we find

$$\gamma_4 = \gamma_{40} e^{-\alpha_{4n} P/kT}. \tag{D5}$$

A value of $\gamma_{40} = 0.16$ is calculated, again in error by a factor of 2 due to errors in g_4 and $(E_c - E_4)$.

Now, if we assume $f_4(\gamma_4) = 1$ at 45°K and 5000 kg cm^{-2}, we can find a first approximation for α_{4n} using Eq. (22), neglecting terms in μ_n and A_n. In this way $\alpha_{4n} = 2.1 \times 10^{-6}$ eV kg^{-1} cm^2. Substituting this in Eq. (D5) at $P = 5000$ kg cm^{-2} we find $\gamma_4 = \gamma_{40} \times 0.08 \sim 0.013$.

Hence, at this pressure and temperature, $f_4(\gamma_4) = 1.01 \pm 0.01$ where the 0.01 error represents an error of a factor of 2 in γ_{40}.

At 45°K and atmospheric pressure, we find that

$$kT d \ln\rho/dP = 1.6 \times 10^{-6} \text{ eV kg}^{-1} \text{ cm}^2,$$

and using

$$f_4(\gamma_4) = 1.13 \pm 10\%,$$

we get

$$\alpha_{4n} = 1.8 \times 10^{-6} \pm 10\% \text{ eV kg}^{-1} \text{ cm}^2,$$

which is in agreement with the value found near 5000 kg cm^{-2}. Thus this approach is reasonable.

(2) At 45°K we find

$$d \ln\rho/dP \sim 10^{-8} \text{ kg}^{-1} \text{ cm}^2.$$

We have seen in Appendix C that

$$d \ln A_n/dP \sim 6 \times 10^{-6} \text{ kg}^{-1} \text{ cm}^2.$$

Thus, the density-of-states factor represents a negligible correction in Eq. (22).

(3) At 45°K the carriers are scattered both by lattice vibrations and ionized impurities. In Appendix C we have seen that the change in lattice scattering mobility with pressure is of the same magnitude as the change in the density-of-states factor and is therefore negligible.

The pressure coefficient of the ionized impurity scattering mobility can be expressed as

$$\frac{d \ln\mu_I}{dP} = \frac{2d \ln K}{dP} + \frac{d \ln}{dP}\left(\frac{m_2}{m_1^{1/2}}\right).$$

However, since the changes in K and in the masses are also of the order of 10^{-6} kg^{-1} cm^2, the pressure coefficient of this scattering mechanism can also be neglected.

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Force on a Moving Dislocation*

ALAN N. STROH†
Laboratory for Insulation Research, Massachusetts, Institute of Technology, Cambridge, Massachusetts
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A Lagrangian formulation is used to discuss the nature of the force on a moving dislocation. Whether or not a Lorentz force appears depends on the definition of force adopted, but it is shown that this force can give rise to no physical effects; a definition which does not introduce it is therefore recommended. The force is given by the usual static expression ($F = \sigma b$) and is independent of the motion of the dislocation.

1. INTRODUCTION

THE nature of the force on a moving dislocation, and especially the existence of the so-called Lorentz force, is still a matter of discussion.[1] We shall argue here that the difficulties associated with the force concept arise because no clear definition of force applicable to a moving dislocation has been given. A definition will be proposed which leads to an unambiguous expression for the force and is consistent with its use in other fields. Before starting on the constructive part of this program, however, we must consider some of the complications which arise when this point is neglected.

The nature of the force on a dislocation at rest has been fully discussed by Eshelby,[2] who has particularly emphasized the need for thorough treatment. His conclusions are that (a) the force should be defined as the derivative of the energy with respect to dislocation displacement; from this it follows that (b) the force in the slip plane is just σb per unit length of the dislocation, where σ is the resolved shear stress. It seems to have been accepted quite uncritically that both statements (a) and (b) apply also in the dynamical case, without realizing that here they are in fact inconsistent. (The question of a Lorentz force does not arise here as it acts normally to the slip plane if it is present at all.) To illustrate this, consider two parallel screw dislocations P and Q in an isotropic medium, P at rest and Q moving with a uniform velocity v in the direction PQ. At any instant the interaction energy of P and Q can depend only on their distance apart and not on their absolute positions; hence, if statement (a) is adopted, the forces each dislocation exerts on the other are equal and opposite. On the other hand, the stress produced by the

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† Deceased.
[1] F. R. N. Nabarro, Phil. Mag. **6**, 1261 (1961).
[2] J. D. Eshelby, Phil. Trans. Roy. Soc. London **A244**, 87 (1951); *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc., New York, 1956), Vol. 3, p. 79.

