Concerning the Notion of "Time Interval" in S-Matrix Theory*

MARVIN L. GOLDBERGER
Palmer Physical Laboratories, Princeton University, Princeton, New Jersey

AND

KENNETH M. WATSON
Department of Physics and Lawrence Radiation Laboratory, University of California, Berkeley, California

(Received May 15, 1962)

The notion of time duration is considered within the framework of S-matrix theory. It is shown that Wigner's definition of the time delay, $-i\hbar[(d\ln S/dE)]$, for a scattering process permits one to define time duration in a "coarse-grained" sense.

I. GENERAL DISCUSSION

THE time development of the state of a quantum mechanical system described by a wave function $\psi(t)$ and Hamiltonian $\mathcal{H}$ is given by the Schrödinger equation,

$$d\psi = -(idt/\hbar)\mathcal{H}\psi. \tag{1}$$

This fundamental dynamic principle determines the change in the state ψ for aribtrarily small time intervals dt. The question has often been raised, however, whether processes of macroscopic measurement may limit one's ability to observe changes in ψ over very short intervals, and hence to verify Eq. (1).

Most experimental situations are concerned with measurements that involve, on the appropriate atomic or nuclear scale, extremely long-time intervals. Such considerations prompted Heisenberg to propose that fundamental theory might yield directly the collision matrix S.[1] Such a theory would provide a relation only between asymptotic states of a system (i.e., the relation between states in the remote past and the remote future) and would presumably not assign physical meaning to the continuous development of the system in time. In addition, one would never speak of the "fluctuations off the energy shell" of a physical system.

Considerable progress has been made in recent years toward the goal proposed by Heisenberg of obtaining the S matrix without a Schrödinger equation.[2] Such studies have not, however, provided a basis for discussing macroscopic time intervals that are essential in common experience and in the processes of making physical measurements. We should like to argue that the S matrix does provide a "coarse-grained" definition of time interval. This coarse-grained notion of interval seems adequate to account for a semiclassical dynamics, and thus for the macroscopic dynamic phenomena of common experience. Whether it is also adequate for all physical processes, or whether it is in fact possible to verify Eq. (1) in detail, is a much deeper question to which we do not adress ourselves here.

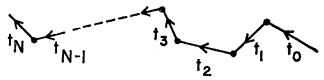

FIG. 1. A sequence of scatterings to illustrate Eq. (6).

The basis for our considerations is provided by Wigner's representation of time delay in a scattering event.[3] If $S(E)$ is a matrix element of the S matrix connecting two precise asymptotic states of an interacting system (we suppress all labels except the total barycentric energy E), the time delay relative to the free-flight time between the states is

$$Q = -i\hbar(d/dE)\ln S(E). \tag{2}$$

To introduce the notion of a sequence of events and time intervals, we suppose a particle undergoes a series of N scatterings as illustrated in Fig. 1. Consistent with the Schrödinger Eq. (1), we may observe the state of the system at times $t_0, t_1, \ldots, t_N$ occurring between scattering events. By this, we mean that t_0 is before the first event, t_1 is after the first but before the second, etc. At a given time t_n $(n=0, 1, \ldots, N)$ we may determine the wave function to be ψ_n. The Schrödinger equation enables us to relate the ψ's as follows:

$$\bar{\psi}_n = \exp[-iK(t_n - t_{n-1})/\hbar]U(t_n, t_{n-1})\bar{\psi}_{n-1}, \tag{3}$$

where K is the kinetic energy operator and $U(t_n, t_{n-1})$ is the unitary operator defined by

$$U(t,t') = \exp[iKt/\hbar]\exp[-i\mathcal{H}(t-t')/\hbar] \times \exp[-iKt'/\hbar], \tag{4}$$

and $\bar{\psi}_n$ is given by

$$\bar{\psi}_n = \exp[iKt_{n-1}/\hbar]\psi_n. \tag{5}$$

If the time intervals $(t_n - t_{n-1})$ are large enough that

* This work was done partly under Air Force contract (M.L.G. and K.M.W.) and partly under the auspices of the U. S. Atomic Energy Commission (K.M.W.).

[1] W. Heisenberg, Z. Physik **120**, 513 and 673 (1943).

[2] We refer here principally to the development of dispersion theory. See, for example, the articles by M. L. Goldberger and G. F. Chew, in *Dispersion Theory and Elementary Particles*, edited by C. DeWitt and R. Omnes (John Wiley & Sons, Inc., New York, 1961).

