which is independent of time and therefore characteristic of the true steady state. Similar considerations apply to the higher-order terms in the steady-state energy.

In a system driven in the steady state by the application of a set of external forces, there are in general two distinct contributions to the energy

$$\langle H\rangle^{(F)}=\sum_j F_j\langle Q_j\rangle^{(F)}+\langle H\rangle^{(0)}. \qquad \text{(B8)}$$

One energy contribution is purely thermal in origin and would be present even if the system were closed, so that no steady-state currents could flow. In the electrical case, for example, such a change in energy might arise from a piling up of the charges at one end of the sample or a polarization of the atoms making up the system. However, the system would still be in canonical equilibrium at the temperature of the thermal reservoir. This thermal energy must be associated with the variable $\dot{q}_i^{(F)}$, since the current $j_i^{(F)}$ would not exist in the closed system. The second-energy contribution is purely dynamic in origin and can exist only in an open system. It would be determined, as in the warm carrier example discussed in Sec. VI, by the requirement that the rate of energy absorption from the applied forces equal the rate of energy dissipation into the thermal reservoir. This energy contribution must be regarded as residing in that part of the system characterized by the variable $j_i^{(F)}$, since this is the only respect in which the steady-state system formally differs from the perturbed equilibrium system.

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Remarks on the Electromagnetic Interactions of Massless Particles*

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A result recently obtained by Case and Gasiorowicz, that a massless particle of spin $s>\frac{1}{2}$ cannot have the usual type of coupling to the electromagnetic field, is examined from a point of view different from that taken by those authors. It is shown that such a particle may, in fact, have derivative couplings of order $2s+1$ or $2s-1$ with the electromagnetic field for s an integer or half-integer, respectively. These couplings cannot be generated by the usual requirement that the equations of motion be invariant under phase transformations, nor can they be present for dimensional reasons in a theory which contains no mass (or characteristic length), these conditions leading to the stated result. However, there is no reason in principle to expect such couplings to be absent if the massless particle interacts also with a charged massive field. An example is given for $s=1$, and the cross section for Coulomb scattering of a massless particle is given for arbitrary s.

I. INTRODUCTION

IN a recent paper,[1] Case and Gasiorowicz showed that a massless particle of spin $s\geqslant 1$ cannot be charged in the usual sense. Their procedure was as follows. The field operator for a free massless particle of integer spin s may be represented by a symmetric traceless tensor $\phi_{\alpha\beta\cdots\sigma}(x)$ of rank s which satisfies the Klein-Gordon equation

$$\Box\phi_{\alpha\beta\cdots\sigma}(x)=0. \qquad (1)$$

In addition, to insure that the field transforms according to the appropriate irreducible representation O_s of the inhomogeneous Lorentz group,[2] it is necessary that $\phi_{\alpha\beta\cdots\sigma}(x)$ satisfy the divergence condition

$$\partial_\alpha\phi_{\alpha\beta\cdots\sigma}(x)=0 \qquad (2)$$

and a generalized gauge condition of the second kind.[1] The conventional (minimal) electromagnetic coupling for the particle may be generated by requiring that the equations of motion be invariant under the simultaneous coordinate-dependent transformations

$$\phi_{\alpha\beta\cdots\sigma}(x)\to e^{ie\chi(x)}\phi_{\alpha\beta\cdots\sigma}(x),$$
$$A_\mu(x)\to A_\mu(x)+\partial_\mu\chi(x),\quad \Box\chi=0. \qquad (3)$$

Equation (1) is then replaced by the familiar result

$$[\partial_\mu-ieA_\mu(x)]^2\phi_{\alpha\beta\cdots\sigma}(x)=0. \qquad (4)$$

Upon rewriting this equation in the form

$$\Box\phi_{\alpha\beta\cdots\sigma}=-j_{\alpha\beta\cdots\sigma}, \qquad (4')$$

one can construct the formal retarded solution for $\phi_{\alpha\beta\cdots\sigma}$,

$$\phi_{\alpha\beta\cdots\sigma}(x)=\phi_{\alpha\beta\cdots\sigma}^{\text{in}}(x)+\int dx'\,D_R(x-x')\,j_{\alpha\beta\cdots\sigma}(x'), \qquad (5)$$

where $D_R(x-x')$ is the retarded Green's function for zero mass, and $\phi_{\alpha\beta\cdots\sigma}^{\text{in}}(x)$ is the asymptotic (free) in-

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[1] K. M. Case and S. G. Gasiorowicz, Phys. Rev. **125**, 1055 (1962).
[2] E. P. Wigner, Ann. Math. **40**, 149 (1939); V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **34**, 211 (1948). Also the more recent review papers of Iu. M. Shirkov, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 861, 1196, 1208 (1957) [translation: Soviet Phys.—JETP **6**, 664, 919, 929 (1958)]; *ibid.* **34**, 717 (1958) [translation: Soviet Phys.—JETP **7**, 493 (1958)]; *ibid.* **35**, 1005 (1958) [translation: Soviet Phys.—JETP **8**, 703 (1959)].

field which satisfies Eqs. (1) and (2). From the observation that $\partial_\alpha j_{\alpha\beta\cdots\sigma}$ is nonzero for the specified coupling, Eqs. (3) and (4), it may then be shown that the asymptotic out-field fails to satisfy the divergence condition of Eq. (2). Since the out-field is again a free field, it is concluded that the specified coupling is inconsistent with Lorentz invariance and zero mass for the particle, hence, $e=0$. A similar result holds for half-integer spins $s>\frac{1}{2}$.