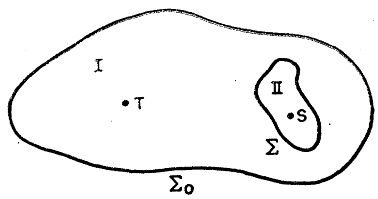

FIG. 1. Elastic solid with singularities S and T.

moving dislocation Q is less than that due to P by a factor $(1-v^2/c^2)^{1/2}$, where c is the transverse velocity of sound[3]; (b) then implies that the forces on the dislocations are unequal. The stress-dependent force as well as the Lorentz force thus needs reconsideration. In the treatment to be given here, (a) will be modified by replacing the energy with the Lagrangian; it will then be found that (b) remains true, and also that no Lorentz force need be introduced.

2. DEFINITION OF FORCE

Consider a mechanical system whose state at any time t may be specified by a finite or infinite number of parameters, of which we take ξ to be a typical representative. The dependence of ξ on time can be found from Hamilton's principle, which tells us to form the Lagrangian $L=L(\xi,\dot{\xi},t)$ and to consider the integral $\int_{t_1}^{t_2} Ldt$; the actual path taken by ξ then makes the integral stationary for all small variations $\delta\xi$ which vanish at times t_1 and t_2. Further, if the system obeys classical mechanics, then $L=T-V$, where T is the kinetic and V the potential energy.

An elastic solid containing a dislocation is just such a mechanical system where the parameters ξ specifying the position of the dislocation are part of the set of parameters specifying the state of the whole system. Let L^S be the Lagrangian of this system. Now consider a similar body in which the dislocation S is absent but some other elastic singularity T is present (this includes the possibility that an external stress may be applied); let L^T be the new Lagrangian. Finally suppose that both S and T are present; the Lagrangian is now not simply L^S+L^T but, in general, contains additional terms; we therefore write

$$L=L^S+L^T+L^i. \tag{1}$$

Because of the presence of the terms L^i, the motion of S given by the Lagrangian L will not be the same as that given by L^S; we may describe the difference by saying that T exerts a force on S. For this force we need therefore not consider the whole Lagrangian L but only the part L^i; this is fortunate because L^i, as we shall show, can be evaluated precisely in linear elastic theory, a circumstance which is not necessarily true for L. Clearly

$$L^i=T^i-V^i \tag{2}$$

in an obvious notation.

[3] F. C. Frank, Proc. Phys. Soc. (London) **A62**, 131 (1949).

If ξ is one of the parameters specifying the position of the dislocation S (and therefore does not occur in L^T), and we may write for small variations $\delta\xi$,

$$\delta\int_{t_1}^{t_2} L^i dt=\int_{t_1}^{t_2} F_\xi\delta\xi dt, \tag{3}$$

then the condition $\delta\int Ldt=0$ gives, on using the Euler-Lagrange equations for the variation of L^S,

$$\int_{t_1}^{t_2}\left(\frac{\partial L^S}{\partial\xi}-\frac{d}{dt}\frac{\partial L^S}{\partial\dot{\xi}}+F_\xi\right)\delta\xi dt=0. \tag{4}$$

Since $\partial L^S/\partial\dot{\xi}$ is the momentum conjugate to ξ of the isolated singularity S, Eq. (4) shows that F_ξ is a force. Hence, Eq. (3) may be taken as defining the force F_ξ, tending to increase ξ, which T exerts on S. An alternative definition will be considered in Sec. 6.

3. THE LAGRANGIAN

The material is assumed to be linear, so that the resultant stresses and strains at any point are the sum of those due to S and T separately; i.e., $\sigma_{ij}=\sigma_{ij}{}^S+\sigma_{ij}{}^T$, $e_{ij}=e_{ij}{}^S+e_{ij}{}^T$. The elastic energy density is $\frac{1}{2}\sigma_{ij}e_{ij}$ giving

$$V^i=\frac{1}{2}\int(\sigma_{ij}{}^Se_{ij}{}^T+\sigma_{ij}{}^Te_{ij}{}^S)d^3x, \tag{5}$$

the integration extending over the whole body. The condition of linearity also implies that the two terms in the integrand are equal, since each is equal to $c_{ijkl}e_{ij}{}^Se_{kl}{}^T$, where c_{ijkl} is the elastic modulus.

If an elastic displacement u_i can be defined, we may derive the strains from it as

$$e_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i}). \tag{6}$$

However, as Eshelby[2] and Kröner[4] have emphasized, an elastic singularity may be characterized as a region in which the compatibility equations are not satisfied and therefore where an elastic displacement cannot be defined. Let us assume that the body can be divided into two regions, I and II separated by a closed surface Σ such that S lies wholly in II and T wholly in I; then u^S exists in I, u^T in II (Fig. 1). From Eqs. (5) and (6) we have

$$\begin{aligned}V^i&=\int_{\mathrm{I}}\sigma_{ij}{}^Tu_{i,j}{}^Sd^3x+\int_{\mathrm{II}}\sigma_{ij}{}^Su_{i,j}{}^Td^3x\\&=\int_{\Sigma_o}\sigma_{ij}{}^Tu_i{}^SdS_j-\int_{\Sigma}\sigma_{ij}{}^Tu_i{}^SdS_j\\&\qquad-\int_{\mathrm{I}}\sigma_{ij,j}{}^Tu_i{}^Sd^3x+\int_{\mathrm{II}}\sigma_{ij}{}^Su_{i,j}{}^Td^3x,\end{aligned} \tag{7}$$

