x'), \quad (11)$$

where

$$I_{\lambda K}(x,x')=\sum_{n=1}^{\infty} I_{\lambda K}{}^{(n)}(x,x')$$

and

$$I_{\lambda K}{}^{(n)}(x,x')$$
$$=K^{-n}\int_{x'}^{x} dx_1\,G_\lambda(Kx,Kx_1)V(x_1)\int_{x'}^{x_1} dx_2\,G_\lambda(Kx_1,Kx_2)V(x_2)\cdots\int_{x'}^{x_{n-2}} dx_{n-1}\,G_\lambda(Kx_{n-2},Kx_{n-1})V(x_{n-1})G_\lambda(Kx_{n-1},Kx').$$

Using the relation

$$W[(x)^{1/2}J_\lambda(x),\quad (x)^{1/2}N_\lambda(x)]=2/\pi, \quad (12)$$

we obtain from Eqs. (7) and (11)

$$F^T(\lambda,K)=\left(\frac{2}{\pi}\right)^{1/2}2^\lambda\Gamma(\lambda+1)K^{-\lambda+1/2}e^{-i\pi(\lambda-1/2)}+\int_0^\infty dx\,F^0(\lambda,K,x)V(x)\Phi^0(\lambda,K,x)$$
$$+\int_0^\infty dx\int_0^x dx'\,F^0(\lambda,K,x)V(x)I_{\lambda K}(x,x')V(x')\Phi^0(\lambda,K,x'). \quad (13)$$

[3] "T" denotes the transpose matrix. The Wronskian of two matrix functions $A(x)$ and $B(x)$ will be defined as

$$W[A(x),B(x)]=A^T(x)dB(x)/dx-dA^T(x)/dxB(x).$$

In the remainder of this section, we shall study the analytic properties of $F^T(\lambda,K)$, using the method developed for the single-channel case by Froissart.[4] He divides the region of integration of Eq. (13) into $\int_0^1 + \int_1^\infty$, or in the case of the third term, into the zones

$$\begin{aligned} &(1)\quad 0<x'<x<1,\\ &(2)\quad 0<x'<1<x,\\ &(3)\quad 1<x'<x. \end{aligned} \tag{14}$$

For an integration parameter smaller than 1, a power series expansion will be used for the integrand. For a range of integration extending to infinity, the integrand is represented as a superposition of exponential functions.

In the two-channel problem, Froissart's argument can be applied in a straightforward manner in case the range of integration is smaller than 1, and also in region (2). However, region (3) requires a careful analysis.

From the explicit form of the kernel $G_\lambda(Kx,Kx')$ and using the fact that G_λ is diagonal, one can show that each element of the Green's function matrix $I_{\lambda K}(x,x')$ is bounded by

$$|[I_{\lambda K}(x,x')]_{ij}| < M_{\lambda K} e^{\nu(x-x')}(x-x'), \tag{15}$$

where $\nu=\max\{|\mathrm{Im}k_1|, |\mathrm{Im}k_2|\}$ and $1<x'<x$. $M_{\lambda K}$ is a positive number depending only on λ and K. Since the potential decreases exponentially, one can immediately find a domain of holomorphy of $F^T(\lambda,K)$ in the K variables:

$$0<|\mathrm{Im}k_1|<\tfrac{1}{2}\mu_0,\quad 0<|\mathrm{Im}k_2|<\tfrac{1}{2}\mu_0. \tag{16}$$

The origin is excluded to avoid the branching point.

In order to enlarge the holomorphy domain beyond that given by Eq. (16), we shall use the method of strip-by-strip continuation.[4,5] Let us first take the Laplace transform of the Green's function.

$$\mathcal{L}(p,p')=\int_1^\infty dx\, e^{-px}\int_1^x dx'\, e^{-p'x'} I_{\lambda K}(x,x'). \tag{17}$$