In the present note, we will re-examine these interesting (though perhaps academic) results from a point of view somewhat different from that of Case and Gasiorowicz, but one which may bring out more clearly the connection between masslessness, Lorentz invariance (specifically, conservation of angular momentum), and the impossibility of the foregoing coupling in the "pure" electrodynamics of the massless particle. The method to be used is essentially the same as that used by DeCelles, Marr, and the author in a recent study of the kinematic structure of vertex functions for massive particles of arbitrary spin.[3] We shall, in particular, examine the consequences of invariance under proper and improper Lorentz transformations for the structure of the matrix elements of the electromagnetic current operator j_μ taken between single-particle states of a massless particle of spin s. The vanishing of these matrix elements would preclude the existence of an effective coupling linear in the electromagnetic field and bilinear in the particle field. It is convenient in this calculation to work in the "brick-wall" coordinate system, in which the 3-momenta of the initial and the final particle are equal in magnitude, but oppositely directed. In this frame, it is easily shown that, for $s\neq\frac{1}{2}$, the matrix elements of the spacelike part of the electromagnetic current are identically zero. The matrix elements of $j_\pm$, the transverse components of the current, vanish as a consequence of the conservation of that component of the total angular momentum along the direction of motion of the particle (the helicity). The matrix elements of j_3 vanish as a result of the properties of the current under space and time inversions, or, alternatively, as a result of current conservation or gauge invariance. On the other hand, the matrix elements of j_0, the fourth component of the current, do not vanish as a consequence of any general symmetry relations. However, from an examination of the transition from the case of finite mass[3] to that of zero mass, it may be shown that the only couplings possible for massless particles of spin s are derivative couplings of orders at least $2s+1$ or $2s-1$ for s an integer or a half-integer, respectively. For dimensional reasons, these couplings must involve at least $2s$ (or $2s-2$) factors of m^{-1}, where m is a mass characteristic of the problem. The result of Case and Gasiorowicz, that a massless particle of spin $s\geqslant 1$ can have no electromagnetic coupling, follows at once from the observation that there is no such mass available in the "pure" electrodynamics of massless particles interacting with the massless electromagnetic field.[4] However, there is no reason in principle to expect such couplings to be absent if the particle in question interacts also with a massive field. The possible scalar and vector-type couplings in this case are obtained as a by-product of our other results. An example of a nonzero effective coupling is given for spin 1, and the cross sections for the Coulomb scattering of a massless particle of arbitrary spin are derived.

[3] L. Durand, III, P. C. DeCelles, and R. B. Marr, Phys. Rev. **126**, 1882 (1962).

II. STRUCTURE OF THE ELECTROMAGNETIC VERTEX FUNCTION FOR MASSLESS PARTICLES

In the present section, we will be concerned with the structure of matrix elements of the electromagnetic current operator j_μ, defined by

$$\Box A_\mu(x)=-j_\mu(x), \tag{6}$$

taken between single-particle states of a massless particle of spin s. Vanishing of these matrix elements precludes an elementary coupling of the particle to the electromagnetic field. It will be convenient to use the assumption of Lorentz invariance of the theory to transform at once to a particular coordinate system, the so-called "brick wall" or Breit frame, in which the incident particle has momentum p in the positive z direction and the final particle, momentum p in the negative z direction. This coordinate system can always be reached by a finite Lorentz transformation unless the initial and final massless particles have colinear momenta in the original coordinate system. This special case adds nothing new in principle, and will be ignored.

The free states of the massless particle of spin s are assumed to transform according to an irreducible unitary representation O_s of the inhomogeneous Lorentz group.[2] Following the conventions of the helicity representation for angular momentum introduced by Jacob and Wick,[5] we will denote by $|ps\lambda\rangle$ the state in which the particle moves with momentum p in the positive z direction, and has helicity (spin projection along p) equal to λ. The helicity is restricted for the massless particle to the two values $\lambda=\pm s$.[2,5] The relative phase of the two possible states may be fixed by specifying the effect on one state of the reflection Y in the $x-z$ plane $(x,y,z\to x,-y,z)$. If $\mathcal{P}$ is the parity reflection $(x,y,z\to -x,-y,-z)$, then

$$Y=e^{-i\pi J_2}\mathcal{P}=\mathcal{P}e^{-i\pi J_2}. \tag{7}$$

[4] There is an interesting question of principle involved in the interactions of massless particles, since the complete absence of a mass, hence, of a unit of length, introduces some difficulty in the definition of asymptotic fields and states: with no standard length (or time), it is difficult to say how far is "far enough" in the determination of asymptotic behavior. The problem can probably be circumvented by the judicious use of wave packets, but is here ignored. The author is indebted to Dr. C. M. Sommerfield for a remark on this problem.