[4] E. Kröner, Z. angew. Phys. **7**, 249 (1959); *Kontinuumstheorie der Versetzungen und Eigenspannungen* (Springer-Verlag, Berlin, 1958).

where Σ_0 is the surface of the body and the positive direction of the normal to Σ is taken to be from region II to I. The integral over Σ_0 represents the work done by the surface tractions of T in producing the displacements of S; that is, it represents a flow of energy into the body from the mechanism that maintains the surface tractions when S is formed in the presence of T. Since the physical behavior of the body must be independent of the particular mechanism by which the surface tractions are maintained, we may suppose that it consists of a conservative system of small inertia (and so negligible kinetic energy); the energy entering the body through Σ_0 must then be equal to the decrease in potential energy of the external mechanism. Hence, if we take V not to be just the potential energy of the body alone but that of the body together with this external mechanism, we may simply omit the integral over Σ_0 in Eq. (7).

Let S be a dislocation lying along a closed curve Γ, and let C be a surface bounded by Γ. Then u_i^S may be taken to be single valued but discontinuous on crossing C, where it will change by a constant amount b_i, the Burgers vector. For Σ we take a tube σ of small radius a enclosing Γ together with surfaces lying close to and on either side of C. Since on σ, u_i is of order $\ln a$, we have

$$\int_\Sigma \sigma_{ij}{}^T u_i{}^S dS_j = -b_i \int_C \sigma_{ij}{}^T dS_j + O(a \ln a). \tag{8}$$

Also

$$\int_{\mathrm{II}} \sigma_{ij}{}^S u_{i,j}{}^T d^3x = O(a), \tag{9}$$

since $\sigma_{ij}{}^S$ behaves like $1/r$ near the dislocation.

Inserting these results into Eq. (7) and letting a tend to zero, we have

$$V^i = b_i \int_C \sigma_{ij}{}^T dS_j - \int \sigma_{ij,j}{}^T u_i{}^S d^3x, \tag{10}$$

the volume integral being taken over the entire body which is cut at the surface C. Here the surface integral is familiar from the static case and represents the work done against the stress system T in giving the two sides of C a relative displacement b_i. The second integral also has a simple physical interpretation if we recall that, by the equations of motion, $-\sigma_{ij,j}{}^T$ is equal to the body forces $X_i{}^T$ acting at any point (the most important of which are the inertial forces $-\rho \ddot{u}_i{}^T$); the integral is thus $\int X_i{}^T u_i{}^S d^3x$ and represents the work done against the body forces of T in forming the displacements of S.

If it were possible to define the material velocities $v_i{}^S$ and $v_i{}^T$ everywhere, we should have for the kinetic energy

$$T^i = \int \rho v_i{}^S v_i{}^T d^3x; \tag{11}$$

but this needs examination since the velocities may not be well defined at a singularity. We again consider the two regions I and II and note that, since $v_i{}^S$ is single valued, region II can be taken as the interior of the tube σ. Near the dislocation, $u_i{}^S$ varies as $\ln r$ and $v_i{}^S$ as $1/r$; hence the contribution to Eq. (11) from region II will be small of order a. Thus Eq. (11) differs by an arbitrarily small amount from the same integral taken over region I alone, in which $v_i{}^S$ is well defined and equal to $\dot{u}_i{}^S$. In general, $v_i{}^T$ will not be defined everywhere in region I, but the following cases, which give no difficulty are physically important and justify our use of Eq. (11): (1) T consists of an external stress only so that $u_i{}^T$ and $v_i{}^T = \dot{u}_i{}^T$ exist everywhere, (2) T is a stationary singularity for which $v_i{}^T = 0$, and (3) $v_i{}^T$ is well defined except in a region making an arbitrarily small contribution to Eq. (11), when this region may be omitted from the integration. In particular we see from the discussion just given for S that this last will be the case if T is a second moving dislocation.

We rewrite Eq. (11) as

$$T^i = \int \rho \dot{u}_i{}^S v_i{}^T d^3x. \tag{12}$$

Also, since the equations of motion for the system T are

$$\sigma_{ij,j}{}^T = \rho \dot{v}_i{}^T \tag{13}$$

if no body forces are present, Eq. (10) becomes

$$V^i = b_i \int_C \sigma_{ij}{}^T dS_j - \int \rho u_i{}^S \dot{v}_i{}^T d^3x. \tag{14}$$

(If T is a moving dislocation, $\dot{v}_i{}^T$ varies as r^{-2} near the dislocation line and it appears that the volume integral in Eq. (14) might diverge logarithmically; this is not so because V^i depends symmetrically on S and T, and we have seen that S gives rise to finite terms only.) According to Eq. (2), we then have the Lagrangian

$$L^i = -b_i \int_C \sigma_{ij}{}^T dS_j + \int \rho \frac{d}{dt}(u_i{}^S v_i{}^T) d^3x. \tag{15}$$

Here the time derivative has been written as d/dt since, as a little consideration will show, it denotes differentiation following the motion of material particle; i.e., a Lagrangian description of the motion is being used.

