Angular Momenta in Relativistic Many-Body Problems

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The problem of the splitting of the total angular momentum for a relativistic system of many free particles is discussed, with the aim of justifying the extension to the relativistic case of the usual nonrelativistic techniques. This is done by introducing external (i.e., center-of-mass) and internal (i.e., relative) coordinates and momenta—it being shown that this is always possible, even for particles with spin. The explicit form of the coordinates and momenta is given for the case of two particles, the general case being obtained by recurrence.

The arguments leading to the choice of coordinates and momenta are thoroughly discussed; this also gives an unambiguous way to prove that the spins must be transformed into the c.m. reference frame, as is usually done.

1. INTRODUCTION

THE present trend of particle physics towards higher energies makes necessary a thorough consideration of the practical consequences of the principles of relativity. In the phenomenological discussion of elementary-particle interactions, widespread use is made of angular momentum analysis; for many-particle systems this can only be accomplished by the well-known techniques of splitting the total angular momentum into several parts, i.e., the spin of the various particles and their relative orbital angular momenta. In nonrelativistic quantum mechanics this is easily done by standard techniques. The nonrelativistic procedure, however, cannot be immediately extended to the relativistic case, since such an extension would lead to partial angular momenta which are not constants of motion. One could then ask whether a nontrivial extension keeping the important features of the nonrelativistic theory exists at all. It is the aim of this paper to show how one can arrive at giving an affirmative answer to that question.

The above problem can be viewed from a group-theoretical standpoint; it then reduces to the Clebsch-Gordan problem for the proper inhomogeneous Lorentz group.[1] This approach is quite general and allows a formal treatment along well-established lines.

In the present paper, a different line of approach is adopted. We look for a canonical transformation leading from the observables of the single particles (position, momentum, and spin) to a set of external observables (c.m. position, total momentum, and total spin) and internal observables (depending on the coupling scheme adopted). The transformation is aimed to give a decomposition of the total angular momentum in strict analogy with the nonrelativistic case. This gives us a clearer insight of how relativity acts in modifying the nonrelativistic results, and yields a sounder operational basis for the whole matter.[2,3]

Section 2 is devoted to a short account of the "canonical" form for the theory of a free relativistic particle with spin, which is the basis for the subsequent treatment. In Sec. 3 a general definition of the external variables is given, aimed at paralleling the description of a free particle. In Sec. 4 explicit forms for the external and internal variables are derived for the simplest case of a system consisting of only two spinless particles. In Sec. 5 allowance is made for the spin of the particles. It is shown that the spins must be transformed to the c.m. reference frame, and that the presence of the spins also modifies the external and internal position vectors. Section 6 deals with the general case of any number of particles with spin. The generalization of the preceding results is given through a recurrence argument. The section ends with a discussion of the main results of the paper.

2. ONE FREE RELATIVISTIC PARTICLE WITH SPIN

The quantum theory of a free relativistic particle with spin can be given a "canonical" form in the sense of Foldy.[4] The present treatment, though leading to the same results, follows a different line of approach, emphasis being given mainly to the axiomatic definition of a "free" particle.[5]

A free particle is defined by the following axioms:

[1] A. J. Macfarlane, Revs. Modern Phys. **34**, 41 (1962).

[2] For a treatment much on the same lines as the present one, compare Chou Kuang-Chao and M. I. Shirokov, Soviet Phys.—JETP **7**, 851 (1958).

[3] Iu. M. Shirokov, Soviet Phys.—JETP **8**, 703 (1959) and G. C. Wick, Ann. Phys. (New York) **18**, 65 (1962), deal with the same problem from a viewpoint which is in some way intermediate between references 1 and 2. For further literature on this subject the reader is referred to the bibliography in references 2 and 3.

[4] L. L. Foldy, Phys. Rev. **102**, 569 (1956).

[5] The case of a spinless particle is treated by E. Fabri and L. E. Picasso (to be published).

(i) The only independent observables are the position vector $\mathbf{q}$, the momentum $\mathbf{p}$, and the spin $\mathbf{s}$, which obey the following commutation rules:

$$\begin{aligned}&[q^i,q^j]=0, \quad [p^i,p^j]=0, \quad [s^i,s^j]=i\epsilon^{ijk}s^k,\\ &[q^i,p^j]=i\delta^{ij}, \quad [q^i,s^j]=0, \quad [p^i,s^j]=0.\end{aligned} \tag{1}$$

(ii) The observables $\mathbf{q}$, $\mathbf{p}$, $\mathbf{s}$ transform like vectors under space rotations.

(iii) Under space translations, $\mathbf{p}$ and $\mathbf{s}$ are invariant; $\mathbf{q}$ transforms like a coordinate vector:

$$[P^i,q^j]=-i\delta^{ij}, \tag{2}$$

P^i being the realization of the generator $\mathfrak{P}^i$ of the translation along the axis x^i.