[5] M. Jacob and G. C. Wick, Ann. Phys. (New York) **7**, 404 (1959).

The momentum and total angular momentum operators $\mathbf{P}$ and $\mathbf{J}$ transform under Y according to

$$Y\mathbf{P}Y^{-1}=(P_1,-P_2,P_3), \tag{8}$$

$$Y\mathbf{J}Y^{-1}=(-J_1,J_2,-J_3). \tag{9}$$

It is therefore evident that the Y transform of the eigenstate $|ps\lambda\rangle$ of P_3 and J_3 is again an eigenstate of those operators with the same eigenvalue p of P_3, but the eigenvalue $-\lambda$ of J_3. Thus, $Y|ps\lambda\rangle$ must be proportional to the state $|p,s,-\lambda\rangle$. Choosing the phase convention for the states to agree with that for massive particles for $\lambda=\pm s$, we write

$$Y|ps\lambda\rangle=\eta_P(-1)^{s-\lambda}|ps,-\lambda\rangle. \tag{10}$$

The constant of proportionality η_P is of unit modulus, and may be identified as the parity factor of the particle in question.[5] States $|p\theta\phi s\lambda\rangle$ in which the particle moves in the θ,ϕ direction with the specified helicity and momentum may now be obtained by applying appropriate rotations to the basic states $|ps,\pm s\rangle$,[5]

$$|p\theta\phi s\lambda\rangle=|\mathbf{p}s\lambda\rangle=R(\phi,\theta,-\phi)|ps\lambda\rangle. \tag{11}$$

We will require in particular only that state in which the motion is in the negative z direction; this may be obtained by a simple rotation through an angle π about the y axis:

$$|p\pi 0 s\lambda\rangle=e^{-i\pi J_2}|ps\lambda\rangle. \tag{12}$$

The properties of the states $|ps\lambda\rangle$ under the antiunitary operation T of Wigner time inversion[6] are also easily determined. Both the momentum and the angular momentum operators $\mathbf{P}$ and $\mathbf{J}$ change sign under T. The helicity operator $\mathbf{J}\cdot\mathbf{P}$ is therefore invariant under T, and the transform of the state $|ps\lambda\rangle$ has the same helicity as the initial state. Since the transformed state also has momentum p in the negative z direction, it is clearly proportional to the state $|p\pi 0s\lambda\rangle$ defined above,

$$\begin{aligned}T|ps\lambda\rangle&=|ps\lambda\rangle_T\\&=\eta_T|p\pi 0s\lambda\rangle\\&=\eta_T e^{-i\pi J_2}|ps\lambda\rangle.\end{aligned} \tag{13}$$

The constant of proportionality η_T is of unit modulus, and is independent of λ and p.[5] We note finally two useful relations which follow from the general representation of T as a complex conjugation K followed by a unitary transformation U, $T=UK$. First, for a state ψ composed of a sum of helicity states,

$$\psi=\textstyle\sum_\lambda a_\lambda|p\theta\phi s\lambda\rangle, \tag{14}$$

the time reversal operation yields

$$T\psi=\textstyle\sum_\lambda a_\lambda{}^*T|p\theta\phi s\lambda\rangle; \tag{15}$$

and second, for arbitrary states ψ, ϕ,

$$(T\psi,T\phi)=(\psi,\phi)^*. \tag{16}$$

It will be convenient to choose a normalization for the plane wave states $|\mathbf{p}s\lambda\rangle$ which corresponds to the Lorentz covariant inner product in Hilbert space,

$$\langle\mathbf{p}'s'\lambda'|\mathbf{p}s\lambda\rangle=(2\pi)^3p_0\delta^3(\mathbf{p}'-\mathbf{p})\delta_{ss'}\delta_{\lambda\lambda'},\quad p_0=|\mathbf{p}|. \tag{17}$$

An integration over the space coordinates is implied in this result. The corresponding completeness relation is

$$|\psi\rangle=\sum_{s,\lambda}\int\frac{d^3p}{(2\pi)^3}\frac{1}{p_0}|\mathbf{p}s\lambda\rangle\langle\mathbf{p}s\lambda|\psi\rangle, \tag{18}$$

the density of states in momentum space being given by the Lorentz invariant quantity $[d^3p/(2\pi)^3p_0]$.[7]

With the foregoing conventions, the single-particle matrix elements of the electromagnetic current operator $j_\mu(x)$ are given in momentum space by

$$\int dx\, e^{-iq\cdot x}\langle\mathbf{p}'s\lambda'|j_\mu(x)|\mathbf{p}s\lambda\rangle=(2\pi)^4\delta^4(p-p'-q)\Gamma_{\lambda'\lambda}{}^{(\mu)}, \tag{19}$$

where, in the brick-wall coordinate system,

$$\Gamma_{\lambda'\lambda}{}^{(\mu)}(p)=\langle ps\lambda'|e^{i\pi J_2}j_\mu(p)|ps\lambda\rangle. \tag{20}$$