4. EVALUATION OF THE FORCE

We must now, according to Eq. (3), find the variation in L_i when the dislocation is given a small displacement $\delta\xi$ from its actual position (curve Γ) to a neighboring position (curve Γ'); let C' be a surface (bounded by Γ') corresponding to C. Consider first the contribution to δL^i from the surface integral in Eq. (15). If the positive direction of the normal to C is reversed and the strip between Γ and Γ' added, we obtain a closed surface $C+C'+\Gamma\Gamma'$ with its normal directed outward every-

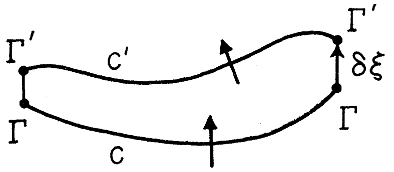

FIG. 2. Displacement of the dislocation from Γ to Γ'.

where (Fig. 2). Hence,

$$-\delta b_i\int_C \sigma_{ij}{}^T dS_j=-b_i\int_{C+C'+\Gamma\Gamma'}\sigma_{ij}{}^T dS_j+b_i\int_{\Gamma\Gamma'}\sigma_{ij}{}^T dS_j$$
$$=-b_i\int\sigma_{ij,j}{}^T d^3x+b_i\int_\Gamma \sigma_{ij}{}^T\epsilon_{jkl}ds_k\delta\xi_l; \qquad (16)$$

the volume integral here, which is to be taken over the interior of $C+C'+\Gamma\Gamma'$, may, by Eq. (13), be written as $-b_i\int\rho\dot{v}_i{}^T d^3x$. If this is added to the variation of the second integral in Eq. (15), the result is the same as if the surface C' were deformed to coincide with $C+\Gamma\Gamma'$. Since the discontinuities across C will then subtract out, we may write

$$\delta L^i=\delta L_1{}^i+\delta L_2{}^i, \qquad (17)$$

where

$$\delta L_1{}^i=b_i\int_\Gamma \sigma_{ij}{}^T\epsilon_{jkl}ds_k\delta\xi_l, \qquad (18)$$

and

$$\delta L_2{}^i=\int\rho\frac{d}{dt}(\delta u_i{}^S v_i{}^T)d^3x_j. \qquad (19)$$

Here $\delta u_i{}^S$ denotes the difference in the displacements due to two dislocations lying along Γ' and Γ and is to be taken single valued everywhere and continuous except on $\Gamma\Gamma'$.

From Eqs. (3) and (18) we obtain a force

$$F=b_i\sigma_{ij}{}^T\epsilon_{jkl}t_k, \qquad (20)$$

per unit length of the dislocation, t_k being a unit vector along the dislocation line. Equation (20) is the usual expression for the force on a dislocation obtained in the static case.[2] As we shall find that $\delta L_2{}^i$ gives rise to no force, the force on a dislocation is thus independent of its motion.

Let σ be a tube of small radius a enclosing Γ and moving with the dislocation velocity $d\xi/dt$ so that Γ remains inside σ throughout the motion. Having thus chosen σ we consider only variations $\delta\xi$ so small that Γ' also lies within σ; then $\delta u_i{}^S$ is bounded, single valued, and continuous outside σ. We first consider the contribution to Eq. (19) from the region outside σ. Noting that ρd^3x, being the mass of a fixed element of the material, is invariant, we have for this region

$$\int\rho\frac{d}{dt}(\delta u_i{}^S v_i{}^T)d^3x=\frac{d}{dt}\int\rho\delta u_i{}^S v_i{}^T d^3x$$
$$+\int_\sigma\rho\delta u_i{}^S v_i{}^T\dot{\xi}_j dS_j, \qquad (21)$$

where the velocity $\dot{\xi}_j$ is measured relative to the material so that the surface integral gives the variation arising from the change in the region of integration. Since, according to Eq. (3), δL^i must be integrated between times t_1 and t_2 at which $\delta\xi$ vanishes, we may omit the term $d/dt\int(\cdots)d^3x$. Then

$$\delta L_2{}^i=\int_\sigma\rho\delta u_i{}^S v_i{}^T\dot{\xi}_j dS_j+\int\rho\frac{d}{dt}(\delta u_i{}^S v_i{}^T)d^3x, \qquad (22)$$

where the volume integral is taken over the interior of σ. Thus, $\delta L_2{}^i$ is seen to depend on conditions near the dislocation line only. Also

$$\int\rho\frac{d}{dt}(\delta u_i{}^S v_i{}^T)d^3x$$
$$=\int\rho\frac{d}{dt}(\delta u_i{}^S)v_i{}^T d^3x+\int\rho\delta u_i{}^S\dot{v}_i{}^T d^3x, \qquad (23)$$

and since near the dislocation $u_i{}^S$ is of order $\log r$ so that $\delta u_i{}^S$ cannot be of order greater than $1/r$, the second integral is small of order a. As $\delta L_2{}^i$ cannot depend on the radius of the tube σ, we may let a tend to zero, giving

$$\delta L_2{}^i=\int_\sigma\rho\delta u_i{}^S v_i{}^T\dot{\xi}_j dS_j+\int\rho v_i{}^T\frac{d}{dt}(\delta u_i{}^S)d^3x. \qquad (24)$$