$\mathcal{L}(p,p')$ is holomorphic in the domain

$$\mathrm{Re}(p+p')>0,\quad \mathrm{Re}p>\nu. \tag{18}$$

We shall then investigate $\mathcal{L}(p,p')$ by using the fact that $I_{\lambda K}(x,x')$ satisfies the following equation (and a similar equation in x'):

$$\left[\frac{d^2}{dx^2}+K^2-\frac{(\lambda^2-\frac{1}{4})}{x^2}-V(x)\right]I_{\lambda K}(x,x')=E\delta(x-x'),$$

where E denotes the unit matrix. Taking the Laplace transforms of these differential equations, we obtain

$$\frac{\partial^2}{\partial p^2}\{(p^2+K^2)\mathcal{L}(p,p')\}-(\lambda^2-\tfrac{1}{4})\mathcal{L}(p,p')-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dq\,\tilde{u}(q)\mathcal{L}(p-q,p')=\frac{\partial^2}{\partial p^2}\left\{\frac{e^{-(p+p')}}{p+p'}\right\} \tag{19a}$$

and

$$(\partial^2/\partial p'^2)\{(p'^2+K^2)\mathcal{L}(p,p')\}-(\lambda^2-\tfrac{1}{4})\mathcal{L}(p,p')-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dq\,\tilde{u}(q)\mathcal{L}(p,\,p'-q)=\frac{\partial^2}{\partial p'^2}\left\{\frac{e^{-(p+p')}}{p+p'}\right\}+D(p,p'), \tag{19b}$$

with

$$\tilde{u}(q)=\int_1^\infty dx\, e^{-qx}x^2V(x)$$

and

$$D(p,p')=\frac{\partial^2}{\partial p'^2}\left\{e^{-p'}\int_0^\infty dx\, e^{-px}\left(\left[\frac{\partial}{\partial x'}I(x,x')\right]_{x'=1}+p'I(x,1)\right)\right\}.$$

It can be shown that $D(p,p')$ satisfies a differential equation similar to Eq. (19) with an entire function as a source. As we shall see, the $D(p,p')$ term will not introduce any new singularities into $\mathcal{L}(p,p')$. Equation (19) will be used to continue $\mathcal{L}(p,p')$ to smaller values of $\mathrm{Re}p$ and $\mathrm{Re}p'$ than those given by Eq. (18). The procedure is to express $\mathcal{L}$ as a function of p or p' in terms of the integral term in which $\mathcal{L}$ is evaluated at $p-q$ or $p'-q$.

Since $u(q)$ is holomorphic in the domain

$$-\mu_0<\mathrm{Re}q<0, \tag{19c}$$

the integrals in Eqs. (19a) and (19b) will be taken along a contour lying in this strip. For $(p+p')=0$, there is a singularity which will have a branch point behavior as a consequence of the derivative terms in Eq. (19). As we continue to the left in p or p', the branch points will move across the strip (19c) and finally, at $(p+p')=-\mu_0$, it will pinch the contour of integration over q against a singularity of $u(q)$. Thus, there will be another singularity at $p+p'=-\mu_0$, and similarly there will be one at every point:

$$p+p'=-m\mu_0, \tag{20}$$

where m is a non-negative integer.

[4] M. Froissart, J. Math. Phys. (to be published).

[5] Throughout the discussion we shall implicitly employ Hartog's theorem. Namely, if, in a domain D, a function of n complex variables is holomorphic in each variable separately, the others held fixed, then it is holomorphic in the n variables simultaneously within D. See S. Bochner and W. T. Martin, *Several Complex Variables* (Princeton University Press, Princeton, New Jersey, 1948), p. 140.

In addition, a singularity will arise if

$$\det(p^2+K^2)=0 \quad \text{or} \quad \det(p'^2+K^2)=0,$$

and this will propagate to give branch points at

$$\begin{aligned} p,p' &= \pm ik_1 - m\mu_0, \\ p,p' &= \pm ik_2 - m\mu_0. \end{aligned} \tag{21}$$

In order to separate the terms in $F^T(\lambda,K)$ which cause singularities, we define a matrix $A(\lambda,K)$ such that

$$\begin{aligned} A^T(\lambda,K) &= F^T(\lambda,-K)2^{-(\lambda-1/2)}\pi^{-1/2}K^{\lambda-1/2}e^{(i\pi/2)(\lambda+1/2)}[\Gamma(\lambda+1)]^{-1} \\ &= 2i/\pi+\tfrac{1}{2}\int_0^\infty xdx\, H_\lambda^{(1)}(Kx)V(x)\{H_\lambda^{(1)}(Kx)+e^{i\pi(\lambda-1/2)}H_\lambda^{(1)}(Kxe^{-i\pi})\} \\ &\quad +\tfrac{1}{2}\int_0^\infty dx\int_0^x dx'\,(xx')^{1/2}H_\lambda^{(1)}(Kx)V(x)I_{\lambda K}(x,x')V(x')\{H_\lambda^{(1)}(Kx')+e^{i\pi(\lambda-1/2)}H_\lambda^{(1)}(Kx'e^{-i\pi})\}. \end{aligned} \tag{22}$$