(iv) The Hamiltonian for the system is

$$E=(p^2+m^2)^{\frac{1}{2}}, \tag{3}$$

where m is a number (the rest mass). This amounts to saying that E gives the realization of the generator $\mathfrak{H}$ of time translations for the system.

It has not been assumed that s^2 is a number, since it is an immediate consequence of axiom (i). From Eq. (1) one can see, indeed, that s^2 commutes with $\mathbf{q}$, $\mathbf{p}$, $\mathbf{s}$ and, therefore, with all observables. From axiom (iii) and Eq. (2) we find with a similar argument that P^i must coincide with p^i (apart for a trivial additive constant).

The realizations J^i of the generators $\mathfrak{J}^i$ of space rotations can be found as follows. From axiom (ii) and the commutation relations for $\mathbf{s}$ we see that $\mathbf{J}-\mathbf{s}$ commutes with $\mathbf{s}$ and is then independent of $\mathbf{s}$. The only solution (apart for a trivial additive constant) for the commutation relations of $\mathbf{J}-\mathbf{s}$ with $\mathbf{q}$ and $\mathbf{p}$ is $\mathbf{q}\times\mathbf{p}$. Thus, we have

$$\mathbf{J}=\mathbf{q}\times\mathbf{p}+\mathbf{s}. \tag{4}$$

We are now ready to discuss the realizations K^i of the generators of Lorentz transformations $\mathfrak{K}^i$.[4] The equations to be satisfied by $\mathfrak{K}^i$ are

$$\begin{aligned}&[\mathfrak{K}^i,\mathfrak{P}^j]=i\delta^{ij}\mathfrak{H}, \quad [\mathfrak{K}^i,\mathfrak{H}]=i\mathfrak{P}^i,\\ &[\mathfrak{K}^i,\mathfrak{J}^j]=i\epsilon^{ijk}\mathfrak{K}^k, \quad [\mathfrak{K}^i,\mathfrak{K}^j]=-i\epsilon^{ijk}\mathfrak{J}^k.\end{aligned} \tag{5}$$

If we put

$$\mathbf{K}=\tfrac{1}{2}(\mathbf{q}E+E\mathbf{q})+\mathbf{G}, \tag{6}$$

we find that $\mathbf{G}$ must commute with $\mathbf{p}$ [from the first of Eqs. (5)]; and so it cannot depend on $\mathbf{q}$, but only on $\mathbf{p}$ and $\mathbf{s}$. From the third of Eqs. (5) we learn only that $\mathbf{G}$ is a vector. If we require that $\mathbf{G}$ is a linear function of $\mathbf{s}$, the last of Eqs. (5) gives two solutions,

$$\mathbf{G}=-\mathbf{s}\times\mathbf{p}/(E\pm m). \tag{7}$$

The minus sign in the denominator, however, must be rejected since it leads to a singularity.[6]

The solution given by Eq. (7) is by no means unique. As we are dealing with an irreducible representation of the Lorentz group (this is a consequence of our axioms), all solutions for $\mathbf{K}$ must be of the form $V\mathbf{K}V^{-1}$, V being a unitary operator function of p^2 and $\mathbf{s}\cdot\mathbf{p}$ only.[7]

It can readily be shown from Eqs. (6) and (7) that

$$\mathbf{s}\cdot\mathbf{p}, \quad m\mathbf{s}+[\mathbf{s}\cdot\mathbf{p}/(E+m)]\mathbf{p} \tag{8}$$

transform like the time and space components of a four-vector. Thus, one arrives at the usual covariant description of spin.

3. EXTERNAL VARIABLES FOR A GENERAL SYSTEM

We shall define "external" variables for a general system in such a way that the description they give of the system parallels that of a free particle. Further ("internal") variables will be needed, of course, to describe the internal degrees of freedom.

Our requirements can be exactly stated as follows:

(i) A c.m. position vector $\mathbf{Q}$, a total momentum $\mathbf{P}$, a total spin $\mathbf{S}$, and a total rest mass M may be defined, with the commutation relations

$$\begin{aligned}&[Q^i,Q^j]=0, \quad [P^i,P^j]=0, \quad [S^i,S^j]=i\epsilon^{ijk}S^k,\\ &[Q^i,P^j]=i\delta^{ij}, \quad [Q^i,S^j]=0, \quad [P^i,S^j]=0,\\ &[Q^i,M]=0, \quad [P^i,M]=0, \quad [S^i,M]=0.\end{aligned} \tag{9}$$

(ii) The realizations of the generators of the Lorentz group are given by the following relations:

$$\begin{aligned}&\mathfrak{P}\rightarrow\mathbf{P}, \quad \mathfrak{H}\rightarrow(P^2+M^2)^{\frac{1}{2}}=E,\\ &\mathfrak{J}\rightarrow\mathbf{Q}\times\mathbf{P}+\mathbf{S}=\mathbf{J},\\ &\mathfrak{K}\rightarrow\tfrac{1}{2}(\mathbf{Q}E+E\mathbf{Q})-\mathbf{S}\times\mathbf{P}/(E+M)=\mathbf{K}.\end{aligned} \tag{10}$$