It will be convenient in the ensuing discussion to use spherical components for the spacelike part of j_μ; these are related to the Cartesian components of $\mathbf{j}=(j_1,j_2,j_3)$ by

$$j_\pm=\mp\tfrac{1}{2}(j_1\pm ij_2),\quad j_3=j_3. \tag{21}$$

We will use the symbol j_0 for the Hermitian fourth component of j_μ. The spacelike part of j_μ transforms under rotations as a polar vector, while j_0 is a scalar. Under the reflection Y in the $x-z$ plane, j_2 changes sign,

$$Yj_\mu Y^{-1}=(j_1,-j_2,j_3,j_0), \tag{22}$$

while under time reversal, $\mathbf{j}$ changes sign,

$$Tj_\mu T^{-1}=(-j_1,-j_2,-j_3,j_0). \tag{23}$$

The restrictions on the matrix elements $\Gamma_{\lambda'\lambda}{}^{(\mu)}$ which follow from the foregoing symmetry properties of j_μ and the free-particle states are easily determined. From the commutation relations of a vector operator with the third component of the total angular momentum[8]

$$[J_3,j_\pm]=\pm j_\pm, \tag{24}$$

we obtain at once the relation

$$\Gamma_{\lambda'\lambda}{}^{(\pm)}=\delta_{\lambda',-\lambda\mp1}\Gamma_{-\lambda\mp1,\lambda}{}^{(\pm)},\quad \lambda,\lambda'=\pm s, \tag{25}$$

which expresses the conservation of helicity (or J_3) at the vertex. For a massless particle, λ and $\lambda'=-\lambda\mp1$

[6] E. P. Wigner, *Group Theory and Its Applications to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959), Chap. 26.

[7] The normalization of the states used in reference 3 differs from that used above by the replacement of p_0 in the defining equations [Eqs. (17), (18)] by (p_0/m). The latter choice is inappropriate in the limit $m\to0$.

[8] See, for example, A. R. Edmonds, *Angular Momentum in Quantum Mechanics*, (Princeton University Press, Princeton, New Jersey, 1957), Chap. 5.

can assume only the values $\pm s$ as indicated. The only spin for which this condition can be fulfilled is $s=\frac{1}{2}$. We thus arrive at the first result, that for massless particles,

$$\Gamma_{\lambda',\lambda}^{(\pm)}=0, \quad s\neq\tfrac{1}{2}. \tag{26}$$

In addition, one can obtain the result

$$\Gamma_{\lambda'\lambda}^{(3)}=0, \quad \text{all } s, \tag{27}$$

either by invoking the condition of current conservation, $\partial\cdot j=0$, which in momentum space yields the relation

$$2p\Gamma_{\lambda'\lambda}^{(3)}=(p_0-p_0')\Gamma_{\lambda'\lambda}^{(0)}=0, \tag{28}$$

or by proving the analog of a theorem proved in reference 3, Eqs. (91) ff., that the matrix elements of the 4 divergence of the electromagnetic current between states of the same single particle vanish independently of the assumption of gauge invariance or current conservation.

It therefore remains only to consider the matrix elements of j_0. Since j_0 transforms as a scalar under rotations,

$$[J_3,j_0]=0, \tag{29}$$

the matrix elements $\Gamma_{\lambda'\lambda}^{(0)}$ are skew diagonal in the helicities (diagonal in J_3)

$$\Gamma_{\lambda'\lambda}^{(0)}=\delta_{\lambda',-\lambda}\Gamma_{-\lambda,\lambda}^{(0)}, \quad \lambda=\pm s. \tag{30}$$

In contrast to the situation which obtained for the matrix elements of $j_\pm$, conservation of J_3 at the vertex does not require that the matrix elements of j_0 vanish; for $\lambda=\pm s$, λ' assumes the allowed values $\mp s$. The invariance of the theory under reflections and time reversal leads to additional symmetry properties of the matrix $\Gamma_{\lambda',\lambda}^{(0)}$. Thus, utilizing the transformation properties of j_μ and the helicity states under the reflection Y in the $x-z$ plane [Eqs. (22), (10)], we obtain

$$\begin{aligned}\Gamma_{\lambda',\lambda}^{(0)}&=\langle ps\lambda'|e^{i\pi J_2}Y^{-1}(Yj_0Y^{-1})Y|ps\lambda\rangle\\&=\eta_P^*\eta_P(-1)^{2s}\langle ps,-\lambda'|e^{i\pi J_2}j_0|ps,-\lambda\rangle\\&=(-1)^{2s}\Gamma_{-\lambda',-\lambda}^{(0)}.\end{aligned} \tag{31}$$

The matrix elements are, furthermore, real. Applying the operation of Wigner time reversal, we obtain from Eqs. (13) to (16) and (23)

$$\begin{aligned}\Gamma_{\lambda',\lambda}^{(0)*}&={}_T\langle ps\lambda'|Te^{i\pi J_2}j_0T^{-1}|ps\lambda\rangle_T\\&=\eta_T^*\eta_T\langle ps\lambda'|e^{2i\pi J_2}j_0e^{-i\pi J_2}|ps\lambda\rangle\\&=\langle ps\lambda'|e^{i\pi J_2}j_0|ps\lambda\rangle\\&=\Gamma_{\lambda',\lambda}^{(0)}.\end{aligned} \tag{32}$$

Thus, $\Gamma_{\lambda',\lambda}^{(0)}$ is a real skew-diagonal matrix in the 2×2 helicity space of the massless particle. We will choose $\Gamma_{-s,s}^{(0)}(p)$ as the single independent nonvanishing matrix element.