To find the force acting at a point P of the dislocation line Γ, we need consider variations $\delta\xi$ which differ from zero near P only; then $\delta u_i{}^S$ will be small except near P. Also, since the force cannot depend on the observer's frame of reference, we may, by superimposing a uniform velocity, take $v_i{}^T$ to be zero at point P; by continuity there will be a neighborhood of P in which $v_i{}^T$ is arbitrarily small. Then, by suitably restricting the range in which $\delta\xi$ differs from zero, we can make $\delta u_i{}^S$ as small as we like outside this neighborhood. Hence we can make $\delta L_2{}^i$ arbitrarily small compared with $\int_\Gamma\delta\xi ds$; Eq. (3) then shows that the corresponding force at P must be zero. Equation (20) thus gives the total force per unit length on the dislocation.

5. IMAGE FORCES

The motion of a dislocation in a finite body will generally differ from that of the same dislocation in an infinite medium; the difference may be described, in analogy with electrostatics, by speaking of an image force acting on the dislocation.[2] The treatment of this force requires certain modifications of Secs. 3 and 4. We denote the stresses and displacements of the dislocation S when situated in an infinite medium by $\sigma_{ij}{}^\infty$ and $u_i{}^\infty$, and their values in the finite body by σ_{ij}, u_i. The image stresses and displacements are then defined by

$$\sigma_{ij}=\sigma_{ij}{}^\infty+\sigma_{ij}{}^I, \quad\text{and}\quad u_i=u_i{}^\infty+u_i{}^I. \qquad (25)$$

The magnitude of the image stress field is to be deter-

mined so that the surface Σ_0 of the body (see Fig. 1) is stressfree; that is on Σ_0

$$\sigma_{ij}{}^{I} dS_j = -\sigma_{ij}{}^{\infty} dS_j. \tag{26}$$

Denoting by III the region outside Σ_0, we have

$$L^{\infty} = T^{\infty} - V^{\infty} = \tfrac{1}{2}\int_{\mathrm{I+II+III}} \rho \dot{u}_i{}^{\infty 2}\, d^3x - \tfrac{1}{2}\int_{\mathrm{I+II+III}} \sigma_{ij}{}^{\infty} u_{i,j}{}^{\infty} d^3x. \tag{27}$$

As is well known, these integrals do not converge for a dislocation lying along an infinite straight line; to avoid this difficulty we shall consider only dislocations lying along closed curves. To simplify the discussion we shall also assume that the dislocation line Γ lies entirely within Σ_0 (i.e., the dislocation does not meet the free surface of the body); then $u_i{}^{\infty}$ has no singularities outside Σ_0 and $u_i{}^{I}$ none within Σ_0. Since

$$L = \tfrac{1}{2}\int_{\mathrm{I+II}} \rho(\dot{u}_i{}^{\infty} + \dot{u}_i{}^{\mathrm{I}})^2\, d^3x - \tfrac{1}{2}\int_{\mathrm{I+II}} (\sigma_{ij}{}^{\infty} + \sigma_{ij}{}^{\mathrm{I}})(u_{i,j}{}^{\infty} + u_{i,j}{}^{\mathrm{I}}) d^3x, \tag{28}$$

$$\begin{aligned} L^{\mathrm{I}} &= L - L^{\infty} \\ &= \int_{\mathrm{I+II}} \rho \dot{u}_i{}^{\infty} \dot{u}_i{}^{\mathrm{I}} d^3x - \tfrac{1}{2}\int_{\mathrm{I+II}} (\sigma_{ij}{}^{\infty} u_{i,j}{}^{\mathrm{I}} + \sigma_{ij}{}^{I} u_{i,j}{}^{\infty}) d^3x \\ &\quad + \tfrac{1}{2}\int_{\mathrm{I+II}} (\rho \dot{u}_i{}^{\mathrm{I}2} - \sigma_{ij}{}^{\mathrm{I}} u_{i,j}{}^{\mathrm{I}}) d^3x \\ &\quad - \tfrac{1}{2}\int_{\mathrm{III}} (\rho \dot{u}_i{}^{\infty 2} - \sigma_{ij}{}^{\infty} u_{i,j}{}^{\infty}) d^3x. \end{aligned} \tag{29}$$