Using the convenient integral representation for $H_\lambda^{(1)}(Kx)$

$$H_\lambda^{(1)}(Kx)=(2K^{-1}/\pi x)^{1/2}e^{-(i\pi/2)(\lambda+1/2)}\left\{e^{iKx}-iK^{-1}\int_{iK}^{-\infty}dp\, e^{px}P'_{\lambda-1/2}(-ipK^{-1})\right\} \tag{23}$$

where $P_{\lambda-1/2}$ is the Legendre function of order $\lambda-\frac{1}{2}$, it is possible to study the behavior of the integrals in Eq. (22) in the domain $1<x'<x$.

For the second term we are led to consider an expression of the form

$$A_{ts}\sim\int_{ik_t}^{-\infty}dp\, P'_{\lambda-1/2}(-ipk_s^{-1})\int_{ik_s}^{-\infty}dp'\, P'_{\lambda-1/2}(-ip'k_s^{-1})\int_1^\infty dx\, e^{px\pm p'x}V_{ts}(x), \tag{24}$$

where $\sim$ indicates that the dominant terms in A_{ts} are proportional to the right-hand side.

From the form of the potentials in Eq. (2), we conclude that

$$\left.\begin{array}{l} A_{11}(\lambda,K) \\ A_{12}(\lambda,K) \\ A_{21}(\lambda,K) \\ A_{22}(\lambda,K) \end{array}\right\} \text{ have branch points at } \left\{\begin{array}{r} 2ik_1=\mu_0 \\ i(k_1\pm k_2)=\mu_0 \\ i(k_2\pm k_1)=\mu_0 \\ 2ik_2=\mu_0, \end{array}\right. \tag{25}$$

respectively. Again, all μ_0's are assumed to be the same, but a modification of this assumption is trivial.

The third term in Eq. (22) leads us to consider

$$\begin{aligned} A_{ts}&\sim\int_{ik_t}^{-\infty}dpP'_{\lambda-1/2}(-ipk_t^{-1})\int_{ik_s}^{-\infty}dp'P'_{\lambda-1/2}(-ipk_s^{-1})\int_1^\infty dx\int_1^x dx'e^{(px\pm p'x')}\{V(x)I_{\lambda K}(x,x')V(x')\}_{ts} \\ &=1/(4\pi^2)\int_{ik_t}^{-\infty}dpP'_{\lambda-1/2}(-ipk_t^{-1})\int_{ik_s}^{-\infty}dp'P'_{\lambda-1/2}(-ipk_s^{-1})\int_{-i\infty}^{i\infty}dq'dq\{u(q)\mathfrak{L}(-p-q,\mp p'-q')u(q')\}_{ts}, \end{aligned} \tag{26}$$

where

$$u(q)=\int_1^\infty dx\, e^{-qx}V(x).$$

From Eqs. (20), (21), and (26), we conclude that

$$\left.\begin{array}{l} A_{11}(\lambda,K) \\ A_{12}(\lambda,K) \\ A_{21}(\lambda,K) \\ A_{22}(\lambda,K) \end{array}\right\} \text{ have branch points at } \left\{\begin{array}{r} \pm 2k_1=(m+1)\mu_0 \\ \pm i(k_1\pm k_2)=(m+1)\mu_0 \\ \pm(k_1\pm k_2)=(m+1)\mu_0 \\ \pm 2ik_2=(m+1)\mu_0, \end{array}\right. \tag{27}$$

respectively.

Analogously to Froissart's results,[4] $A(\lambda,K)$ will have singularities in the λ plane. These arise from the integration of Eq. (22) over regions where x or x' is smaller than 1. In particular, if

$$\sigma_{ij}(\mu)\rightarrow\sum_n a_n\mu^{-c_n}\quad\text{as}\quad\mu\rightarrow\infty\quad\text{and}\quad c_n\geqslant\epsilon_0,$$

then the Mellin transform of $\sigma_{ij}(\mu)$ has poles at $2-c_n$. Thus, $A(\lambda,K)$ will have poles on the real λ axis at

$$\lambda=-n-\tfrac{1}{2}\sum_{s=1}^{r} l_s c_s,\tag{28}$$

where n, l_s, and r are non-negative integers. In particular, if $\sigma_{ij}(\mu)=O(e^{-\mu})$ as $\mu\rightarrow\infty$, or if $\sigma_{ij}(\mu)$ is holomorphic at ∞, then the poles lie at the negative half integers.