The foregoing argument exhausts the restrictions on the matrix elements of j_μ imposed by the general properties of j_μ and the single-particle states under proper and improper Lorentz transformations. Conservation of the third component of the total angular momentum requires that the matrix elements of $j_\pm$ vanish identically in the brick-wall coordinate system for massless particles of spins $s\neq\frac{1}{2}$. The matrix elements of j_3 vanish for all spins for massless or massive particles as a consequence of the reflection and time inversion symmetries, independently of the (sufficient) condition of current conservation.[3] (The weak form of the theorem is of interest in connection with the possible interactions of particles of arbitrary spin with general vector currents.) On the other hand, the general symmetry properties of Lorentz invariance, and invariance under space and time inversions are not sufficient, even when coupled with zero mass for the particle in question, to insure that the matrix elements of j_0 vanish. It is therefore clear that the result of Case and Gasiorowicz, which requires that $\Gamma_{\lambda',\lambda}^{(\mu)}(p)\equiv0$, depends in an essential manner on the type of coupling which was assumed. It will be shown in the following sections that the only allowed electromagnetic interactions for massless particles with $s>\frac{1}{2}$ correspond to derivative couplings of orders higher than can be generated by the combined phase and gauge transformations of Eqs. (3). It is furthermore clear that no such couplings are permitted unless the theory contains a mass, introduced either in the electromagnetic coupling itself, or through the interaction of the particle in question with a massive field. To this discussion we now turn.

III. VERTEX FUNCTIONS FOR MASSIVE PARTICLES AND THE ZERO-MASS LIMIT

The structure of the single-particle matrix elements of j_μ for massive particles of arbitrary spin was studied in detail in reference 3. In the present section, we shall begin with those results, and consider the transition to the zero-mass case, thereby obtaining some additional information about the latter. Since it has already been established [Eqs. (26) and (27)] that the matrix elements of $\mathbf{j}$ vanish identically for a massless particle with $s\neq\frac{1}{2}$, we will confine our attention to the matrix elements of j_0. The helicity space for the massive particle has dimension $2s+1$, $-s\leq\lambda\leq s$, and the matrix $\Gamma_{\lambda',\lambda}^{(0)}$ is consequently a square $(2s+1)\times(2s+1)$ matrix. From reference 3, Eqs. (87) to (89), we obtain the following symmetry properties of $\Gamma_{\lambda',\lambda}^{(0)}$, derivable from the properties of j_0 and the basic states under proper and improper Lorentz transformations:[7]

$$\Gamma_{\lambda',\lambda}^{(0)}=\delta_{\lambda',-\lambda}\Gamma_{-\lambda,\lambda}^{(0)}, \quad -s\leq\lambda\leq s; \quad (J_3) \tag{33}$$

$$\Gamma_{\lambda',\lambda}^{(0)}=(-1)^{2s}\Gamma_{-\lambda',-\lambda}^{(0)}; \quad (Y) \tag{34}$$

$$\Gamma_{\lambda',\lambda}^{(0)*}=\Gamma_{\lambda',\lambda}^{(0)}. \quad (T) \tag{35}$$

Thus, $\Gamma_{\lambda',\lambda}^{(0)}$ is a real skew-diagonal matrix in the helicity space, having $s+\frac{1}{2}$ or $s+1$ independent elements accordingly as s is a half-integer or integer, respectively.

By extending the two-dimensional helicity space for a massless particle of spin s to a $(2s+1)$-dimensional space, with the stipulation that all matrix elements vanish for which $|\lambda|, |\lambda'| < s$, we may regard the results of Sec. II as a special case of those above.

It was shown in reference 3, by general methods, that the vertex functions $\Gamma_{\lambda',\lambda}^{(0)}(p)$ for a particle of nonzero mass may be written in terms of Wigner $3j$ symbols[9] (symmetrized vector coupling coefficients) and certain reduced matrix elements $Q_J(p)$ in the form

$$\Gamma_{\lambda',\lambda}^{(0)}(p)=(-1)^{2s}\sum_{J=0,\text{ even}}\begin{pmatrix} s & J & s \\ \lambda' & 0 & \lambda \end{pmatrix} Q_J(p). \quad (36)$$