On using Green's lemma and Eq. (13), the third integral becomes

$$\tfrac{1}{2}\int_{\mathrm{I+II}} \rho \frac{d}{dt}(\dot{u}_i{}^{\mathrm{I}} u_i{}^{\mathrm{I}}) d^3x - \tfrac{1}{2}\int_{\Sigma_0} \sigma_{ij}{}^{\mathrm{I}} u_i{}^{\mathrm{I}} dS = \tfrac{1}{2}\frac{d}{dt}\int_{\mathrm{I+II}} \rho \dot{u}_i{}^{\mathrm{I}} u_i{}^{\mathrm{I}} d^3x - \tfrac{1}{2}\int_{\Sigma_0} \sigma_{ij}{}^{\mathrm{I}} u_i{}^{\mathrm{I}} dS_j; \tag{30}$$

similarly, the fourth integral is

$$-\tfrac{1}{2}\frac{d}{dt}\int_{\mathrm{III}} \rho \dot{u}_i{}^{\infty} u_i{}^{\infty} d^3x - \tfrac{1}{2}\int_{\Sigma_0} \sigma_{ij}{}^{\infty} u_i{}^{\infty} dS_j. \tag{31}$$

Since the terms $(d/dt)\int(\cdots)d^3x$ contribute nothing to $\delta\int L dt$, we may omit them and, using Eq. (26), take

$$L^{\mathrm{I}} = \int_{\mathrm{I+II}} \rho \dot{u}_i{}^{\infty} \dot{u}_i{}^{\mathrm{I}} d^3x - \tfrac{1}{2}\int_{\mathrm{I+II}} (\sigma_{ij}{}^{\infty} u_{i,j}{}^{\mathrm{I}} + \sigma_{ij}{}^{\mathrm{I}} u_{i,j}{}^{\infty}) d^3x + \tfrac{1}{2}\int_{\Sigma_0} (\sigma_{ij}{}^{\mathrm{I}} u_i{}^{\infty} + \sigma_{ij}{}^{\infty} u_i{}^{\mathrm{I}}) dS_j. \tag{32}$$

The expression (32) for L shows that it depends symmetrically and bilinearly on the quantities labeled by I and ∞. Also, according to Eq. (26), the I quantities are proportional to the ∞ quantities; hence, the actual variation L^{I} will be just twice the variation we should obtain if we treated the I quantities as constant during the variation: a result we shall require later.

Equation (32) may now be reduced to a simpler form though one in which the symmetry is not apparent. First we note that the integrals over region II become vanishingly small as the volume of this region tends to zero, and therefore may be omitted. Using Green's lemma, Eq. (13), and the relation

$$\sigma_{ij}{}^{\infty} u_{i,j}{}^{\mathrm{I}} = \sigma_{ij}{}^{I} u_{i,j}{}^{\infty}, \tag{33}$$

we have

$$L^{\mathrm{I}} = \int_{\mathrm{I}} \rho \frac{d}{dt}(u_i{}^{\infty} \dot{u}_i{}^{\mathrm{I}}) d^3x + \int_{\Sigma} \sigma_{ij}{}^{\mathrm{I}} u_i{}^{\infty} dS_j + \tfrac{1}{2}\int_{\Sigma_0} (\sigma_{ij}{}^{\infty} u_i{}^{\mathrm{I}} - \sigma_{ij}{}^{\mathrm{I}} u_i{}^{\infty}) dS_j; \tag{34}$$

and again using Green's lemma and Eqs. (13) and (33), the integral over Σ_0 is

$$\int_{\Sigma_0} (\sigma_{ij}{}^{\infty} u_i{}^{\mathrm{I}} - \sigma_{ij}{}^{\mathrm{I}} u_i{}^{\infty}) dS_j = \int_{\Sigma} (\sigma_{ij}{}^{\infty} u_i{}^{\mathrm{I}} - \sigma_{ij}{}^{\mathrm{I}} u_i{}^{\infty}) dS_j + \int_{\mathrm{I}} \rho \frac{d}{dt}(\dot{u}_i{}^{\infty} u_i{}^{\mathrm{I}} - \dot{u}_i{}^{\mathrm{I}} u_i{}^{\infty}) d^3x. \tag{35}$$

Hence,

$$L^{\mathrm{I}} = \tfrac{1}{2}\int_{\mathrm{I}} \rho \frac{d}{dt}(u_i{}^{\infty} \dot{u}_i{}^{\mathrm{I}}) d^3x + \tfrac{1}{2}\int_{\Sigma} \sigma_{ij}{}^{\mathrm{I}} u_i{}^{\infty} dS_j + \tfrac{1}{2}\int_{\mathrm{I}} \rho \frac{d}{dt}(\dot{u}_i{}^{\infty} u_i{}^{\mathrm{I}}) d^3x + \tfrac{1}{2}\int_{\Sigma} \sigma_{ij}{}^{\infty} u_i{}^{\mathrm{I}} dS_j. \tag{36}$$