Singularities of the matrix $A(\lambda,K)$ are confined to the point set given by Eqs. (27) and (28), where there will be branch points and poles, respectively. (Of course, there will be an essential singularity at infinity in each variable.) Except for the origin, this describes the analytic behavior of the modified Jost matrix A as a function of the three variables λ, k_1, and k_2. In the next section we shall discuss the behavior of the matrix near the origin of the (k_1,k_2) space. This turns out to hold particular interest since, for physical values, k_1 and k_2 are related by

$$k_1{}^2=ak_2{}^2+b,\tag{29}$$

where a and b are the mass ratio and threshold energy difference, respectively. Renninger has carried out a detailed analysis of the threshold behavior.[6]

4. BEHAVIOR AT THE ORIGIN

In the preceding section we found that the branch point in the Laplace transform of the potential led to singularities of the S matrix in (k_1,k_2) space. We will now restrict ourselves to sufficiently small neighborhood of the origin,

$$|k_1|<\tfrac{1}{2}\mu_0,\quad |k_2|<\tfrac{1}{2}\mu_0,\tag{30}$$

where these singularities do not appear, and study the analytic behavior at the origin.

From the relation

$$\begin{aligned}J_\lambda(Kx')N_\lambda(Kx)-N_\lambda(Kx')J_\lambda(Kx)&=(\sin\pi\lambda)^{-1}\\&\times\{J_\lambda(Kx')J_{-\lambda}(Kx)-J_{-\lambda}(Kx')J_\lambda(Kx)\},\end{aligned}\tag{31}$$

it is evident that $G_\lambda(Kx,Kx')$ and $I_{\lambda K}(x,x')$ are entire functions of K^2 for every set of x, x' and λ. Thus, the matrix $A(\lambda,K)$ takes the form

$$\begin{aligned}A^T(\lambda,K)&=2i/\pi+\int_0^\infty xdx\, H_\lambda{}^{(1)}(Kx)V(x)J_\lambda(Kx)\\&\times\int_0^\infty dx\int_0^x dx'\,(xx')^{1/2}H_\lambda{}^{(1)}(Kx)\\&\qquad\times V(x)I_{\lambda K}(x,x')V(x')J_\lambda(Kx)\\&=K^{-\lambda}B^T(\lambda,K)K^\lambda.\end{aligned}\tag{32}$$

[6] G. H. Renninger, Princeton University (to be published) and (Private communication).

Here, $B(\lambda,K)$ can be written as

$$B(\lambda,K)=C(\lambda,K^2)+D(\lambda,K^2)K^{2\lambda},\tag{33}$$

where $C_{ij}(\lambda,K^2)$ and $D_{ij}(\lambda,K^2)$ are even functions of K, and they are holomorphic if K lies in the domain (30).

We define $-K$ as $e^{-i\pi}K$.

Then the S matrix of Eq. (9) can be written as

$$\begin{aligned}S(\lambda,K)&=K^{1/2}A^{-1}(\lambda,K)A(\lambda,-K)K^{-1/2}\\&=K^{-(\lambda-1/2)}B^{-1}(\lambda,K)B(\lambda,-K)K^{(\lambda-1/2)}.\end{aligned}\tag{34}$$

These formulas display in the exponents the precise branching property of S at the origin. By using its components, one can study the behavior of each element.

5. HOLOMORPHY OF S

While in Sec. 3 we discussed meromorphy of the S matrix, we present here some special results concerning holomorphy. Let us introduce a new wave function,

$$\Psi(\lambda,K,x)=\Phi(\lambda,K,x)[F^T(\lambda,-K)]^{-1}K^{+1/2}.\tag{35}$$

Then for $\mathrm{Re}\lambda>0$,

$$\Psi_{ij}(\lambda,K,x)\rightarrow a_{ij}x^{\lambda+1/2}\quad\text{as}\quad x\rightarrow 0,\tag{36a}$$

$$\begin{aligned}\Psi(\lambda,K,x)\rightarrow\tfrac{1}{2}\{&e^{-iKx}K^{-1/2}\\&-e^{iKx-i\pi(\lambda-1/2)}K^{-1/2}S(\lambda,K)\}.\end{aligned}\tag{36b}$$