The $Q_J(p)$ represent multipole form factors corresponding to the charge $(J=0)$, electric quadrupole $(J=2)$, and higher electric 2^J-pole moments of the particle in question. The state $|ps\lambda\rangle$ is obtained for a particle of mass m by the application of a Lorentz transformation to the rest state $|s,\lambda\rangle$,

$$|ps\lambda\rangle=e^{-i\zeta K_3}|s,\lambda\rangle, \quad \zeta=\sinh^{-1}(p/m), \quad (37)$$

where K_3 the generator of the ordinary Lorentz transformations along the z axis. Using this information, it was possible to obtain the limiting form of $Q_J(p)$ for $(p/m)\rightarrow 0$. With the normalization of the states $|ps\lambda\rangle$ used in reference 3, $Q_J(p)$ was shown to approach zero as $(p/m)^J$ times a series in even powers of (p/m). These results did not differentiate between possible over-all factors of (p_0/m) and unity which will be important in later arguments $[p_0^2=p^2+m^2, m\neq 0]$. A single factor (p_0/m) must in fact be present in the functions $Q_J(p)$ for particles of integer spin. It is not possible in this case to construct a quantity with the transformation properties of j_0 [fourth component of a 4-vector] using the spin indices alone; the relevant quantity must be proportional to the 4-vector $(p+p')$, hence must contain an over-all factor $(p_0+p_0')/m=2p_0/m$ in the brick-wall frame.[10] This factor is absent in the case of half-integer spins, corresponding to the fact that appropriate 4-vectors can be constructed using the spinor indices alone, as, e.g., through the introduction of the Dirac matrix γ_μ for $s=\frac{1}{2}$. Finally, the change to the normalization of the states $|ps\lambda\rangle$ defined by Eqs. (17) and (18) introduces an extra factor of m in the functions $Q_J(p)$, and we obtain the limiting forms for $(p/m)\rightarrow 0$, $m\neq 0$,

$$\begin{aligned} Q_J(p)&\sim p_0(p/m)^J, \quad s=j=\text{integer}; \quad (38)\\ Q_J(p)&\sim m(p/m)^J, \quad s=j+\tfrac{1}{2}. \end{aligned}$$

This behavior corresponds to an effective 2^J-pole contribution to the operator j_0 which involves at least $J+1$ differentiations of the field operators for integer spins, and at least J differentiations for half-integer spins.[11] The factor m^{-J} insures that the resulting operator has the proper dimensions.

At this point, we will depart from our general methods, and assume specifically that the operator j_μ, in accordance with the usual Lagrangian theories, is bilinear in the field operator for the particle in question, and contains no other operators. We will, in particular, exclude for the moment electromagnetic couplings induced by the interaction of the particle with other fields. The 2^J-pole coupling will furthermore be assumed to contain only a finite number of derivatives. The general properties of the functions $Q_J(p)$ stated above can then be derived without the use of the transformation in Eq. (37), and will remain valid in the limit $m=0$ for which the transformation is singular. The number of possible types of electromagnetic coupling may be decreased in this limit as a consequence of the additional gauge conditions which must then be satisfied by the field operators[1] (or at least by matrix elements of the field operators), but no new possibilities are introduced. The restrictions on the interactions are in fact easily determined by inverting the multipole decomposition in Eq. (36), and noting that all matrix elements except $\Gamma_{\mp s,\pm s}^{(0)}$ must vanish for $m=0$. As a consequence of the last condition, the functions $Q_J(p)$ are no longer independent; those for $J<2j$ are simple numerical multiples of $Q_{2j}(p)$:

$$\begin{aligned} Q_J(p)&=2(2J+1)\begin{pmatrix} s & s & J \\ -s & s & 0 \end{pmatrix}\Gamma_{-s,s}^{(0)}(p)\\ &=\left[\frac{2J+1}{4j+1}\right]^{\frac{1}{2}}\begin{pmatrix} s & s & J \\ -s & s & 0 \end{pmatrix}\\ &\quad\times\begin{pmatrix} s & s & 2j \\ -s & s & 0 \end{pmatrix}^{-1} Q_{2j}(p), \quad J\text{ even}, \quad (39) \end{aligned}$$

where $j=s$ or $s-\frac{1}{2}$ for s integer or half-integer, respectively. In particular then, the multipole form factors for a given s all have the same momentum dependence. Since the simplest coupling associated with $Q_{2j}(p)$ involves $2j+1$ differentiations of the field operators for integer spins, or $2j$ differentiations for half-integer spins, it must also, for dimensional reasons, involve a factor M^{-2j} or M^{-2j+1}, where M is a mass characteristic of the problem. It there exists no such mass, the couplings must vanish except for spins 0 and $\frac{1}{2}$. On the other hand, if the massless particle interacts with a massive field, there is no general argument which will rule out induced couplings of the above type for

[9] See reference 8, Chap. 3.

[10] The extra factor of (p_0/m) is absent in the matrix elements of $j_\pm$. Quantities with the correct transformation properties can in this case be constructed using the spin matrices and the gradient operators which must be present: the matrix elements behave for $(p/m)\rightarrow 0$ as $(p/m)^J$, J odd (reference 3).