The conditions (10) give a unique, obvious expression for $\mathbf{P}$ and M; also, $\mathbf{S}$ is uniquely found if $\mathbf{Q}$ is given. Thus, the only problem is to find $\mathbf{Q}$. The solution [which is unique under conditions (9)] is the following:

$$\begin{aligned}\mathbf{Q}=\tfrac{1}{2}(\mathbf{K}E^{-1}+E^{-1}\mathbf{K})+\mathbf{J}\times\mathbf{P}/M(E+M)\\ -[(\mathbf{K}\times\mathbf{P})\times\mathbf{P}]/EM(E+M).\end{aligned} \tag{11}$$

[No ambiguity arises from the quotient form of the last two terms in the right-hand side (rhs) of Eq. (11), as the numerators commute with the denominators.] As a matter of fact, in order to derive the form of $\mathbf{Q}$ given by Eq. (11) we only need the commutation relations of $\mathbf{Q}$ with $\mathbf{P}$ and M; the remaining conditions of (9) follow.

Equation (11) requires some explanation; in its rhs we find $\mathbf{K}$, $\mathbf{P}$, $\mathbf{J}$, E, M, all of which are understood to be known expressions in terms of the "elementary" ob-

[6] The double sign is also given in reference 2 without discussion. For massless particles, both solutions coincide and the singularity disappears due to the $\mathbf{p}$ in the numerator; the representation of the Lorentz group one gets in that case, however, is reducible. We will not dwell on this point here.

[7] In reference 2 there is a claim for the uniqueness of $\mathbf{K}$. This is probably to be meant in the sense that the original form of $\mathbf{K}$ can be recovered if V is used to redefine $\mathbf{q}$ and $\mathbf{s}$. It is not obvious that such an operation should be physically inessential. For a discussion on this point see reference 5.

servables of the system. This is only meaningful if we know that our system is composed of noninteracting subsystems, and if we know the representatives of the generators of the Lorentz group for each constituent of the system in terms of its own variables. For instance, if our system is a collection of free particles $\mathbf{P}$ will obviously be the sum of the momenta of all the particles, and a similar result will also hold for the other generators. In the following we will always be concerned with such kind of systems.

Since we have left room for internal variables, our general system is not "elementary" in the sense of Newton and Wigner.[8] M and S^2 are not numbers, because they do not commute with the internal variables; this means that the representation of the Lorentz group for the system is generally reducible. Nevertheless, we can make use of (10), much on the same lines as analogous formulas can be used for an elementary system.

4. EXTERNAL AND INTERNAL VARIABLES FOR TWO SPINLESS PARTICLES[9]

In this section a system of two free spinless particles will be considered. Quantities relating to a single particle will be labeled with the indices 1, 2, respectively. The generators of the Lorentz group for the system are given by[10]

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2, \quad E=E_1+E_2, \tag{12}$$

$$\mathbf{J}_0=\mathbf{q}_1\times\mathbf{p}_1+\mathbf{q}_2\times\mathbf{p}_2, \tag{13}$$

$$\mathbf{K}_0=\mathbf{K}_1+\mathbf{K}_2=\tfrac{1}{2}(\mathbf{q}_1E_1+E_1\mathbf{q}_1+\mathbf{q}_2E_2+E_2\mathbf{q}_2). \tag{14}$$

The total rest mass M is given by

$$M^2=E^2-P^2=m_1^2+m_2^2+2E_1E_2-2\mathbf{p}_1\cdot\mathbf{p}_2. \tag{15}$$

By directly inserting (12) to (15) into Eq. (11), one gets the expression for $\mathbf{Q}_0$. Since we are not interested in its explicit form, its derivation is deferred to Appendix 1, where we will also give the proof of Eq. (7) of BFI. $\mathbf{S}_0$, as given by

$$\mathbf{S}_0=\mathbf{J}_0-\mathbf{Q}_0\times\mathbf{P}, \tag{16}$$

can be easily computed.

We are now ready to look for the internal variables. We try to construct a set of canonical variables for the system by adjoining to $\mathbf{Q}_0$, $\mathbf{P}$ a pair of vectors $\mathbf{q}_0$ and $\mathbf{p}$. We feel that we should add the further requirement that $\mathbf{S}_0$ can be expressed in the form

$$\mathbf{S}_0=\mathbf{q}_0\times\mathbf{p}. \tag{17}$$

This, however, is not necessary; the single condition that $\mathbf{p}$ be a vector insures that its conjugate variable $\mathbf{q}^0$ is also a vector. Then Eq. (17) is automatically satisfied.