As in Sec. 3, the second integral reduces to an integral over a surface C bounded by the dislocation line Γ. In the last two integrals, since $\dot{u}_i{}^{\infty}$ and $\sigma_{ij}{}^{\infty}$ are single valued, the surface Σ may be contracted into the tube about σ. Then these integrals are

$$\tfrac{1}{2}\frac{d}{dt}\int_{\mathrm{I}} \rho \dot{u}_i{}^{\infty} u_i{}^{\mathrm{I}} d^3x + \tfrac{1}{2}\int_{\sigma} \rho \dot{u}_i{}^{\infty} u_i{}^{\mathrm{I}} \xi_j dS_j + \tfrac{1}{2}\int_{\sigma} \sigma_{ij}{}^{\infty} u_i{}^{\mathrm{I}} dS_j, \tag{37}$$

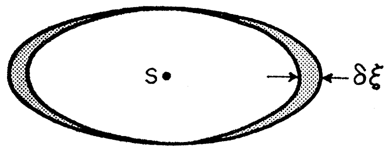

FIG. 3. Displacement of the tube σ about the dislocation S.

where $\dot{\xi}_j$ is the velocity of the surface σ; $(d/dt)\int(\cdots)d^3x$, makes no contribution to $\delta\int L^{\mathrm{I}}dt$. We may take the radius a of σ so small that the variation of u_i^{I} across σ is negligible. Then, breaking σ up into elements of sufficiently small length, the integrals over σ are

$$\tfrac{1}{2}\sum u_i^{\mathrm{I}}\int(\rho\dot{u}_i^{\infty}\dot{\xi}_j+\sigma_{ij}^{\infty})dS_j; \tag{38}$$

the integral here represents the flow of momentum into each section of σ, and this must be zero since it is easily seen that the momentum inside σ tends to zero with the radius a. Thus

$$L^{\mathrm{I}}=-\tfrac{1}{2}b_i\int_C\sigma_{ij}^{\mathrm{I}}dS_j+\tfrac{1}{2}\int\rho\frac{d}{dt}(u_i^{\infty}v_i^{\mathrm{I}})d^3x. \tag{39}$$

This differs from Eq. (15) by a factor of 1/2. However, when we form the variation, σ_{ij}^T is constant but σ_{ij}^I is not and this, as we have seen, gives a further factor of 2 in the variation of Eq. (39). Hence, we obtain the same expression for the force as before [Eq. (20)], namely,

$$F^{\mathrm{I}}=b_i\sigma_{ij}^I\epsilon_{jkl}t_k. \tag{40}$$

6. THE LORENTZ FORCE

Our definition has led to a force that is independent of the dislocation velocity, and we have found no terms representing a Lorentz force. However, if we choose a different but equally legitimate definition we shall find a Lorentz force arising. Equation (3) is equivalent to defining the force as

$$F_\xi=\frac{\partial L^i}{\partial\xi}-\frac{d}{dt}\frac{\partial L^i}{\partial\dot{\xi}}; \tag{41}$$

we might instead have defined the force to be equal to the term $\partial L^i/\partial\xi$ only, and then $\partial L^i/\partial\dot{\xi}$ would represent an additional contribution to the momentum of the dislocation. This, of course, would not effect our deductions as to the motion of the dislocation. It is nevertheless interesting to evaluate the force according to this definition. We must still consider the variation in L^i for small displacements $\delta\xi$, but now the dislocation velocity $\dot{\xi}$ is to be treated as an independent variable. Equation (17) remains valid and δL_1^i leads to the force (20) as before; we also have the expression (24) for δL_2^i. Taking the radius a of the tube σ to be small compared with the radius of curvature of the dislocation line Γ, we may substitute in the first integral in Eq. (24)

$$\delta u_i^S=(\partial u_i^S/\partial\xi_k)\delta\xi_k=-u_{i,k}^S\delta\xi_k. \tag{42}$$

Assuming also $\delta\xi$ to be small compared with the distance in which the change in v^T is appreciable, we may in the volume integral keep the dislocation fixed and give the region of integration, the interior of the tube σ, an equal and opposite displacement $-\delta\xi$. We have then to evaluate the integral over the shaded area in Fig. 3, obtaining for this term

$$-\int\rho v_i^T\frac{du_i^S}{dt}\delta\xi_jdS_j=\int\rho v_i^Tu_{i,k}^S\dot{\xi}_k\delta\xi_jdS_j, \tag{43}$$

since with $\dot{\xi}_j$ constant, du_i^S/dt can be due only to the change in position of the dislocation. Hence

$$\begin{aligned}\delta L_2^i&=\int(\dot{\xi}_k\delta\xi_j-\dot{\xi}_j\delta\xi_k)\rho v_i^Tu_{i,k}^SdS_j\\&=\int_\sigma\rho v_i^Tu_{i,k}^SdS_j\epsilon_{jkl}\epsilon_{lmn}\dot{\xi}_n\delta\xi_m.\end{aligned} \tag{44}$$