[11] For example, the 2^J-pole contribution to j_0 may be written in the nonrelativistic limit as an operator independent of the spin matrices, multiplied by the scalar formed by contracting J gradient operators with the symmetric spin-tensor operator of rank J, the whole being properly symmetrized.

arbitrary s. For spin 0, a charge coupling of the usual type is possible. For spin $\frac{1}{2}$ the charge coupling is zero [compare, Eq. (38) with $J=0$, $m=0$], but a direct coupling to the transverse components of the electromagnetic field is possible. The only nonvanishing matrix elements of $j_{\pm}$ are seen from Eq. (25) to be $\Gamma_{-\frac{1}{2},-\frac{1}{2}}{}^{(+)}$ and $\Gamma_{\frac{1}{2},\frac{1}{2}}{}^{(-)}$. From the general properties of $j_{\pm}$ and the single-particle states under the time and space inversion T and Y, one easily shows that the matrix elements are real, and that

$$\Gamma_{-\frac{1}{2},-\frac{1}{2}}{}^{(+)}=-\Gamma_{\frac{1}{2},\frac{1}{2}}{}^{(-)}.$$

The functions were shown in reference 3 to approach zero linearly in p for $p\to 0$, and to correspond (for massive particles) to a magnetic dipole type of coupling. We note finally that if the spin $\frac{1}{2}$ particle interacts with a massive field, it may also possess an induced charge coupling.

IV. DISCUSSION AND EXAMPLES

The essential role of zero mass in the paper of Case and Gasiorowicz[1] is clear. The requirement of the invariance of the theory of massless particles interacting with the electromagnetic field under the simultaneous phase and gauge transformations in Eqs. (3) leads to couplings involving at most one derivative of the field operators, Eqs. (4). The discussion in Sec. III indicates that the only possible coupling for spin s involves $2s+1$ derivatives for s an integer, and $2s-1$ dreivatives for s a half-integer; consequently, the charge parameter e in the phase transformation in Eq. (3) must vanish in a consistent theory for $s>\frac{1}{2}$, and no electromagnetic couplings of the usual kind can occur.[1] This does not preclude the introduction of a coupling of the type Q_{2j}, but such a coupling necessarily introduces a length (mass) into the theory. The arguments of Sec. III can be extended to cover also the electromagnetic couplings induced by the interaction of the massless particle with a massive charged field, but the possible Yukawa type couplings to such a field are again somewhat unusual. The structure of the vertex functions describing the interaction of a massless particle with a vector meson field is the same as that derived above for the matrix elements of j_μ. The corresponding structure of the vertex functions for couplings to scalar or pseudoscalar fields follows with minor changes from that deduced for j_0. Such couplings are all, therefore, of the derivative type associated with Q_{2j}.

It may be of interest to write down the simplest examples, the direct electromagnetic couplings for spins 0, $\frac{1}{2}$, and 1. For $s=0$, the equations of motion for the electromagnetic and particle fields are given by the familiar relations

$$\Box A_\mu=-ie\phi^*\overleftrightarrow{\partial}_\mu\phi+2e^2A_\mu\phi^*\phi, \tag{40}$$

$$(\partial_\mu-ieA_\mu)^2\phi=0. \tag{41}$$

For $s=\frac{1}{2}$, one has the Dirac equation for zero mass, and the standard equations of motion,

$$\Box A_\mu=ie\bar{\psi}\gamma_\mu\psi, \tag{42}$$

$$\gamma_\mu(\partial_\mu-ieA_\mu)\psi=0. \tag{43}$$

This coupling for spin $\frac{1}{2}$ leads for massive particles to the result for the single-particle matrix elements of $j_\mu=-ie\bar{\psi}\gamma_\mu\psi$,[12]

$$\langle p'|j_\mu|p\rangle=-ie\bar{u}(p')[\gamma_\mu F_1(q^2)+(\kappa/2m)F_2(q^2)\sigma_{\mu\nu}(p'-p)_\nu]u(p), \tag{44}$$

where F_1 and F_2 are the Dirac and Pauli form factors for the particle in question, and $q=p'-p$. For $m\to 0$, the second term must vanish, $\kappa\to 0$, if there is no other mass in the problem. Furthermore, we must then have $F_1(q^2)=1$, all q^2, and

$$\langle p'|j_\mu|p\rangle\to -ie\bar{u}(p')\gamma_\mu u(p). \tag{45}$$

It is illuminating to specialize this result to the brick-wall coordinate system and reduce the spinors to two component form. The properly normalized spinors for a representation with $\boldsymbol{\gamma}=-i\gamma_4\boldsymbol{\alpha}$ are of the form

$$u_\lambda(p)=\left(\frac{p}{2}\right)^{1/2}\begin{pmatrix}\chi_\lambda\\ \sigma_3\chi_\lambda\end{pmatrix}, \tag{46}$$

where the χ's are normalized Pauli spinors. Using the helicity representation for the spin, one obtains

$$\langle -\mathbf{p},\lambda'|j_0|\mathbf{p},\lambda\rangle=0, \tag{47.1}$$

$$\langle -\mathbf{p},\lambda'|\mathbf{j}|\mathbf{p}\lambda\rangle=i\chi_{\lambda'}{}^\dagger\sigma_2(\mathbf{p}\times\boldsymbol{\sigma})\chi_\lambda, \tag{47.2}$$

leading to the expected results, Eqs. (25), (26), and (38) for $m=0$. If the massless particle interacts with a charged massive field, there can also be an induced Pauli-type coupling as in Eq. (44).