The proof runs as follows: A unitary transformation U must exist, mapping the set $\mathbf{q}_1$, $\mathbf{p}_1$, $\mathbf{q}_2$, $\mathbf{p}_2$ in that order onto the set $\mathbf{Q}_0$, $\mathbf{P}$, $\mathbf{q}_0$, $\mathbf{p}$. Since U preserves the vector character of all canonical coordinates, it must be a scalar. Therefore, it commutes with $\mathbf{J}_0$. By transforming Eq. (13) by U one gets

$$\mathbf{J}_0=\mathbf{Q}_0\times\mathbf{P}+\mathbf{q}_0\times\mathbf{p}, \tag{18}$$

and Eq. (17) follows.

We restrict ourselves to a $\mathbf{p}$ which is a function of $\mathbf{p}_1$ and $\mathbf{p}_2$ only. Then it must be of the form

$$\mathbf{p}=Y_1\mathbf{p}_1-Y_2\mathbf{p}_2, \tag{19}$$

with Y_1, Y_2 scalar functions of $\mathbf{p}_1$, $\mathbf{p}_2$. It can be shown that $\mathbf{p}$, as defined by these conditions, is essentially unique; we get for $\mathbf{p}$ the momentum of particle 1 in the c.m. reference frame (see Appendix 2 for a detailed proof). The arbitrariness in $\mathbf{p}$ is reduced to a canonical transformation leaving $\mathbf{Q}_0$ and $\mathbf{P}$ invariant; then $\mathbf{p}$ can only be multiplied by a scalar function of p^2. We will take no advantage, however, of this arbitrariness.

With our choice, the explicit form of $\mathbf{p}$ is given by Eq. (19) with

$$Y_{1,2}=\tfrac{1}{2}\mp\frac{M(E_1-E_2)+m_1^2-m_2^2}{2M(E+M)}. \tag{20}$$

The derivation of $\mathbf{q}_0$ becomes at this point merely a computational task; since in the following we will not need an explicit form for $\mathbf{q}_0$, we defer to Appendix 2 the exact steps of the computation.

5. TWO PARTICLES WITH SPIN

In this section we give the extension of our previous results to the case of two particles with spin.

The total angular momentum of the system is now given by

$$\mathbf{J}=\mathbf{q}_1\times\mathbf{p}_1+\mathbf{q}_2\times\mathbf{p}_2+\mathbf{s}_1+\mathbf{s}_2=\mathbf{J}_0+\mathbf{s}_1+\mathbf{s}_2. \tag{21}$$

After Eq. (18), an obvious decomposition of $\mathbf{J}$ is the following:

$$\mathbf{J}=\mathbf{Q}_0\times\mathbf{P}+\mathbf{q}_0\times\mathbf{p}+\mathbf{s}_1+\mathbf{s}_2. \tag{22}$$

The four terms into which $\mathbf{J}$ is decomposed are all constants of the motion; it is also apparent that the commutation requirements are all satisfied.

The extension we are looking for appears to be quite straightforward: $\mathbf{Q}_0$ and $\mathbf{P}$ would be the external variables, $\mathbf{q}_0$ and $\mathbf{p}$ the internal ones, the spin taking no part in the transformation. This solution, however, is not consistent with the general form (11) for $\mathbf{Q}$. From

$$\mathbf{K}=\mathbf{K}_0-\mathbf{s}_1\times\mathbf{p}_1/(E_1+m_1)-\mathbf{s}_2\times\mathbf{p}_2/(E_2+m_2), \tag{23}$$

[8] T. D. Newton and E. P. Wigner, Revs. Modern Phys. 21, 400 (1949).

[9] This case has been already dealt with in our earlier paper [B. Barsella and E. Fabri, Phys. Rev. 126, 1561 (1962)]. This paper is subsequently referred to as BFI.

[10] We use throughout this section the subscript "0" for all those operators which will change expression for the spin-nonzero case. This will be of use later.

and Eq. (21) one gets

$$\mathbf{Q}=\mathbf{Q}_0-\frac{(E_1'-m_1)\mathbf{s}_1\times\mathbf{P}+(E+M)\mathbf{s}_1\times\mathbf{p}}{M(E+M)(E_1+m_1)}+\frac{\mathbf{s}_1\cdot(\mathbf{p}\times\mathbf{P})}{EM(E+M)(E_1+m_1)}\mathbf{P}-\frac{(E_2'-m_2)\mathbf{s}_2\times\mathbf{P}-(E+M)\mathbf{s}_2\times\mathbf{p}}{M(E+M)(E_2+m_2)}-\frac{\mathbf{s}_2\cdot(\mathbf{p}\times\mathbf{P})}{EM(E+M)(E_2+m_2)}\mathbf{P}, \quad (24)$$

where

$$E_1'=(p^2+m_1^2)^{\frac{1}{2}}, \quad E_2'=(p^2+m_2^2)^{\frac{1}{2}}. \quad (25)$$

$\mathbf{Q}$ can be written in the more compact form,

$$Q^i=Q_0{}^i+\Omega_1{}^{ij}s_1{}^j+\Omega_2{}^{ij}s_2{}^j. \quad (26)$$

It should be noted that Ω_1 and Ω_2 depend only on the momenta $\mathbf{p}$ and $\mathbf{P}$, and on the rest masses of the particles; Ω_1 goes into Ω_2 and vice versa through an interchange of the particles. (Under such an operation, $\mathbf{P}\rightarrow\mathbf{P}$ and $\mathbf{p}\rightarrow-\mathbf{p}$.)