Neglecting terms small of order $(a\ \ln a)$ this can be written as

$$\delta L_2^i=\int_\sigma\epsilon_{jkl}\frac{\partial}{\partial x_k}(\rho v_i^Tu_i^S\epsilon_{lmn}\dot{\xi}_n\delta\xi_m)dS_j, \tag{45}$$

where to make u_i^S single valued we introduce a cut along σ, this cut being a curve differing from Γ only by a displacement of order a. By Stokes's theorem the surface integral can be replaced by an integral along its boundary, i.e., the cut; if this is done, the cut will be traversed twice in opposite senses, the corresponding values of u_i^S differing by b_i. Hence,

$$\delta L_2^i=b_i\int_\Gamma\rho v_i^T\epsilon_{jmn}\dot{\xi}_n\delta\xi_mdS_j, \tag{46}$$

or there is a force,

$$F_l=\rho b_iv_i^T\dot{\xi}_j\epsilon_{jkl}t_k \tag{47}$$

per unit length of the dislocation line, where $\dot{\xi}_j$ is the velocity of the dislocation relative to the material. This is the Lorentz force acting perpendicular to the direction of the dislocation motion, and with the present definition should be added to Eq. (20) to give the resultant force. In addition, there will be a momentum whose rate of change is also given by Eq. (47); we shall not

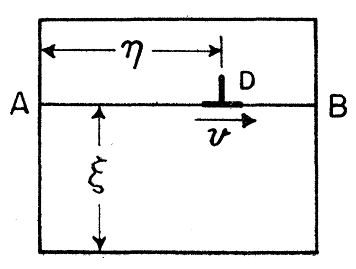

FIG. 4. A dislocation moving across the solid.

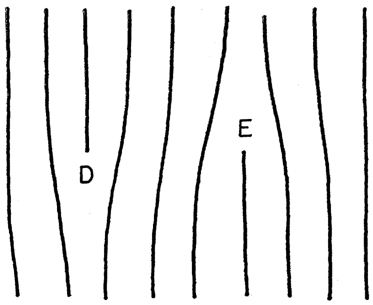

FIG. 5. Lattice deformation near two dislocations of opposite sign.

attempt to evaluate this momentum in the general case though a special case will be considered shortly.

Eshelby[2] and Nabarro[1] have considered the Lorentz force in a simple case (straight dislocations moving with constant velocity). As definition of force they take $\partial T/\partial\xi$, but since in their case the potential energy V is zero, this is equivalent to defining force as $\partial L/\partial\xi$; thus they find a Lorentz force. Their argument is as follows: Consider a slip plane AB crossing a unit cube of the material (Fig. 4), the slip plane containing n dislocations moving with velocity V. The relative velocity of the material on either side of the slip plane is nbV, so that if the mean material velocity in the slip plane is v, the two parts of the body have velocities $(v+\frac{1}{2}nbV)$ and $(v-\frac{1}{2}nbV)$. Hence, the kinetic energy is

$$T=\tfrac{1}{2}\rho\xi(v+\tfrac{1}{2}nbV)^2+\tfrac{1}{2}\rho(1-\xi)(v-\tfrac{1}{2}nbV)^2, \qquad (48)$$

giving the force tending to increase ξ as $\partial T/\partial\xi=n\rho bvV$. Taking the force per dislocation, we obtain an expression for Lorentz force agreeing with Eq. (47).

But terms in $\dot{\xi}$ have been omitted from T so that the additional momentum is missed. (True, $\dot{\xi}=0$, but this value must be substituted after and not before the differentiation.) It is simplest now to consider the dislocations individually. Suppose a dislocation D, a distance η from the edge, climbs a distance a from one atomic plane to the next. If no diffusion occurs and the dislocation does not leave a trail of defects behind it, the atomic plane of atoms AD must be given a displacement b as the dislocation climbs,[5] (see Fig. 5) and so the velocity of this plane will not be v but $(v+b\dot{\xi}/a)$. Hence, the strip AD gives an additional contribution to the kinetic energy [Eq. (48)] of amount $\frac{1}{2}\rho\eta a[(v+b\dot{\xi}/a)^2-v^2]$, and we obtain a corresponding momentum $(\partial T/\partial\dot{\xi})_{\delta\xi/dt=0}=\rho bv\eta$ per dislocation. Since $\dot{\eta}=V$, we see that the rate of change of this momentum is just equal to the Lorentz force.

5. DISCUSSION

We have seen that two different definitions of force can be given: either Eq. (3) [or equivalently Eq. (41)] may be used, or force may be defined as $\partial L^i/\partial\xi$. These two definitions are equally legitimate and must lead to the same physical results in every case; in particular neither can give rise to any effects dependent on the absolute velocity of the body. On the other hand, the second definition, $F_\xi=\partial L^i/\partial\xi$, introduces the Lorentz force which apparently depends on the absolute velocity and is cancelled by the rate of change of the additional momentum which must now be introduced; there appears to be no advantage in introducing two terms which always cancel with one another in different places. It is preferable therefore to retain our first definition [Eq. (3)] when the force on a dislocation is given by the simple expression [Eq. (20)] in every case.

[5] Nabarro (see reference 1) suggests that consideration of the transport of matter can be avoided by assuming dislocations of both signs climbing in the same direction to be present. But if D and E are dislocations of opposite sign (Fig. 5). the strip of material DE between them has still to be given a displacement b as they climb, and this is equivalent to the case we are considering.