The first essentially new case is that of the complex massless vector field ϕ_α, $s=1$. A nonzero coupling, which necessitates the introduction a mass $M\neq 0$ in the theory, may be obtained by adding to the free Lagrangian a term

$$\mathcal{L}'=(ie'/M^2)f_{\alpha\mu}{}^*F_{\mu\nu}f_{\nu\alpha}, \tag{48}$$

where

$$f_{\alpha\beta}=\partial_\alpha\phi_\beta-\partial_\beta\phi_\alpha, \tag{49.1}$$

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{49.2}$$

We remark that the resulting Lagrangian can be made invariant under the phase and gauge transformation in Eq. (3) by replacing ∂_σ in Eq. (49.1) by $(\partial-ieA)_\sigma$. However, the only consistent choice of e for the massless field is $e=0$. The equations of motion which follow from Eq. (48) are

$$\Box A_\mu=-j_\mu=(ie'/M^2)\partial_\nu[f_{\alpha\nu}{}^*f_{\mu\alpha}-f_{\alpha\mu}{}^*f_{\nu\alpha}], \tag{50}$$

$$\Box\phi_\beta=-J_\beta=(ie'/M^2)\partial_\alpha[F_{\beta\nu}f_{\nu\alpha}-F_{\alpha\nu}f_{\nu\beta}]. \tag{51}$$

[12] G. Salzman, Phys. Rev. 99, 973 (1955).

We remark in particular that the currents j_μ and J_β are both conserved,

$$\partial\cdot j=0, \quad \partial\cdot J=0. \tag{52}$$

Consequently, if we follow Case and Gasiorowicz[1] and write for the asymmptotic out-field,[4]

$$\phi_\beta^{\text{out}}=\phi_\beta^{\text{in}}-\int dx'\, D(x-x')J_\beta(x'), \tag{53}$$

we find that the out-field satisfies the divergence condition

$$\partial_\beta\phi_\beta^{\text{out}}=0$$

if the in-field does also. This was not the case for the current generated by the phase transformation, Eqs. (4), and gave Case and Gasiorowicz their proof. We note finally that the conserved quantity associated with J_β vanishes if the fields disappear at spatial infinity,[13]

$$\int d^3x\, J_0(x)=-(ie'/M^2)\int d^3x\, \nabla\cdot[(\partial_0 A_\nu-\partial_\nu A_0)\times(\partial_\nu\boldsymbol{\phi}-\nabla\phi_\nu)-(\nabla A_\nu-\partial_\nu\mathbf{A})\times(\partial_\nu\phi_0-\partial_0\phi_\nu)]=0. \tag{54}$$

As a final example, it may be of interest to give the cross sections for the Coulomb scattering of a massless particle of arbitrary spin. The calculation may be performed trivially using the methods of reference 3, Eqs. (129) ff., with appropriate changes in the normalization of the states. For spin 0, one obtains the familiar Rutherford cross section,

$$d\sigma/d\Omega=(e^2/2p)^2\sin^{-4}(\tfrac{1}{2}\theta)=\sigma_R, \quad s=0. \tag{55}$$

For spin $\frac{1}{2}$,[14]

$$d\sigma/d\Omega=\sigma_R\cos^2(\tfrac{1}{2}\theta), \quad s=\tfrac{1}{2}. \tag{56}$$

The vanishing of the cross section for $\theta=\pi$ corresponds to the vanishing of the matrix element of j_0 in the brick-wall coordinate frame, Eq. (47.1). For spins $s>\frac{1}{2}$, we introduce the notation for the brick-wall matrix elements

$$\Gamma_{-s,s}^{(0)}=2ep(p/M)^J q_J(p/M), \tag{57}$$

where $J=2s$ for s an integer, and $J=2(s-1)$ for s a half-integer. For the direct coupling of minimal complexity, the dimensionless form factor $q_J(p/M)$ is a constant. Noting that the brick-wall momentum p is given in terms of the particle momenta in an arbitrary Lorentz frame by $4p^2=(p-p')^2=4pp'\sin^2(\frac{1}{2}\theta)$, we obtain for the cross section

$$d\sigma/d\Omega=\sigma_R\left[q_J\left(\frac{p}{M}\sin(\tfrac{1}{2}\theta)\right)\right]^2\times(p/M)^{2J}\sin^{2J}(\tfrac{1}{2}\theta), \quad s\neq\tfrac{1}{2}. \tag{58}$$

The cross section is finite for $\theta=0$ for $s\geq 1$.

[13] Compare reference 1, footnote 5.

[14] Equation (30) in reference 1 is incorrect.