The form (24) of $\mathbf{Q}$ determines that of $\mathbf{S}$ as defined by

$$\mathbf{S}=\mathbf{J}-\mathbf{Q}\times\mathbf{P}. \quad (27)$$

Our aim is now to express $\mathbf{S}$ in terms of internal variables $\mathbf{q}$, $\mathbf{p}$, $\mathbf{s}_1'$, and $\mathbf{s}_2'$:

$$\mathbf{S}=\mathbf{q}\times\mathbf{p}+\mathbf{s}_1'+\mathbf{s}_2'. \quad (28)$$

It is immediately apparent that $\mathbf{s}_1'$, $\mathbf{s}_2'$ must differ from $\mathbf{s}_1$ and $\mathbf{s}_2$ since these latter do not commute with $\mathbf{Q}$. Because of the symmetry of the role played by the two particles, we may restrict ourselves to considering $\mathbf{s}_1'$.

We search for an $\mathbf{s}_1'$ of the form

$$s_1'^i=a_1{}^{ij}s_1{}^j, \quad (29)$$

with $a_1{}^{ij}$ a function of $\mathbf{p}$, $\mathbf{P}$, m_1, m_2 only. If $\mathbf{s}_1'$ is to represent a spin, the matrix $a_1{}^{ij}$ must be an orthogonal unimodular matrix. If $\mathbf{s}_1'$ is to commute with $\mathbf{Q}$ we must have

$$0=[Q^i,s_1'^j]=[Q_0{}^i,a_1{}^{jk}]s_1{}^k+\Omega_1{}^{il}a_1{}^{jr}[s_1{}^l,s_1{}^r]=i(\partial a_1{}^{jk}/\partial P^i+\Omega_1{}^{il}a_1{}^{jr}\epsilon^{lrk})s_1{}^k. \quad (30)$$

Thus, we are led to a differential equation for a_1:

$$\partial a_1{}^{jk}/\partial P^i+\Omega_1{}^{il}a_1{}^{jr}\epsilon^{lrk}=0. \quad (31)$$

Equation (31) has only one solution. To see this, assume for a moment that two solutions exist. It can be easily shown that one of them differs from the other through multiplication by an orthogonal matrix not a function of $\mathbf{P}$; but no such matrix exists (apart from the identity) as one cannot define a rotation if only one vector, $\mathbf{p}$, is available.

It can be verified that the solution of Eq. (31) is just that matrix which gives the explicit transformation law of $\mathbf{s}_1$ to the c.m. reference frame.[11]

We are now left with the problem of finding a good $\mathbf{q}$. By analogy with Eq. (26) we guess a solution,

$$q^i=q_0{}^i+\omega_1{}^{ij}s_1{}^j+\omega_2{}^{ij}s_2{}^j. \quad (32)$$

If $\mathbf{q}$ is to commute with $\mathbf{s}_1'$ we must have for ω_1

$$\partial a_1{}^{jk}/\partial p^i+\omega_1{}^{il}a_1{}^{jr}\epsilon^{lrk}=0. \quad (33)$$

Equation (33) can be solved to yield

$$\omega_1{}^{ij}=-\tfrac{1}{2}\epsilon^{jks}a_1{}^{rk}\partial a_1{}^{rs}/\partial p^i. \quad (34)$$

$\mathbf{q}$ is then uniquely defined; we have only to check that all properties we have required for it are true. $\mathbf{q}$ obviously has the right commutation relations with $\mathbf{P}$ and $\mathbf{p}$. Only computational skill is required in order to verify the commutation relations of the components of $\mathbf{q}$ with each other and with those of $\mathbf{Q}$. We should still verify that Eq. (28) is satisfied. This, however, follows by an obvious extension of the analogous result of Sec. 4.

6. MANY-PARTICLE SYSTEMS. CONCLUDING REMARKS

For a system containing more than two particles the preceding arguments may be carried over through a recurrence method. The basis for this is given by our having defined external variables for a general system in strict parallel with the variables of a free particle.

