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Leading Regge Trajectory in the $\lambda\varphi^4$ Theory

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In a field-theoretic model of pion-pion scattering it is shown that the leading Regge pole approaches $l=0$ as the squared energy approaches $\pm\infty$.

CHEW and Frautschi have suggested that high-energy elastic scattering is governed by a leading Regge trajectory $\alpha(t)$ such that $\alpha(0)=1$.[1] The function $\alpha(t)$ gives the path of a pole in the complex angular momentum plane for a crossed process with total squared energy t, as explained by Chew and Frautschi,[1] Blankenbecler and Goldberger,[2] and Frautschi *et al.*[3] For the uncrossed elastic scattering channel in which we consider the high-energy limit, t is the squared momentum transfer.

There is no $J=1$ bound state at zero energy in the crossed channel if, in accord with the ideas of Blankenbecler and Goldberger[2] and Frautschi *et al.*,[3] the leading Regge pole has a signature which insures the vanishing of the residues at odd integral angular momenta.

In the region relevant to small-angle scattering at high energies, that is for small negative t, the derivative $\alpha'(t)$ must be positive. An apparent difficulty with the whole picture is the fact that, if $\alpha(t)$ continues to decrease with decreasing t, there will be an S-wave ghost pole when $\alpha(t)=0$, at some negative value of t. In potential scattering the trajectories $\alpha_i(t)$ always go negative for sufficiently negative t, and it is natural to assume this will happen for elementary particles also. Frautschi, Gell-Mann, and Zachariasen[3] express the hope that the residue of this ghost will vanish, though the signature argument no longer applies. However, it is very hard to see how this may come about in a calculation.

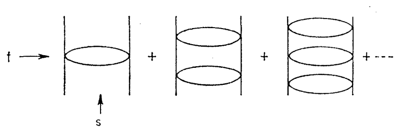

FIG. 1. The ladder graphs. s is the squared energy for the process in which the behavior in the l plane is to be examined. When we look at the high-energy limit in the crossed channel, s and t are to be interchanged.

In this note we point out that for the ladder graphs in pion-pion scattering the leading Regge pole in fact approaches $l=0$, rather than some negative value, in the limit $t\to-\infty$. The ghost pole, therefore, moves to infinity, barring some strange behavior of the function $\alpha(t)$.

We consider the $\lambda\varphi^4$ theory, and the sum of the graphs of Fig. 1. The sum of these graphs is given by the solution to the Bethe-Salpeter equation with a kernel which is appropriate to the two-pion bubble in the $\lambda\varphi^4$ theory. The behavior of the Bethe-Salpeter scattering amplitude for complex angular momenta has been discussed elsewhere at some length.[4] Here we use the result that the Regge poles are given by the zeros of a Fredholm determinant,

$$D(l,s)=\mathrm{Det}[1-K(l,s)]=0. \tag{1}$$

$K(l,s)$ is an integral operator in a two-dimensional space, labeled by continuous indices q, ω where the ranges are $0<q<\infty$; $-\infty<\omega<\infty$,

$$\begin{aligned}\langle q,\omega|K(l,s)|q'\omega'\rangle=&\frac{-i}{(2\pi)^3}[q^2+m^2-i\epsilon-(\omega-\tfrac{1}{2}\sqrt{s})^2]^{-1}\\&\times[q^2+m^2-i\epsilon-(\omega+\tfrac{1}{2}\sqrt{s})^2]^{-1}\int_{4m^2}^{\infty}d\xi\,\rho_{\mathrm{B.A.}}(\xi)\\&\times Q_l\left(\frac{q^2+q'^2+\xi-i\epsilon-(\omega-\omega')^2}{2qq'}\right).\end{aligned} \tag{2}$$

Here $\rho_{\mathrm{B.A.}}(\xi)$ is the second-order spectral function for the dispersion relation for the bubble diagram of Fig. 1, in the $\lambda\varphi^4$ theory,

$$\rho_{\mathrm{B.A.}}(\xi)=(\lambda^2/32\pi^2)[(\xi-4m^2)/\xi]^{\frac{1}{2}}. \tag{3}$$

The kernel defined in Eq. (2) is to be inserted into the

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[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **8**, 41 (1962).
[2] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).
[3] S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).
[4] B. W. Lee and R. F. Sawyer, preceding paper [Phys. Rev. **127**, 2266 (1962)].

series expansion for $D(l,s)$,

$$D(l,s)=1-\mathrm{Tr}K(l,s)-(1/2)\,\mathrm{Tr}K^2(l,s)+(1/2)[\mathrm{Tr}K(l,s)]^2+\cdots. \quad (4)$$

Using the spectral function from Eq. (3), one finds that the integrals implied in the traces in Eq. (4) fail to converge for any value of l. This is because $\rho_{\mathrm{B.A.}}(\xi)$ goes to a constant as ξ approaches infinity. For any $\rho(\xi)$ which goes to zero at infinity the integrals in the expansion of Eq. (4) will converge for $\mathrm{Re}l\geq 0$.