[3] E. P. Wigner, Phys. Rev. **98**, 145 (1955). In the general case when $|S| \neq 1$, Eq. (2) must be replaced by the equation, $Q = \mathrm{Re}[-\hbar d\ln S(E)/dE]$. [See L. Eisenbud, dissertation, Princeton University, June, 1948 (unpublished) where the concept of time delay was first discussed; and F. T. Smith, Phys. Rev. **118**, 349 (1960).] For simplicity of presentation, we shall use Eq. (2) in our discussion, it being remembered that the results are generalized by merely replacing the expression on the right-hand side of (2) by its real part.

the system is close to the energy shell between scatterings, we may set

$$U(t_n,t_{n-1})=S_n, \tag{6}$$

the S matrix for the nth scattering event. By definition $S=U(\infty,-\infty)$, so what we are assuming is that all irrelevant transients have disappeared already at finite times. In this case Eqs. (3) and (6) permit us to write

$$\begin{aligned}\bar{\psi}_N&=\exp[-i(K/\hbar)(t_N-t_0)]\prod_{n=1}^{N}U(t_n,t_{n-1})\bar{\psi}_0\\&=\exp[-i(K/\hbar)(t_N-t_0)]\prod_{n=1}^{N}S_n\bar{\psi}_0,\end{aligned} \tag{7}$$

a relation which connects the initial state ψ_0, prior to any of the scatterings, to the final state ψ_N following all of the scatterings. The S matrix for the entire process has the form

$$S=\prod_{n=1}^{N}S_n. \tag{8}$$

Thus, if a complex process involves a sequence of interactions sufficiently separated in time that Eqs. (6) are valid, the S matrix for the complete event *factors* into a product of S matrices for the separate interactions. This group property of the S matrix is essential for the subsequent discussion.

It is reasonable to suppose—and we shall do so—that Eq. (8) is a general property of the S matrix, valid even in theories not based on the Schrödinger Eq. (1).[4] We shall also suppose that it is meaningful to talk about the free-flight time of wave packets over macroscopic intervals defined by

$$\delta t_{\text{free}}=d/v, \tag{9}$$

where d is the distance traveled and v is the packet velocity. In the S-matrix theory, we expect the recurrence relation

$$\begin{aligned}\bar{\psi}_n&=s_n\bar{\psi}_{n-1},\\ s_n&=\exp[-i(K/\hbar)(t_n-t_{n-1})]S_n,\end{aligned} \tag{10}$$

to determine the wave functions $\bar{\psi}_n$ at times intermediate between the interactions. We emphasize that the conditions of Eqs. (8) through (10) are assumed to be valid quite independently of whether or not there is a Schrödinger equation.

Now the Wigner relation, Eq. (2), and our expression for the S matrix, Eq. (8), enable us to write the total time delay Q as

$$Q=-i\hbar(d/dE)\ \ln S(E)=\textstyle\sum_n Q_n, \tag{11}$$

where

$$Q_n=-i\hbar(d/dE)\ \ln S_n \tag{12}$$

is the time delay for the nth interaction. From expressions (8) and (12) we find that the time duration associated with the nth scattering is

$$\delta t_n\equiv t_n-t_{n-1}=(d_n/v)+Q_n, \tag{13}$$

where d_n is the distance traveled and v is the asymptotic velocity.[5]

This expression, Eq. (13), is our fundamental result. It provides a *definition* of the time duration for each of the interactions illustrated in Fig. 1. It therefore permits us to attach a *time label* to points on the orbit of the scattered particle. This definition of time interval and the recurrence relation, Eqs. (10), provide a *macroscopic dynamic principle* limited by the coarseness of our time mesh.

The factorization (8) of the S matrix is rigorous, of course, only in the limit that the scattering interactions are separated by very large distances. From the present viewpoint, time duration is defined for real processes only to within a precision determined by the accuracy with which the factorization (8) approximates the exact S matrix. If this were a correct point of view, then limitations on the processes of measurement would not permit a more precise definition of time duration.

It is interesting to rewrite Eqs. (10) and (13) in a "Schrödinger-like" form. First, we express Eq. (3) as

$$\begin{aligned}\delta t_n&=(d_n/v)[1-(iv/d_n)\hbar(d/dE)\ \ln S_n]\\&=(\delta t_n)_{\text{free}}[1-(i\hbar/d_n)(d/dp)\ \ln S_n],\end{aligned} \tag{14}$$

where we have used the relations $v=dE/dp$, and $(\delta t_n)_{\text{free}}=d_n/v$, the free-flight time.[6] Thus, δt_n is expressed in terms of the free-flight time and the S-matrix element for the nth scattering S_n.

Next, we write from Eqs. (10),

$$\delta\bar{\psi}_n\equiv\bar{\psi}_n-\bar{\psi}_{n-1}=[\exp\{-i\delta t_n[K/\hbar+i(\ln S_n)/(\delta t_n)]\}-1]\bar{\psi}_{n-1}, \tag{15}$$

which is equivalent to

$$i\hbar(\partial\bar{\psi}/\partial t)=[K+i\hbar(\ln S_n/\delta t_n)]\hbar. \tag{16}$$

Since $\ln S_n$ and δt_n are constant in each interval, by integrating Eq. (16) over an interval, we recover Eq. (10).

This result, Eq. (16), has the formal appearance of a Schrödinger equation with a potential $i\hbar(\ln S_n)/\delta t_n$ that depends only on the free-flight time and on S_n. It is rigorously equivalent to the Schrödinger equation in the asymptotic regions between scatterings; of

[4] In general case, the product (8) is a matrix product, involving sums over spin substates, etc. For simplicity, we suppose our system to be simple enough that (7) is an algebraic product. Then in the general case, Eq. (2) applies to each matrix element of a given S_n in Eq. (8).