Consider a system of n particles. It can be split into two subsystems of, say, n_1 and $n_2=n-n_1$ particles. For each subsystem external variables can be defined, say, $\mathbf{Q}_1$, $\mathbf{P}_1$, $\mathbf{S}_1$ and $\mathbf{Q}_2$, $\mathbf{P}_2$, $\mathbf{S}_2$. Since these external variables have the same properties as those of a free particle the procedure of Sec. 5 can be applied to give $\mathbf{Q}$, $\mathbf{P}$, $\mathbf{S}$, $\mathbf{q}$, $\mathbf{p}$, $\mathbf{S}_1'$, $\mathbf{S}_2'$ in terms of them. The same can be done on each subsystem, and so on until final subsystems each consisting of a single particle are reached.

There is a wide variety of splittings, leading to the various couplings for the angular momenta, in full analogy with the nonrelativistic case.

We have thus reached our goal, which was to show that the relativistic treatment of angular momentum follows a procedure quite similar to the nonrelativistic one. The main result—and not an obvious one—is the possibility of defining external and internal position vectors. The explicit form for these coordinates is quite complicated even in the simplest cases. Happily, this causes no trouble in the usual analysis of angular distributions and polarizations, where only momenta and spins are of interest. We have also clarified in what way the spins are to be dealt with, by explaining why the c.m. transform is to be used.

As a practical rule only the following changes are required in the usual nonrelativistic procedure:

[11] See, for example, V. I. Ritus, Soviet Phys.—JETP **13**, 240 (1961) Sec. 4. In this paper we use a slightly modified form, which is found in Appendix 3.

(i) The internal momenta must be evaluated using the relativistic formulas.

(ii) The spins must be transformed to the center of mass of the next bigger subsystem.

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APPENDIX 1. DERIVATION OF Q_0

By the procedure outlined in Sec. 4 the following form of $\mathbf{Q}_0$ is immediately found:

$$\begin{aligned}\mathbf{Q}_0=\tfrac{1}{4}(&\mathbf{q}_1E_1E^{-1}+\mathbf{q}_2E_2E^{-1}+E^{-1}\mathbf{q}_1E_1+E^{-1}\mathbf{q}_2E_2\\ &+E_1\mathbf{q}_1E^{-1}+E_2\mathbf{q}_2E^{-1}+E^{-1}E_1\mathbf{q}_1+E^{-1}E_2\mathbf{q}_2)\\ &+\frac{(\mathbf{q}_1\times\mathbf{p}_1+\mathbf{q}_2\times\mathbf{p}_2)\times\mathbf{P}}{M(E+M)}\\ &-\frac{[(\mathbf{q}_1E_1+E_1\mathbf{q}_1+\mathbf{q}_2E_2+E_2\mathbf{q}_2)\times\mathbf{P}]\times\mathbf{P}}{2ME(E+M)}.\end{aligned}\tag{A1}$$

If one wants a form of $\mathbf{Q}_0$ in terms of one-particle variables the only thing to do is to substitute for $\mathbf{P}$, E, and M their expressions (12) and (15). This, however, is not the form we gave in BFI; this latter can be found as follows.

First, transform Eq. (A1) into

$$\begin{aligned}\mathbf{Q}_0=\tfrac{1}{2}\{&(\mathbf{q}_1E_1+\mathbf{q}_2E_2)E^{-1}\\ &+(\mathbf{q}_1\times\mathbf{p}_1+\mathbf{q}_2\times\mathbf{p}_2)\times[\mathbf{P}/M(E+M)]\\ &-[(\mathbf{q}_1E_1+\mathbf{q}_2E_2)\times\mathbf{P}]\\ &\times[\mathbf{P}/EM(E+M)]\}+\text{H.c.}\end{aligned}\tag{A2}$$

Second, express $\mathbf{p}_1$ and $\mathbf{p}_2$ in terms of $\mathbf{p}$ and $\mathbf{P}$ by inverting Eqs. (12) and (19) as follows:

$$\begin{aligned}\mathbf{p}_1&=\mathbf{p}+Y_2\mathbf{P},\\ \mathbf{p}_2&=-\mathbf{p}+Y_1\mathbf{P},\end{aligned}\tag{A3}$$

(we recall that from the definition of Y_1, Y_2 the relation $Y_1+Y_2=1$ follows). By inserting Eqs. (A3) into (A2) and by using the following alternative forms for Y_1 and Y_2,

$$Y_{1,2}=E_{2,1}/E\pm\mathbf{p}\cdot\mathbf{P}/E(E+M),\tag{A4}$$

one easily arrives at

$$\begin{aligned}\mathbf{Q}_0=\frac{1}{2}\bigg\{&(\mathbf{q}_1E_1+\mathbf{q}_2E_2)E^{-1}\\ &+\left[(\mathbf{q}_1-\mathbf{q}_2)\times\left(\mathbf{p}-\frac{\mathbf{p}\cdot\mathbf{P}}{E(E+M)}\mathbf{P}\right)\right]\\ &\times[\mathbf{P}/M(E+M)]\bigg\}+\text{H.c.},\end{aligned}\tag{A5}$$

which is Eq. (7) of BFI.