We, therefore, adopt a cutoff procedure which replaces $\rho_{\mathrm{B.A.}}(\xi)$ by

$$\rho(\xi)=\rho_{\mathrm{B.A.}}(\xi)\xi^{-\eta}, \quad 0<\eta<1. \quad (5)$$

The parameter η is to be allowed to approach zero at the end of the calculation. Now the integral in (2) fails to converge when $l=-\eta$. It is easy to continue in the l plane, however, separating out a pole of $K(l,s)$ at $l=-\eta$,

$$\langle q,\omega|K(l,s)|q',\omega'\rangle=\frac{1}{l+\eta}(qq')^{-\eta+1}\times\frac{\pi\Gamma(1-\eta)}{\Gamma(\frac{3}{2}-\eta)}\frac{\lambda^2}{32\pi^2}[q^2+m^2-i\epsilon-(\omega-\tfrac{1}{2}\sqrt{s})]^{-1}\times[q^2+m^2-i\epsilon-(\omega+\tfrac{1}{2}\sqrt{s})]^{-1}+R(l,s,q,\omega,q',\omega'). \quad (6)$$

The function R is analytic in the l plane to the right of $\mathrm{Re}l=-1$.

From Eqs. (1) and (6) it follows that the continuation of $D(l,s)$ has a simple pole at $l=-\eta$. The proof that the perturbation series for $D(l,s)$ converges in a region $\mathrm{Re}l>l_0$, where $l_0<-\eta$, proceeds as in Lee and Sawyer.[4] The fact that the singularity in $D(l,s)$ at $l=-\eta$ is a simple pole follows from the factorable nature of the residue in Eq. (6) [of the form $f_1(q,\omega)f_2(q',\omega')$] and from elementary properties of determinants (see reference 4).

$D(l,s)$ can, therefore, be written in the form,

$$D(l,s)=1-[f(s)/(l+\eta)]-g(s,l), \quad (7)$$

where $g(s,l)$ is analytic in the half l plane $\mathrm{Re}l>-1$. As in Lee and Sawyer[4] it can be shown that $f(s)$ and $g(s,l)$ both approach zero as s approaches $\pm\infty$. The Regge pole is given by the solution to

$$l=-\eta+f(s)+(l+\eta)g(s,l). \quad (8)$$

As s approaches $\pm\infty$ we see that a Regge pole approaches $l=-\eta$. There is no solution to the right of $l=-\eta$ in this limit; we are dealing with the leading trajectory.

As η approaches zero the end of the leading trajectory (that is, the value at $s=\pm\infty$), therefore, moves to $l=0$ through negative values. The ghost evidently moves to $s=-\infty$ in this limit.

On the validity of our approximation there is not much which can be said. The fact that the lowest-order spectral function for the two-pion exchange approaches a constant does not mean, of course, that the true spectral function for the Bethe-Salpeter kernel, coming from the sum of the appropriate irreducible graphs, behaves in the same way. Furthermore our limiting process in the parameter η is dubious. Without some such procedure, however, we have no way to compute the trajectories. We have no assurance, even if the limit $\eta\rightarrow 0$ exists in some sense, that the character of the singularities in the l plane does not change completely when $\eta=0$. What we can show is that if η is very small the ghost is at very large negative s.

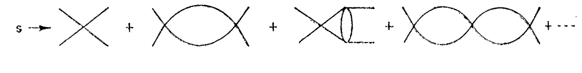

FIG. 2. The bubble diagrams.

It is worth noting that our connection between the behavior of the spectral function at infinity and the value of l at which the Regge trajectory begins is the exact analog of a result in the nonrelativistic case with a superposition of Yukawa potentials. Using the same weight function [Eq. (5)] for the superposition,

$$V(r)=\frac{1}{r}\int_{4m^2}^{\infty}d\xi\,\rho(\xi)e^{-\xi^{\frac{1}{2}}r}, \quad (9)$$

one finds again that the leading trajectory begins at $l=-\eta$. There is one difference; in potential theory this is true only in the range $\frac{1}{2}<\eta<1$. For $\eta<1/2$ the method fails to converge, the potential being more singular than r^{-2} at the origin. In potential theory, therefore, the trajectories must begin to the left of $l=-1/2$, a known result.[5] In field theory the end point can apparently move to $l=0$.