[5] When the system is degenerate, we must suppose (just as in the Schrödinger theory) that an observation of the state is made at each time t_n. Then the S_n in Eq. (12) is that matrix element of the S matrix associated with the observed transition.

[6] For a three-dimensional array of scatterers, the index n may be replaced by a "coarse-grained" coordinate $\mathbf{r}$ and an S-matrix $S(\mathbf{r},\mathbf{p})$ defined. Then Eq. (14) takes the appealing form $\delta t_n=(\delta t_n)_{\text{free}}(1-i\hbar\nabla_r\cdot\nabla_p\ \ln S)$.

course, it does not describe in the Schrödinger sense the wave function in the "near-field region" where the scattering occurs.

Our description has been rather schematic and we now illustrate it with two examples.

II. A RESONANCE REACTION

Consider the excitation of a resonance level of a nucleus A by a γ ray. The nucleus decays to a lower excited state by γ-ray emission, and then to its ground state by neutron emission. The reaction envisaged is then

$$\gamma+A\rightarrow A^*\rightarrow A^{**}+\gamma'\rightarrow A'+\gamma'+n. \tag{17}$$

The total energy is $E=\epsilon_\gamma+W_0$, where ϵ_γ is the incident γ-ray energy and W_0 is the energy of nucleus A. The energies of the excited nuclei A^*, A^{**} are W_1 and W_2, respectively. The energy of the emitted γ ray is ϵ_γ', and that of the neutron ϵ_n.

The S-matrix element for the process is

$$\langle\epsilon_n,\epsilon_{\gamma'},A'|S|\epsilon_\gamma,A\rangle.$$

We assume that each virtual state is sufficiently long-lived that the one-level resonance formula may be used. Then[7]

$$\begin{aligned}\langle\epsilon_n,\epsilon_{\gamma'},A'|S|\epsilon_\gamma,A\rangle&=\frac{a_n}{E-(W_2+\epsilon_{\gamma'})+i\Gamma_n/2}\cdot\frac{-i\Gamma_\gamma}{E-W_1+i\Gamma_\gamma/2}\\&\equiv S_2S_1.\end{aligned} \tag{18}$$

The quantities a_n, Γ_n, and Γ_γ are considered to be slowly varying functions of energy. We find then from Eq. (12):

$$\begin{aligned}Q_1&=\mathrm{Re}\{-i\hbar d\ \ln S_1/dE\}\\&=\mathrm{Re}\left\{\frac{i\hbar}{E-W_1+i\Gamma_\gamma/2}\right\},\\Q_2&=\mathrm{Re}\{-i\hbar d\ \ln S_2/dE\}\\&=\mathrm{Re}\left\{\frac{i\hbar}{E-(W_2+\epsilon_{\gamma'})+i\Gamma_n/2}\right\}.\end{aligned} \tag{19}$$

Very close to resonance, these expressions become

$$\begin{aligned}Q_1&=2\hbar/\Gamma_\gamma,\\Q_2&=2\hbar/\Gamma_n,\end{aligned} \tag{20}$$

as expected.

In this simple example we have predicted the expected lifetime of each state from the S matrix without involving a Schrödinger equation. We must emphasize, however, that for a real process the factorization of the S matrix implied by Eq. (18) is only approximate; we have neglected nonresonant contributions to S. According to the point of view proposed above, it is only to within the accuracy of Eq. (18) that the separate time intervals for the two decay steps have physical meaning.

III. THE SEMICLASSICAL LIMIT

Let us now imagine that the separate scattering regions illustrated in Fig. 1 coalesce into one region of continuous interaction. We shall suppose that the forces acting on the particle vary slowly over a wavelength $1/p$, so the orbit of the particle may be defined as in classical mechanics.

In the usual Schrödinger description we would represent the interaction by a potential $V(r)$, and find the wave function in the W.K.B. approximation. We shall show that in this case, Eq. (16) is equivalent to the Schrödinger equation.

To the extent that we disregard spatial derivatives of the potential, we may treat the kinetic-energy operator K and the potential V as commuting operators. Then[8]

$$\begin{aligned}\exp[-i(t_n-t_{n-1})(K+V)]\approx&\exp[-it_nK]\\&\times\exp[-i(t_n-t_{n-1})V]\exp[it_{n-1}K],\end{aligned}$$

and from Eq. (6),

$$S_n=\exp\{i-i\delta t_nV\}.$$

Hence,

$$i(\ln S_n)/\delta t_n=V, \tag{21}$$

and within this limit, Eq. (16) is equivalent to the Schrödinger equation.[9]

[7] See, for example, Sec. 8.B in our forthcoming book *Collision Theory* (John Wiley & Sons, Inc., New York, 1962), where such two-step resonance processes are discussed.

[8] It is easy to write down the correction terms that arise from the noncommutativity of K and V. It is clear that they involve powers of δt_n and spatial derivatives V. See, for example, M. L. Goldberger and E. N. Adams, J. Chem. Phys. **20**, 240 (1952).

[9] An explicit derivation of Eq. (21) in the W.K.B. approximation is given in Sec. 8E of reference 7.