APPENDIX 2. DERIVATION OF p AND q_0

We want to prove that $\mathbf{p}$ must be of the form (19), with Y_1, Y_2 given either by Eq. (20) or by Eq. (A4). It will be expedient to use a different symbol for the $\mathbf{p}$ we are looking for, say $\mathbf{p}'$, and write

$$\mathbf{p}'=Z_1\mathbf{p}_1-Z_2\mathbf{p}_2,\tag{A6}$$

with Z_1 and Z_2 unknown functions of $\mathbf{p}_1$, $\mathbf{p}_2$. The $\mathbf{p}$ given by (19) will be considered as a known function of $\mathbf{p}_1$, $\mathbf{p}_2$ as well as will Y_1, Y_2; our aim is to show that $\mathbf{p}'$ cannot essentially differ from $\mathbf{p}$, if $\mathbf{Q}_0$ is to be the one we already know.

We have the following differential identities:

$$\begin{aligned}\frac{\partial}{\partial p_1{}^i}&=\frac{\partial}{\partial P^i}+Z_1\frac{\partial}{\partial p'^i}+\left(\frac{\partial Z_1}{\partial p_1{}^i}p_1{}^j-\frac{\partial Z_2}{\partial p_j{}^i}p_2{}^j\right)\frac{\partial}{\partial p'^j},\\ \frac{\partial}{\partial p_2{}^i}&=\frac{\partial}{\partial P^i}-Z_2\frac{\partial}{\partial p'^i}+\left(\frac{\partial Z_1}{\partial p_2{}^i}p_1{}^j-\frac{\partial Z_2}{\partial p_2{}^i}p_2{}^j\right)\frac{\partial}{\partial p'^j}.\end{aligned}\tag{A7}$$

By subtraction we find that

$$\partial^i=M^{ij}\partial/\partial p'^j,\tag{A8}$$

where

$$\partial^i=\partial/\partial p_1{}^i-\partial/\partial p_2{}^i,\tag{A9}$$

$$M^{ij}=Z\delta^{ij}+p_1{}^j\partial^iZ_1-p_2{}^j\partial^iZ_2,\tag{A10}$$

and

$$Z=Z_1+Z_2.\tag{A11}$$

Equation (A8) can be inverted as follows:

$$\partial/\partial p'^i=N^{ij}\partial^j,\tag{A12}$$

where

$$N^{ij}=(1/Z)\delta^{ij}+p_1{}^jw_1{}^i-p_2{}^jw_2{}^i,\tag{A13}$$

with

$$\begin{aligned}w_1{}^i&=-\frac{1}{\bar{\Delta}Z}\{[Z-(p_2{}^r\partial^rZ_2)]\partial^iZ_1-(p_2{}^r\partial^rZ_1)\partial^iZ_2\},\\ w_2{}^i&=-\frac{1}{\bar{\Delta}Z}\{[Z+(p_1{}^r\partial^rZ_1)]\partial^iZ_2-(p_1{}^r\partial^rZ_2)\partial^iZ_1\},\end{aligned}\tag{A14}$$

where finally

$$\begin{aligned}\bar{\Delta}=Z^2&+Z(p_1{}^r\partial^rZ_1)-Z(p_2{}^r\partial^rZ_2)\\ &-(p_1{}^r\partial^rZ_1)(p_2{}^s\partial^sZ_2)\\ &+(p_2{}^r\partial^rZ_1)(p_1{}^s\partial^sZ_2)=\frac{1}{Z}\|M^{ij}\|.\end{aligned}\tag{A15}$$

By summing Eqs. (A7) we get

$$\partial/\partial P^i=\tfrac{1}{2}(D^i-R^{ij}N^{jk}\partial^k),\tag{A16}$$

where

$$D^i=\partial/\partial p_1{}^i+\partial/\partial p_2{}^i,\tag{A17}$$

$$R^{ij}=p_1{}^jD^iZ_1-p_2{}^jD^iZ_2.\tag{A18}$$

In the momentum representation, $Q_0{}^i$ is represented by $i\partial/\partial P^i$. It would be quite natural to identify our previous expression (A5) for $Q_0{}^i$ with the one given by (A16) after multiplication by i. This would not be quite correct, however, since the differential operator (A16) is not Hermitian. All things proceed properly if the comparison is done between (A16) and the "H.c." part of $Q_0{}^i$ in Eq. (A5) without the factor $\frac{1}{2}$.