Up to this point we have avoided claiming that the complete scattering amplitude in the $\lambda\varphi^4$ theory is an analytic function of l. Our arguments have applied to the sum of the diagrams of Fig. 1 only. In a conventional $\lambda\varphi^4$ theory there exist sets of diagrams like those of Fig. 2 which give pure S-wave scattering amplitudes and which cannot be continued into the l plane. There seem to be two possible viewpoints toward these terms.

(a) One may assume that the complete scattering amplitude should be an analytic function of l, compute the amplitudes for higher l in some approximation such as ours, and continue analytically to $l=0$. This amounts to discarding the graphs of Fig. 2.

(b) If the graphs of Fig. 2 are to be included, which seems the more reasonable course, the scattering amplitude may be written in the form,

$$f(s,t)=\sum(2l+1)P_l[1+2t/(s-4)]f(l,s)+g(s), \quad (10)$$

where $f(l,s)$ is analytic in l except at the Regge poles. Now when we look at the high-energy limit in the

[5] T. Regge, Nuovo cimento **14**, 951 (1959); **18**, 974 (1960). (See also reference 3.)

crossed channel we see that the terms, $g(t)$, are energy independent. For all values of t such that the first Regge pole is at a value of l greater than zero, therefore, the term from $g(t)$ in Eq. (10) will be dominated by the Regge term. The conclusion of the foregoing work is that in the high-energy limit the crossed bubble diagrams of Fig. 2 will always be dominated by a Regge pole from the ladder graphs of Fig. 1.

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Spherical Lattice Gas

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The spherical model of a lattice gas has recently been treated by W. Pressman and J. B. Keller, utilizing the relationship between the grand partition function of the spherical lattice gas and the partition function of the spherical model of a ferromagnet. A phase transition in three dimensions was found for the lattice gas, but no description of the transition region was possible. Here we give some features of the canonical partition function for the same model. The equation of state is obtained for all densities and temperatures, and predicts that the pressure remains constant in the transition region. The particle correlations and fluctuations are calculated, and their behavior in this region is shown to be consistent with a phase transition. A physical interpretation of the normal modes of the spherical model indicates that the model contains the general aspects of clustering of particles. The origin of an anomalous behavior of the model, mentioned by Pressman and Keller, consisting of the pressure becoming negative for all temperatures at sufficiently large specific volumes, is described and a method for eliminating it is introduced.

I. INTRODUCTION

A RECENT paper by Pressman and Keller describes the behavior of this simplified model of an imperfect gas. The description is made by using the well-known relation between the partition function for the Ising model of a ferromagnet and the grand partition function of the lattice gas.[2] These relations hold as well for the spherical model (abbreviated SM henceforth) of both lattice gas and ferromagnet. The SM of a ferromagnet, introduced by Berlin and Kac,[3] leads to a partition function which has been evaluated by them for one, two, and three dimensions. It then becomes a straightforward matter to apply their results to the grand partition function for the SM of a lattice gas, and this was done in reference 1. Since the SM of a ferromagnet exhibits spontaneous magnetization in three dimensions below a critical temperature, a phase transition is found for the SM of a lattice gas in three dimensions. However, the transition region itself is not described by this method, so that one does not know how the pressure behaves in this region. Also this formal identification does not give any indication of the physical nature of the correspondence between lattice gas and the SM of a lattice gas.

The spherical model of a lattice gas was treated by one of us (HAG) some time ago as part of a doctoral dissertation.[4] Here, the canonical partition function was used, in order to be able to describe the behavior of the system right through the transition region and in order to have available a better understanding of the mechanism responsible for the transition. The present work is devoted to pointing out some features that were obtained in the description via the canonical ensemble which are not revealed by the work of Pressman and Keller. In particular, the canonical ensemble predicts that the *pressure remains constant inside the transition region*, and that the fluctuations and correlations have the proper over-all behavior to be expected in this region. In addition, one can see clearly the origin of the anomalous effect of the pressure going negative at sufficiently large specific volume for all temperatures and how a simple modification of the model allows removing this nonphysical behavior, without affecting the qualitative aspects of the phase transition.

In Sec. II, the canonical partition function for the SM is introduced. Some connections between the lattice gas and the SM of the lattice gas are discussed. Section III contains the evaluation of the partition function for all densities by the method of steepest descents. Since the formal development parallels that already re-

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[1] W. Pressman and J. B. Keller, Phys. Rev. **120**, 22 (1960).
[2] C. N. Yang and T. D. Lee, Phys. Rev. **87**, 410 (1952).
[3] T. H. Berlin and M. Kac, Phys. Rev. **86**, 821 (1952).
[4] H. A. Gersch, Ph.D. thesis, The Johns Hopkins University, 1953 (unpublished).