Equation (1) reads

$$\{d^2/dx^2+K^2-(\lambda^2-\tfrac{1}{4})/x^2-V\}\Psi=0,$$

and taking the adjoint gives

$$(d^2/dx^2)\Psi^\dagger+\Psi^\dagger\{K^{\dagger 2}-(\lambda^{*2}-\tfrac{1}{4})/x^2-V\}=0,$$

which implies

$$\{W[\Psi^*,\Psi]\}'=(\lambda^2-\lambda^{*2})x^{-2}\Psi^\dagger\Psi-\Psi^\dagger(K^2-K^{\dagger 2})\Psi.\tag{37}$$

If K is physical and λ is real, then Ψ^* and Ψ are solutions of the same equation. Evaluating the Wronskian at the origin for positive λ, we obtain

$$W[\Psi^*,\Psi]=0,\tag{38}$$

which is true for all x. From Eqs. (36b) and (38) follows the relation

$$(i/2)(S^\dagger S-E)=0.$$

This implies $S^\dagger S=E$ for all positive real λ with K physical. Thus, we have shown that elastic unitarity holds in this (unphysical) region, just as in the one-channel case. This tells us that in any domain of holomorphy connected to the real positive λ axis and physical K, we have the extended unitarity relation:

$$S^\dagger(\lambda^*,K^*)S(\lambda,K)=E.\tag{39}$$

If now λ becomes complex in the right half λ plane, Eq. (37) is integrable from 0 to ∞ and gives

$$(\mathrm{Im}\lambda)^{-1}[S^\dagger Se^{2\pi\mathrm{Im}\lambda}-E]_{ij}>0.\tag{40}$$

Therefore, in the quadrant $\mathrm{Re}\lambda > 0$, $\mathrm{Im}\lambda < 0$,

$$[S^\dagger S]_{jj} < e^{-2\pi \mathrm{Im}\lambda} = e^{2\pi|\mathrm{Im}\lambda|}. \tag{41}$$

Thus, the positive diagonal elements of $S^\dagger S$ are bounded. Since the two diagonal elements are $|S_{11}|^2+|S_{12}|^2$ and $|S_{21}|^2+|S_{22}|^2$, each element of the S matrix is bounded in this quadrant, for physical K.

6. CONCLUSIONS

It has been shown that Froissart's work on single-channel scattering can be generalized to the many-channel case. The singularities of the Jost matrix were discussed in detail and appear in Eqs. (27) and (28). Since the S matrix is given by the quotient of the two Jost matrices, the singularities of S will include the origin, those in Eqs. (27) and (28) and poles which arise at points where the determinant of one Jost matrix vanishes. Thus, in particular, for a potential of type (2), $S(\lambda,K)$ is meromorphic in the entire λ plane except for an essential singularity at infinity.

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Regge Poles and High-Energy Limits in Field Theory

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It is shown that the Bethe-Salpeter scattering amplitude in the ladder approximation is meromorphic in the complex angular momentum half-plane, $\mathrm{Re}l > -3/2$. There is always at least one Regge pole in this region.

The T matrix element for complex l can be written in the form, $N_l D_l^{-1}$, where N_l and D_l have convergent perturbation expansions. D_l has only a right-hand cut in the squared energy variable, with a branch point at the elastic scattering threshold and at each production threshold. N_l has a left-hand cut, and in addition a right-hand cut beginning from the first three-particle threshold. The Regge poles are zeros of D_l. Much of the information about the trajectories of Regge poles is contained in the lowest-order expression for D_l. The general properties of the trajectories are the same as for the case of scattering from a Yukawa potential. For sufficiently small coupling constant a Regge trajectory $\alpha(s)$ may apparently be expanded in a perturbation series, valid except near thresholds in s.

The connection between the Regge poles of the ladder graphs and the high-energy behavior of the "strip" graphs is discussed. In the $\lambda\varphi^3$ theory it is shown that the second-order expression for the leading Regge trajectory, for the sum of the ladder graphs, determines the leading term in the high-energy limit of the nth order strip graph. This relationship has been checked in fourth-order perturbation theory, and is evidence for the consistency of a perturbation approach to the calculation of Regge trajectories.

1. INTRODUCTION

It has been suggested recently that the ideas of Regge,[1,2] concerning certain asymptotic properties of potential scattering amplitudes, may be applicable in elementary particle physics.[3–6] Their applicability depends on the nature of the behavior of elementary particle scattering amplitudes in the complex angular momentum plane. Though a certain domain of analyticity in the l plane follows from assuming the validity of the Mandelstam representation,[7,8] it is doubtful that

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