By first comparing terms containing $q_r{}^i$ $(r=1, 2)$ in (A5) with those containing $\partial/\partial p_r{}^i$ in (A16) (i is the free index), we easily get

$$Z_r = ZY_r. \tag{A19}$$

The remaining term in Eq. (A5) is

$$\left(p^i - \frac{p^r P^r}{E(E+M)} P^i\right) \frac{P^k}{M(E+M)} (q_1{}^k - q_2{}^k), \tag{A20}$$

and so we must have

$$\frac{P^k}{M(E+M)} \left(p^i - \frac{p^r P^r}{E(E+M)} P^i\right) = (Y_1 - Y_2)\delta^{ik} - R^{ij} N^{jk}. \tag{A21}$$

It can be shown, from the definition of Y_1, that

$$\frac{P^k}{M(E+M)} \left(p^i - \frac{p^r P^r}{E(E+M)} P^i\right) = -P^k \frac{\partial Y_1}{\partial P^i}. \tag{A22}$$

The rhs of (A21) can be transformed by using (A19) and the following identity:

$$Y_2 \frac{\partial Y_1}{\partial p_1{}^i} + Y_1 \frac{\partial Y_2}{\partial p_2{}^i} = \frac{\partial Y_1}{\partial P^i} \Delta, \tag{A23}$$

where

$$\Delta = 1 + P^r \partial^r Y_1 = \bar{\Delta}_{Z=1}. \tag{A24}$$

Then Eq. (A21) becomes

$$\frac{\partial Y_1}{\partial P^i} = \frac{\Delta}{\bar{\Delta}} Z[Z - (p_2{}^r \partial^r Z)] \frac{\partial Y_1}{\partial P^i} + \frac{Z}{\bar{\Delta}} [Y_1 + (p_2{}^r \partial^r Y_1)] \times \left(Y_2 \frac{\partial Z}{\partial p_1{}^i} + Y_1 \frac{\partial Z}{\partial p_2{}^i} \right). \tag{A25}$$

This is trivially satisfied by $Z=1$; this shows that $\mathbf{p}$ is a good choice. But, are there other good choices? The answer is given by transforming Eq. (A25) into a simpler form. We make use of the identities

$$Y_2 \frac{\partial}{\partial p_1{}^i} + Y_1 \frac{\partial}{\partial p_2{}^i} = \frac{\partial}{\partial P^i} + \Delta \frac{\partial Y_1}{\partial P^i} P^r \frac{\partial}{\partial p^r}, \tag{A26}$$

$$\partial^i = \frac{\partial}{\partial p^i} + (\partial^i Y_1) P^r \frac{\partial}{\partial p^r}. \tag{A27}$$

Then we find that

$$\partial Z/\partial P^i = 0, \tag{A28}$$

i.e., Z depends on p^2 only, Q.E.D.

Let us now look for $\mathbf{q}_0$. Eq. (A12) must be used, with the form of N^{ij} resulting from putting $Z=1$ in Eq. (A13). We find that

$$w_1{}^i = -w_2{}^i = -(1/\Delta)\partial^i Y_1 = -\partial Y_1/\partial p^i, \tag{A29}$$

and, therefore,

$$q_0{}^i = \tfrac{1}{2}[q_1{}^i - q_2{}^i - (\partial Y_1/\partial p^i) P^j (q_1{}^j - q_2{}^j)] + \text{H.c.} \tag{A30}$$

By explicitly performing the derivative of Y_1 with respect to p^i, one gets

$$\mathbf{q}_0 = \frac{1}{2}\left\{ (\mathbf{q}_1 - \mathbf{q}_2) + [(\mathbf{q}_1 - \mathbf{q}_2)\cdot \mathbf{P}] \times \left[\frac{\mathbf{P}}{M(E+M)} - 4\frac{(m_1{}^2 - m_2{}^2)E + M\mathbf{p}\cdot\mathbf{P}}{E[M^4 - (m_1{}^2 - m_2{}^2)^2]} \mathbf{p} \right] \right\} + \text{H.c.}, \tag{A31}$$

which is Eq. (8) of BFI. It is not difficult to assure oneself that $\mathbf{q}_0$ satisfies all the usual commutation relations, and this completes the problem.

APPENDIX 3. THE FORM OF $a_1{}^{ij}$

Since $a_1{}^{ij}$ represents a rotation, it can be written in the form

$$a_1{}^{ij} = \delta^{ij} \cos\omega + u^i u^j (1 - \cos\omega) + \epsilon^{ijk} u^k \sin\omega, \tag{A32}$$

where the unit vector u^i defines the axis of the rotation, and ω is the rotation angle. The axis of the rotation is oriented in the direction $\mathbf{p}\times\mathbf{P}$; so

$$\mathbf{u} = \mathbf{p}\times\mathbf{P}/|\mathbf{p}\times\mathbf{P}|. \tag{A33}$$

As to ω, we have from Eq. (32) of reference 11

$$\tan(\omega/2) = \frac{|\mathbf{p}\times\mathbf{P}|}{(E+M)(E_1{}' + m_1) + \mathbf{p}\cdot\mathbf{P}}. \tag{A34}$$